SOLUZIONE

① La funzione non è nè pari nè dispari.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\omega x)\,dx$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{sen}(k\omega x)\,dx$$

dove $L = 2 \quad \omega = \frac{\pi}{L} = \frac{\pi}{2}$.

$$a_0 = \frac{1}{2\cdot 2}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\left\{\int_{-2}^{0} 1\cdot dx + \int_{0}^{2}(2+x)\,dx\right\} =$$

$$= \frac{1}{4}\left\{\left[x\right]_{-2}^{0} + \left[2x+\frac{x^2}{2}\right]_{0}^{2}\right\} = \frac{1}{4}\left\{0+2+4+2-0-0\right\} =$$

$$= \frac{1}{\cancel{4}}\cdot\cancel{8}^{\,2} = 2$$

$$a_k = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(k\frac{\pi}{2}x\right)dx =$$

$$= \frac{1}{2}\left\{\int_{-2}^{0} 1\cdot\cos\left(k\frac{\pi}{2}x\right)dx + \int_{0}^{2}(2+x)\cos\left(k\frac{\pi}{2}x\right)dx\right\} =$$

$$= \frac{1}{2}\left\{\left[\frac{\text{sen}\left(k\frac{\pi}{2}x\right)}{k\frac{\pi}{2}}\right]_{-2}^{0} + \left[(2+x)\frac{\text{sen}\left(k\frac{\pi}{2}x\right)}{k\frac{\pi}{2}}\right]_{0}^{2} - \int_{0}^{2}\frac{\text{sen}\left(k\frac{\pi}{2}x\right)}{k\frac{\pi}{2}}dx\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\overset{0}{\cancel{\text{sen}(0)}} - \overset{0}{\cancel{\text{sen}(-k\pi)}}}{k\frac{\pi}{2}}\right] + \left[\frac{4\cdot \overset{0}{\cancel{\text{sen}(k\pi)}} - 2\,\overset{0}{\cancel{\text{sen}(0)}}}{k\frac{\pi}{2}}\right] - \left[-\frac{\cos\left(k\frac{\pi}{2}x\right)}{\left(k\frac{\pi}{2}\right)^2}\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cos(k\pi) - \cos(0)}{\left(k\frac{\pi}{2}\right)^2}\right\} = \frac{1}{\cancel{2}_{1}}\cdot\frac{\cancel{4}^{2}}{k^2\pi^2}\left((-1)^k - 1\right) = \frac{2}{k^2\pi^2}\left((-1)^k - 1\right)$$

$$b_k = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(k\frac{\pi}{2}x\right)dx =$$

$$= \frac{1}{2}\left\{\int_{-2}^{0} 1\cdot\sin\left(k\frac{\pi}{2}x\right)dx + \int_{0}^{2}(2+x)\sin\left(k\frac{\pi}{2}x\right)dx\right\} =$$

$$= \frac{1}{2}\left\{\left[-\frac{\cos\left(k\frac{\pi}{2}x\right)}{k\frac{\pi}{2}}\right]_{-2}^{0} + \left[(2+x)\left(-\frac{\cos\left(k\frac{\pi}{2}x\right)}{k\frac{\pi}{2}}\right)\right]_{0}^{2} - \int_{0}^{2} -\frac{\cos\left(k\frac{\pi}{2}x\right)}{k\frac{\pi}{2}}dx\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-\cos(0) + \cos(-k\pi)}{k\frac{\pi}{2}}\right] + \left[\frac{4(-\cos(k\pi)) - 2(-\cos(0))}{k\frac{\pi}{2}}\right] + \left[\frac{\sin\left(k\frac{\pi}{2}x\right)}{\left(k\frac{\pi}{2}\right)^2}\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{-1 + (-1)^k}{k\frac{\pi}{2}} + \frac{-4(-1)^k + 2}{k\frac{\pi}{2}} + \frac{\cancel{\sin(k\pi)} - \cancel{\sin(0)}}{\left(k\frac{\pi}{2}\right)^2}\right\} =$$

$$=\frac{1}{2}\left\{\frac{-1+2+(1-4)(-1)^k}{k\frac{\pi}{2}}\right\}=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{k\pi}\left(1-3(-1)^k\right)=\frac{1}{k\pi}\left(1-3(-1)^k\right)$$

Lo sviluppo è:

$$S_f(x)=2+\sum_{k=1}^{\infty}\frac{2}{k^2\pi^2}\left((-1)^k-1\right)\cos\left(k\frac{\pi}{2}x\right)+\frac{1}{k\pi}\left(1-3(-1)^k\right)\sin\left(k\frac{\pi}{2}x\right)$$

② La funzione non è né pari né dispari.

