SOLUZIONE

① Lo schema è monostep esplicito e quindi è stabile. Per studiarne la convergenza bisogna stabilire se è consistente.

Errore locale di discretizzazione

$$\tau(x, \ ) = \Delta(x,y) - \phi(x,y)$$

dove $\Delta(x,y) = f(x,y) + \frac{1}{2}(f_x(x,y) + f_y(x,y)\cdot f(x,y))\cdot h + O(h^2)$

$$\phi(x,y) = 6f(x \ , y \ ) - 5f(x \ +4h, \ y \ +4h f(x \ , y \ )) \simeq$$

$$\simeq 6f(x,y) - 5f(x,y) - 5\left[f_x(x,y)\cdot 4 + f_y(x,y)\cdot 4f(x,y)\right]\cdot h$$

$$+ O(h^2) = f(x,y) - 20\left[f_x(x,y) + f_y(x,y)f(x,y)\right]\cdot h + O(h^2)$$

$$\Rightarrow \tau(x,y) = \cancel{f(x,y)} + \frac{1}{2}(f_x(x,y) + f_y(x,y)\cdot f(x,y))\cdot h + O(h^2) +$$

$$- \cancel{f(x,y)} + 20(f_x(x,y) + f_y(x,y)f(x,y))\cdot h + O(h^2) =$$

$$= \frac{41}{2}(f_x(x,y) + f_y(x,y)f(x,y))h + O(h^2)$$

Il metodo è convergente ma non del 2° ordine

$$\begin{cases} x_0 = -\frac{1}{2} \\ \eta_0 = 1 \end{cases}$$

$$\begin{cases} x_1 = -\frac{1}{2} + \frac{1}{2} = 0 \\ \eta_1 = \eta_0 + \frac{1}{2}\left[6f(x_0,\eta_0) - 5f(x_0 + 4\frac{1}{2}, \eta_0 + 4\frac{1}{2}f(x_0,\eta_0))\right] = \end{cases}$$

$$= 1+\frac{1}{2}\left[6f\left(-\frac{1}{2},1\right)-5f\left(-\frac{1}{2}+2,\,1+2f\left(-\frac{1}{2},1\right)\right)\right]=$$

$$=1+\frac{1}{2}\left[\cancel{6}^{3}\cdot\frac{3}{\cancel{2}}-5f\left(\frac{3}{2},4\right)\right]=$$

$$\boxed{f\left(-\frac{1}{2},1\right)=-3\left(-\frac{1}{2}\right)\cdot 1=\frac{3}{2}}$$

$$=1+\frac{1}{2}\left[9-5(-18)\right]=$$

$$f\left(\frac{3}{2},4\right)=-3\left(\frac{3}{\cancel{2}}\right)\cdot\cancel{4}^{2}=-18$$

$$=1+\frac{1}{2}\cdot 99=\frac{101}{2}$$

$$\begin{cases} x_2=0+\frac{1}{2}=\frac{1}{2} \\ \eta_2=\eta_1+\frac{1}{2}\left[6f(x_1,\eta_1)-5f\left(x_1+2,\,\eta_1+2f(x_1,\eta_1)\right)\right]= \end{cases}$$

$$=\frac{101}{2}+\frac{1}{2}\left[6f\left(0,\frac{101}{2}\right)-5f\left(2,\,\frac{101}{2}+2f\left(0,\frac{101}{2}\right)\right)\right]=$$

$$=\frac{101}{2}+\frac{1}{2}\left[-5f\left(2,\frac{101}{2}\right)\right]=$$

$$\boxed{f\left(0,\frac{101}{2}\right)=0}$$

$$=\frac{101}{2}+\frac{1}{2}\left[-5\cdot(-303)\right]=$$

$$f\left(2,\frac{101}{2}\right)=-3\cdot\cancel{2}\cdot\frac{101}{\cancel{2}}=$$

$$=-$$

$$=\frac{101}{2}+\frac{1}{2}\cdot 1515=\frac{1616}{2}=808$$

(2) Metodo di Eulero $\rightarrow \begin{cases} \underline{\eta}_{j+1}=\underline{\eta}_j+hf(x_j,\underline{\eta}_j) \\ \underline{\eta}_0=\underline{y}_0 \end{cases}$

$$\begin{cases} x_0=2 \\ \underline{\eta}_0=\begin{bmatrix}0\\1\end{bmatrix} \end{cases} \qquad \underline{f}=\begin{bmatrix} xy_1-y_2 \\ y_1-\frac{y_2}{x} \end{bmatrix}$$

$$\begin{cases} x_1=2+1/2=\frac{5}{2} \\ \underline{\eta}_1=\underline{\eta}_0+\frac{1}{2}f(x_0,\underline{\eta}_0)=\begin{bmatrix}0\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix} x_0\eta_{01}-\eta_{02} \\ \eta_{01}-\frac{1}{x_0}\eta_{02} \end{bmatrix}= \end{cases}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \cdot 0 - 1 \\ 0 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/4 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix}$$

$$\begin{cases} x_2 = \frac{5}{2} + \frac{1}{2} = 3 \\ \eta_2 = \eta_1 + \frac{1}{2} f(x_1, \eta_1) = \begin{bmatrix} -1/2 \\ 3/4 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 \eta_{11} - \eta_{12} \\ \eta_{11} - \frac{1}{x_1} \eta_{12} \end{bmatrix} = \end{cases}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{5}{2} \cdot \left(-\frac{1}{2}\right) - \frac{3}{4} \\ -\frac{1}{2} - \frac{2}{5}\left(\frac{3}{4}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - 1 \\ \frac{3}{4} - \frac{2}{5} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{3}{2} \\ \frac{7}{20} \end{bmatrix}$$

