SOLUZIONE

1) $\eta_{k+2} - (\gamma-3)\,\eta_{k+1} = h\left[\left(2+\frac{\gamma}{5}\right) f(x_k,\eta_k) - 2\frac{\gamma}{5} f(x_{k+1},\eta_{k+1})\right]$

$a_2 = 1 \qquad\qquad b_2 = 0$

$a_1 = 3-\gamma \qquad\qquad b_1 = -2\frac{\gamma}{5}$

$a_0 = 0 \qquad\qquad b_0 = 2+\frac{\gamma}{5}$

$$\rho(x) = \sum_{j=0}^{k} a_j x^j = \sum_{j=0}^{2} a_j x^j = (3-\gamma)x + x^2 = 0$$

$$x\left[(3-\gamma)+x\right] = 0 \to \begin{cases} x = 0 \\ x = \gamma - 3 \end{cases}$$

$$|\gamma-3| \le 1 \to -1 \le \gamma - 3 \le 1 \to 2 \le \gamma \le 4$$

2) $\eta_{i+2} - \frac{4}{3}\eta_{i+1} + \frac{1}{3}\eta_i = h\left[\frac{2}{3} f(x_{i+1},\eta_{i+2})\right]$

$a_2 = 1 \qquad\qquad b_2 = \frac{2}{3}$

$a_1 = -\frac{4}{3} \qquad\qquad b_1 = 0$

$a_0 = \frac{1}{3} \qquad\qquad b_0 = 0$

Stabilità $\to \sum_{j=0}^{2} a_j x^j = \frac{1}{3} - \frac{4}{3}x + x^2 = 0 \to$

$\to 3x^2 - 4x + 1 = 0$

$$x_{1/2} = \frac{4 \pm \sqrt{16-12}}{6} = \begin{cases} 1 \\ \frac{1}{3} \end{cases}$$

Il metodo è stabile.

consistenza $\to$ $\tau(x,h) = \frac{1}{h}\sum_{j=0}^{k} a_j\, y(x+jh) - \sum_{j=0}^{k} b_j\, y'(x+jh)$

$$= \frac{1}{h}\left\{\frac{1}{3}y(x) - \frac{4}{3}y(x+h) + y(x+2h)\right\} - \frac{2}{3}y'(x+2h) \simeq$$

$$\simeq \frac{1}{h}\left\{\frac{1}{3}\cancel{y(x)} - \frac{4}{3}\left[\cancel{y(x)} + y'(x)\cdot h + y''(x)\frac{h^2}{2} + O(h^3)\right] +\right.$$

$$\left. + \left[\cancel{y(x)} + y'(x)\cdot 2h + y''(x)\cdot\frac{\cancel{4}^{2}h^2}{\cancel{2}} + O(h^3)\right]\right\} +$$

$$-\frac{2}{3}\left[y'(x) + y''(x)2h + O(h^2)\right] =$$

$$= -\frac{4}{3}\cancel{y'(x)} + \cancel{2y'(x)} - \cancel{\frac{2}{3}y'(x)} - \frac{2}{3}y''(x)\cdot h + \cancel{2y''(x)}\cdot h +$$

$$-\frac{4}{3}y''(\cancel{x})\cdot h + O(h^2) = O(h^2)$$ è del 2° ordine

(3) lo schema è multistep, esplicito a uno stadio.

$$\eta_{k+2} - \frac{3}{2}\eta_{k+1} + \gamma\eta_k = h\left[\frac{1}{2}f(x_k, \eta_k)\right]$$

$a_0 = \gamma$ $\qquad$ $b_0 = \frac{1}{2}$

$a_1 = -\frac{3}{2}$ $\qquad$ $b_1 = b_2 = 0$

$a_2 = 1$

STABILITÀ $\to$ $\sum_{j=0}^{2} a_j x^j = \gamma - \frac{3}{2}x + x^2 \to 2x^2 - 3x + 2\gamma = 0$

$$x_{1/2} = \frac{3 \pm \sqrt{9-16\gamma}}{4} \to \gamma = \frac{1}{2} \to x_1 = 1,\ x_2 = 1/2 \quad \text{stabile}$$

$$\to \gamma = \frac{3}{4} \to x_{1/2} = \frac{3 \pm i\sqrt{3}}{4}$$

$$\left| \frac{3 \pm i\sqrt{3}}{4} \right| = \sqrt{\frac{9}{16} + \frac{3}{16}} = \sqrt{\frac{3}{4}} < 1 \quad \text{stabile}$$