$L=3 \quad \omega=\frac{\pi}{3}$

$$a_0=\frac{1}{2\cdot 3}\int_{-3}^{3}(2x+5)\,dx=\frac{1}{6}\left[x^2+5x\right]_{-3}^{3}=\frac{1}{6}\left[\cancel{9}+15-\cancel{9}+15\right]=$$

$$=\frac{1}{\cancel{6}}\cdot\cancel{30}^{5}=5$$

$$a_k=\frac{1}{3}\int_{-3}^{3}(2x+5)\cos\left(k\frac{\pi}{3}x\right)dx=$$

$$=\frac{1}{3}\left\{\left[(2x+5)\frac{\operatorname{sen}\left(k\frac{\pi}{3}x\right)}{k\frac{\pi}{3}}\right]_{-3}^{3}-\int_{-3}^{3}2\cdot\frac{\operatorname{sen}\left(k\frac{\pi}{3}x\right)}{k\frac{\pi}{3}}dx\right\}=$$

$$=\frac{1}{3}\left\{\left[\frac{\cancel{11\cdot\operatorname{sen}(k\pi)}+\cancel{\operatorname{sen}(-k\pi)}}{k\frac{\pi}{3}}\right]-2\left[-\frac{\cos\left(k\frac{\pi}{3}x\right)}{\left(k\frac{\pi}{3}\right)^2}\right]_{-3}^{3}\right\}=$$

$$=\frac{1}{3}\left\{+2\left(\frac{\cos(k\pi)-\cos(-k\pi)}{\left(k\frac{\pi}{3}\right)^2}\right)\right\}=\frac{2}{3}\left\{\frac{\cancel{(-1)^k}-\cancel{(-1)^k}}{\left(k\frac{\pi}{3}\right)^2}\right\}=0$$

$$b_k = \frac{1}{3}\int_{-3}^{3} (2x+5)\sin\left(k\frac{\pi}{3}x\right)dx =$$

$$= \frac{1}{3}\left\{\left[(2x+5)\left(-\frac{\cos\left(k\frac{\pi}{3}x\right)}{k\frac{\pi}{3}}\right)\right]_{-3}^{3} - \int_{-3}^{3} 2\left(-\frac{\cos\left(k\frac{\pi}{3}x\right)}{k\frac{\pi}{3}}\right)dx\right\} =$$

$$= \frac{1}{3}\left\{\frac{11\cdot(-\cos(k\pi)) + (-\cos(-k\pi))}{k\frac{\pi}{3}} + 2\left[\frac{\operatorname{sen}\left(k\frac{\pi}{3}x\right)}{\left(k\frac{\pi}{3}\right)^2}\right]_{-3}^{3}\right\} =$$

$$= \frac{1}{3}\left\{\frac{-12\cdot(-1)^k}{k\frac{\pi}{3}} + 2\left[\frac{\cancel{\operatorname{sen}(k\pi)} - \cancel{\operatorname{sen}(-k\pi)}}{\left(k\frac{\pi}{3}\right)^2}\right]\right\} =$$

$$= \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{k\pi}\left(-12(-1)^k\right) = -\frac{12}{k\pi}(-1)^k$$

Quindi:

$$Sf(x) = 5 - \sum_{k=1}^{\infty} \frac{12}{k\pi}(-1)^k \sin\left(k\frac{\pi}{3}x\right)$$

③ La funzione è pari, quindi $b_k = 0$ e

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{e} \quad a_k = \frac{2}{L}\int_0^L f(x)\cos(k\omega x)\,dx$$

$L = 2 \qquad \omega = \frac{\pi}{2}$

$$a_0=\frac{1}{2}\int_0^2 \cos\frac{\pi}{6}x\cdot dx=\frac{1}{2}\left[\frac{\operatorname{sen}\frac{\pi}{6}x}{\frac{\pi}{6}}\right]_0^2=$$

$$=\frac{1}{2}\left(\frac{\operatorname{sen}\frac{\pi}{3}-\cancel{\operatorname{sen}(0)}}{\frac{\pi}{6}}\right)=\frac{1}{2}\cdot\frac{\overset{3}{\cancel{6}}}{\pi}\cdot\frac{\sqrt{3}}{2}=\frac{3}{\pi}\frac{\sqrt{3}}{2}$$

$$a_k=\frac{2}{2}\int_0^2 \cos\frac{\pi}{6}x\cdot\cos\left(k\frac{\pi}{2}x\right)dx=$$

$$\boxed{\cos(\alpha)\cos(\beta)=\frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)}$$

