Soluzione

①

$$A \cdot B = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 4 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ \frac{2}{3} & 8 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + \frac{1}{2}\cdot\frac{2}{3} + 2\cdot 1 & 1\cdot 6 + \frac{1}{2}\cdot 8 + 2\cdot 1 \\ 0\cdot 4 + 1\cdot\frac{2}{3} + 4\cdot 1 & 0\cdot 6 + 1\cdot 8 + 4\cdot 1 \\ 2\cdot 4 + 2\cdot\frac{2}{3} + 1\cdot 1 & 2\cdot 6 + 2\cdot 8 + 1\cdot 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{19}{3} & 12 \\ \frac{14}{3} & 12 \\ \frac{7}{3} & 29 \end{bmatrix}$$

$$\underline{c}^T \underline{d} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 1\cdot 2 + 2\cdot 1 + 3\cdot 2 = 10$$

$$\underline{c}\,\underline{d}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 & 1\cdot 1 & 1\cdot 2 \\ 2\cdot 2 & 2\cdot 1 & 2\cdot 2 \\ 3\cdot 2 & 3\cdot 1 & 3\cdot 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 6 & 3 & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 4 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 4 \\ 2 & 2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\cdot 1 + \frac{1}{2}\cdot 0 + 2\cdot 2 & 1\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 + 2\cdot 2 & 1\cdot 2 + \frac{1}{2}\cdot 4 + 2\cdot 1 \\ 0\cdot 1 + 1\cdot 0 + 4\cdot 2 & 0\cdot\frac{1}{2} + 1\cdot 1 + 4\cdot 2 & 0\cdot 2 + 1\cdot 4 + 4\cdot 1 \\ 2\cdot 1 + 2\cdot 0 + 1\cdot 2 & 2\cdot\frac{1}{2} + 2\cdot 1 + 1\cdot 2 & 2\cdot 2 + 2\cdot 4 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 6 \\ 8 & 9 & 8 \\ 4 & 5 & 13 \end{bmatrix}$$

2)

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i| = \frac{1}{4} + \frac{3}{2} + 2 = \frac{15}{4}$$

$$\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{2}\right)^2 + 2^2} = \sqrt{\frac{1}{16} + \frac{9}{4} + 4} = \sqrt{\frac{101}{16}} =$$

$$= \frac{1}{4}\sqrt{101} \simeq 2,5125$$

$$\|\underline{x}\|_\infty = \max_{i=1,\dots,n} |x_i| = \max\left\{\frac{1}{4}, \frac{3}{2}, 2\right\} = 2$$

$$\|\underline{y}\|_1 = \sum_{i=1}^{n} |y_i| = \sqrt{10} + \sqrt{17} + 1 \simeq 8,2854$$

$$\|\underline{y}\|_2 = \sqrt{\sum_{i=1}^{n} |y_i|^2} = \sqrt{10 + 17 + 1} = \sqrt{28} = 2\sqrt{7} \simeq 5,2915$$

$$\|\underline{y}\|_\infty = \max_{i=1,\dots,n} |y_i| = \max\{\sqrt{10}, \sqrt{17}, 1\} = \sqrt{17} \simeq 4,1231$$

) $$\|\underline{w}\|_\infty = \max_{i=1,\dots,n} |w_i| = \max\{|\alpha|, 0, 1\} =$$

$$= \begin{cases} 1 & \text{se} \quad 1 > |\alpha| \to -1 < \alpha < 1 \\ |\alpha| & \text{se} \quad |\alpha| \geq 1 \to \alpha \leq -1 \wedge \alpha \geq 1 \end{cases}$$

$$\|\underline{w}\|_2 = \sqrt{\sum_{i=1}^{n} |w_i|^2} = \sqrt{|2|^2+0^2+1^2} = \sqrt{2^2+1}$$

- $$\|\underline{w}\|_1 = \sum_{i=1}^{n} |w_i| = |2|+0+1 = |2|+1$$

Le due norme 1 e 2 sono contemporaneamente unitarie se e solo se $2=0$.