CONSISTENZA $\to \tau(x,h) = \frac{1}{h} \sum_{j=0}^{r} a_j y(x+jh) - \sum_{j=0}^{r} b_j y'(x+jh) =$

$$= \frac{1}{h} \left\{ \gamma y(x) - \frac{3}{2} y(x+h) + y(x+2h) \right\} - \frac{1}{2} y'(x) =$$

$$\simeq \frac{1}{h} \left\{ \gamma y(x) - \frac{3}{2} \left[y(x) + y'(x)h + y''(x)\frac{h^2}{2} + O(h^3) \right] + \right.$$

$$\left. + \left[y(x) + y'(x)2h + y''(x)\frac{4h^2}{2} + O(h^3) \right] \right\} - \frac{1}{2} y'(x) =$$

$$= \frac{1}{h}\left(\gamma - \frac{3}{2} + 1\right) y(x) + \left(\cancel{-\frac{3}{2}} + \cancel{2} - \cancel{\frac{1}{2}}\right) y'(x) + \left(-\frac{3}{4} + 2\right) y''(x) h \quad \Rightarrow$$

$\gamma = \frac{1}{2} \to \frac{1}{2} - \frac{3}{2} + 1 = 0 \to$ consistente

$\gamma = \frac{3}{4} \to \frac{3}{4} - \frac{3}{2} + 1 = \frac{1}{4} \neq 0 \to$ non consistente

Per $\gamma = \frac{1}{2}$ il metodo è convergente, per $\gamma = \frac{3}{4}$ no.

④ $\eta_{i+2}+\eta_i = h[5f(x_i,\eta_i)]$

$a_2=1$
$a_1=0$
$a_0=1$

STABILITÀ $\to \sum_{j=0}^{r} a_j x^j = 1+x^2=0 \to x_{1/2}=\pm i$

$|i|=1 \to$ stabile

⑤ $\eta_{k+2}-\frac{3}{2}\eta_{k+1}+\frac{1}{2}\eta_k = h\left[\frac{1}{2}f(x_k,\eta_k)\right]$

$a_2=1$ $\qquad b_0=\frac{1}{2}$
$a_1=-\frac{3}{2}$ $\qquad b_1=b_2=0$
$a_0=\frac{1}{2}$

STABILITÀ $\to \sum_{j=0}^{r} a_j x^j = \frac{1}{2}-\frac{3}{2}x+x^2 \to 2x^2-3x+1=0$

$x_{1/2}=\frac{3\pm\sqrt{9-8}}{4} = \begin{cases}1\\ 1/2\end{cases}$ stabile

CONSISTENZA $\to \tau(x,h)=\frac{1}{h}\sum_{j=0}^{r} a_j y(x+jh) - \sum_{j=0}^{r} b_j y'(x+jh) =$

$=\frac{1}{h}\left\{\frac{1}{2}y(x)-\frac{3}{2}y(x+h)+y(x+2h)\right\}-\frac{1}{2}y'(x) \simeq$

$\simeq \frac{1}{h}\left\{\cancel{\frac{1}{2}y(x)}-\frac{3}{2}\left[\cancel{y(x)}+y'(x)h+y'(x)\frac{h^2}{2}\right]+\left[\cancel{y(x)}+y'(x)2h+y''(x)2h^2\right]\right.$
$\left.+O(h^3)\right\}-\frac{1}{2}y'(x) =$

$$= \left(-\cancel{\frac{3}{2}} + \cancel{2} - \cancel{\frac{1}{2}}\right) y'(x) + \left(-\frac{3}{4} + 2\right) y''(x) h + O(h^2) = O(h)$$

è consistente del 1° ordine.