3) $f(x, y) = -y^2 + 2$

$$\frac{\partial f}{\partial y} = -2y$$

non limitata ma continua. Quindi la soluzione esiste ed è unica in un intorno di $x_0 = 0$.

$$\begin{cases} x_0 = 0 \\ \eta_0 = 1 \end{cases}$$

$$f(0, 1) = -1 + 2 = 1$$

$$\begin{cases} x_1 = 0 + \frac{1}{2} = \frac{1}{2} \\ \eta_1 = \eta_0 + h f(x_0, \eta_0) = 1 + \frac{1}{2} f(0, 1) = 1 + \frac{1}{2} \cdot 1 = \frac{3}{2} \end{cases}$$

$$\begin{cases} x_2 = \frac{1}{2} + \frac{1}{2} = 1 \\ \eta_2 = \eta_1 + h f(x_1, \eta_1) = \frac{3}{2} + \frac{1}{2} f\left(\frac{1}{2}, \frac{3}{2}\right) = \end{cases}$$

$$= \frac{3}{2} + \frac{1}{2}\left(-\frac{1}{4}\right) = \frac{11}{8} \qquad f\left(\frac{1}{2}, \frac{3}{2}\right) = -\frac{9}{4} + 2 = -\frac{1}{4}$$

$$\begin{cases} x_3 = 1 + \frac{1}{2} = \frac{3}{2} \\ \eta_3 = \eta_2 + \frac{1}{2} f(x_2, \eta_2) = \frac{11}{8} + \frac{1}{2} f\left(1, \frac{11}{8}\right) = \frac{11}{8} + \frac{1}{2} \cdot \frac{7}{64} = \frac{183}{128} \end{cases} \qquad f\left(1, \frac{11}{8}\right) = -\frac{121}{64} + 2 = \frac{7}{64}$$

4) $z_1 = y$
$z_2 = y'$

$$\underline{f} = \begin{bmatrix} z_2 \\ (z_2 - 1)x - z_1 \end{bmatrix}$$

$$\begin{cases} z_1' = z_2 \\ z_2' = (z_2 - 1)x - z_1 \\ z_1\left(\frac{1}{2}\right) = 1 \quad z_2\left(\frac{1}{2}\right) = 0 \end{cases}$$

$$\begin{cases} x_0 = \frac{1}{2} \\ \underline{\eta}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} x_1 = \frac{1}{2} + \frac{1}{2} = 1 \\ \underline{\eta}_1 = \underline{\eta}_0 + h \underline{f}(x_0, \underline{\eta}_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \eta_{02} \\ (\eta_{02} - 1)x_0 - \eta_{01} \end{bmatrix} = \end{cases}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ (0-1) \cdot \frac{1}{2} - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ -\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix}$$

$$\begin{cases} x_2 = 1 + \frac{1}{2} = \frac{3}{2} \\ \underline{\eta}_2 = \underline{\eta}_1 + h\,\underline{f}(x_1, \underline{\eta}_1) = \begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \eta_{12} \\ (\eta_{12}-1)\,x_1 - \eta_{11} \end{bmatrix} = \end{cases}$$

$$= \begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -3/4 \\ \left(-\frac{3}{4}-1\right)\cdot 1 - 1 \end{bmatrix} = \begin{bmatrix} 1-3/8 \\ -\frac{3}{4}-\frac{11}{8} \end{bmatrix} = \begin{bmatrix} 5/8 \\ -\frac{17}{8} \end{bmatrix}$$

5) a) è monostep, esplicito a 2 stadi

b) è multistep, esplicito a 1 stadio

6) a) essendo monostep esplicito è stabile $\forall\, \alpha, \beta \in \mathbb{R}$.

$$\tau(x,y) = \Delta(x,y) - \phi(x,y)$$

$$\Delta(x,y) = f(x,y) + \frac{1}{2}\left(f_x(x,y) + f_y(x,y)\, f(x,y)\right) h + O(h^2)$$

$$\phi(x,y) = 2\alpha\left[3f(x,y) + f(x+2\beta h,\, y + 2\beta h f(x,y))\right] \simeq$$

$$\simeq 2\alpha \Big\{ 3f(x,y) + f(x,y) + \left(f_x(x,y) + f_y(x,y) f(x,y)\right) 2\beta \cdot h +$$

$$+ O(h^2) \Big\} = 8\alpha f(x,y) + \quad \left(f_x(x,y) + f_y(x,y) f(x,y)\right) h$$

$$+ O(h^2)$$

$$\Rightarrow \tau(x,y) = (1-8\alpha) f(x,y) + \left(\frac{1}{2} - 4\alpha\beta\right)\left(f_x(x,y) + f_y(x,y)\, f(x,y)\right) h$$

$$+ O(h^2)$$

Il metodo è convergente se $1-8\alpha=0 \to \alpha=\frac{1}{8}$

ed è del secondo ordine se, posto $\alpha=\frac{1}{8}$,

$\frac{1}{2}-4\alpha\beta=\frac{1}{2}-\frac{\beta}{2}=0 \to \beta=1$.