④ Il prodotto scalare

- $$\langle \underline{v}_2, \underline{v}_1 \rangle = \underline{v}_2^T \underline{v}_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1\cdot 1 + (-1)\cdot 2 + 0 \cdot 1 =$$

$$= -1 \neq 0$$ $\underline{v}_2$ non è ortogonale a $\underline{v}_1$

- $$\langle \underline{v}_2, \underline{v}_3 \rangle = \underline{v}_2^T \underline{v}_3 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1\cdot(-1) + (-1)\cdot 0 + 0\cdot 1 =$$

$$= -1 \neq 0$$ $\underline{v}_2$ non è ortogonale a $\underline{v}_3$

- GRAM-SCHMIDT

$$\tilde{\underline{q}}_k = \underline{v}_k - \sum_{j=1}^{k-1} r_{jk} \underline{q}_j \qquad r_{jk} = \langle \underline{q}_j, \underline{v}_k \rangle$$

$$q_k = \frac{\tilde{q}_k}{\|\tilde{q}_k\|}$$

- $\boxed{k=1}$ $$\tilde{\underline{q}}_1 = \underline{v}_1 - \cancel{\sum_{j=1}^{0} r_{j1} q_j} = \underline{v}_1 \qquad \|\tilde{\underline{q}}_1\| = \|\underline{v}_1\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\rightarrow \underline{q}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{6}}{6}\\ \frac{\sqrt{6}}{3}\\ \frac{\sqrt{6}}{6}\end{bmatrix}$$

$$\boxed{=2} \quad \tilde{\underline{q}}_2 = \underline{v}_2 - \sum_{j=1}^{1} r_{j2}\underline{q}_j = \underline{v}_2 - r_{12}\underline{q}_1 = \circledast$$

$$r_{12} = \langle \underline{q}_1, \underline{v}_2 \rangle = \begin{bmatrix}\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6}\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{\sqrt{6}}{6} - \frac{\sqrt{6}}{3} = -\frac{\sqrt{6}}{6}$$

$$\circledast = \begin{bmatrix}1\\-1\\0\end{bmatrix} + \frac{\sqrt{6}}{6}\begin{bmatrix}\sqrt{6}/6\\ \sqrt{6}/3\\ \sqrt{6}/6\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix} + \begin{bmatrix}\frac{1}{6}\\ \frac{1}{3}\\ \frac{1}{6}\end{bmatrix} = \begin{bmatrix}7/6\\ -2/3\\ 1/6\end{bmatrix}$$

$$\|\tilde{\underline{q}}_2\| = \sqrt{\frac{49}{36} + \frac{4}{9} + \frac{1}{36}} = \sqrt{\frac{11}{6}}$$

$$\underline{q}_2 = \sqrt{\frac{6}{11}}\begin{bmatrix}7/6\\ -2/3\\ 1/6\end{bmatrix} = \begin{bmatrix}\frac{7\sqrt{6}}{6\sqrt{11}}\\ -\frac{2}{3}\frac{\sqrt{6}}{\sqrt{11}}\\ \frac{1}{6}\frac{\sqrt{6}}{\sqrt{11}}\end{bmatrix} = \begin{bmatrix}\frac{7}{66}\sqrt{66}\\ -\frac{2}{33}\sqrt{66}\\ \frac{1}{66}\sqrt{66}\end{bmatrix}$$

$$\boxed{=3} \quad \tilde{\underline{q}}_3 = \underline{v}_3 - \sum_{j=1}^{2} r_{j3}\underline{q}_j = \underline{v}_3 - r_{13}\underline{q}_1 - r_{23}\underline{q}_2 \quad \circledast\circledast$$

$$r_{13} = \langle \underline{q}_1, \underline{v}_3 \rangle = \begin{bmatrix}\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6}\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = -\frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{6} = 0$$

$$r_{23} = \langle \underline{q}_2, \underline{v}_3 \rangle = \begin{bmatrix}\frac{7}{66}\sqrt{66} & -\frac{2}{3}\sqrt{66} & \frac{1}{66}\sqrt{66}\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = -\frac{7}{66}\sqrt{66} + \frac{1}{66}\sqrt{66} =$$

$$= -\frac{1}{11}\sqrt{66}$$

$$** = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \cancel{0 \cdot \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}} + \frac{1}{11}\sqrt{66} \begin{bmatrix} 7\sqrt{66}/66 \\ -2\sqrt{66}/66 \\ \sqrt{66}/66 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{7}{11} \\ -\frac{4}{11} \\ \frac{1}{11} \end{bmatrix} = \begin{bmatrix} -\frac{4}{11} \\ -\frac{4}{11} \\ \frac{12}{11} \end{bmatrix}$$

$$\|\tilde{q}_3\| = \sqrt{\frac{16}{121} + \frac{16}{121} + \frac{144}{121}} = \sqrt{\frac{176}{121}} = \frac{4\sqrt{11}}{11}$$

$$q_3 = \frac{11}{4\sqrt{11}} \begin{bmatrix} -\frac{4}{11} \\ -\frac{4}{11} \\ \frac{12}{11} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{11}}{11} \\ -\frac{\sqrt{11}}{11} \\ \frac{3\sqrt{11}}{11} \end{bmatrix}$$