$$=\int_0^2\frac{1}{2}\left[\cos\left(\left(\frac{\pi}{6}+k\frac{\pi}{2}\right)x\right)+\cos\left(\left(\frac{\pi}{6}-k\frac{\pi}{2}\right)x\right)\right]dx=$$

$$=\frac{1}{2}\left\{\left[\frac{\operatorname{sen}\left(\frac{\pi}{6}+k\frac{\pi}{2}\right)x}{\frac{\pi}{6}+k\frac{\pi}{2}}+\frac{\operatorname{sen}\left(\frac{\pi}{6}-k\frac{\pi}{2}\right)x}{\frac{\pi}{6}-k\frac{\pi}{2}}\right]_0^2\right\}=$$

$$=\frac{1}{2}\left\{\frac{\operatorname{sen}\left(\frac{\pi}{6}+k\frac{\pi}{2}\right)\cdot 2-\cancel{\operatorname{sen}(0)}}{\frac{\pi}{6}+k\frac{\pi}{2}}+\frac{\operatorname{sen}\left(\frac{\pi}{6}-k\frac{\pi}{2}\right)2-\cancel{\operatorname{sen}(0)}}{\frac{\pi}{6}-k\frac{\pi}{2}}\right\}=$$

$$=\frac{1}{2}\left\{\frac{\operatorname{sen}\left(\frac{\pi}{3}+k\pi\right)}{\frac{\pi}{6}-k\frac{\pi}{2}}+\frac{\operatorname{sen}\left(\frac{\pi}{3}-k\pi\right)}{\frac{\pi}{6}-k\frac{\pi}{2}}\right\}=$$

$$\boxed{\operatorname{sen}\left(\frac{\pi}{3}\pm k\pi\right)=\operatorname{sen}\left(\frac{\pi}{3}\right)\cos(k\pi)\pm\cancel{\operatorname{sen}\overset{=0}{(k\pi)}\cos\left(\frac{\pi}{3}\right)}=\frac{\sqrt{3}}{2}(-1)^k}$$

$$= \frac{1}{2}\left\{ \frac{\frac{\sqrt{3}}{2}(-1)^k}{\frac{\pi}{6}+k\frac{\pi}{2}} + \frac{\frac{\sqrt{3}}{2}(-1)^k}{\frac{\pi}{6}-k\frac{\pi}{2}} \right\} =$$

$$= \frac{\sqrt{3}}{2\cdot 2}\left\{ \frac{\left(\frac{\pi}{6}-\cancel{k\frac{\pi}{2}}\right)(-1)^k + \left(\frac{\pi}{6}+\cancel{k\frac{\pi}{2}}\right)(-1)^k}{\frac{\pi^2}{36}-k^2\frac{\pi^2}{4}} \right\} = \frac{\sqrt{3}}{4}\left\{ \frac{\frac{\pi}{3}(-1)^k}{\frac{\pi^2}{36}-k^2\frac{\pi^2}{4}} \right\} =$$

$$= \frac{\sqrt{3}}{4}\left\{ \frac{\frac{\pi}{3}(-1)^k}{\frac{\pi^2-9k^2\pi^2}{36}} \right\} = \frac{\sqrt{3}}{4}\left\{ \frac{\frac{\pi}{3}(-1)^k}{\frac{\pi^2(1-9k^2)}{36}} \right\} =$$

$$= \frac{\sqrt{3}}{\cancel{4}} \cdot \frac{\overset{9\ 3}{\cancel{36}}}{\pi^2(1-9k^2)} \cdot \frac{\pi}{\cancel{3}}(-1)^k = \frac{3\sqrt{3}\,\cancel{\pi}}{\pi^2(1-9k^2)}(-1)^k = \frac{3\sqrt{3}}{\pi(1-9k^2)}(-1)^k$$

Il denominatore non si annulla per valori interi di k.

Quindi $f(x) = \frac{3\sqrt{3}}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-9k^2)}\cos\left(k\frac{\pi}{3}x\right)\right)$

4) la funzione è pari infatti $f(-x) = f(x)$.

In generale per la forma complessa si ha

$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ e $c_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\omega x}\,dx$

con $L = \frac{1}{2}$ e $\omega = 2\pi$

In questo caso

$$c_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{\frac{1}{2}}\int_0^{1/2} (-2+x)\,dx =$$

$$= 2\left[-2x + \frac{x^2}{2}\right]_0^{1/2} = 2\left[-1 + \frac{1}{8} + 0 - 0\right] = \cancel{2}\left(-\frac{7}{\cancel{8}_4}\right) = -\frac{7}{4}$$

$$c_k = \frac{1}{2\cdot\frac{1}{2}}\int_{-\frac{1}{2}}^{1/2} f(x)\, e^{-ik2\pi x}\,dx =$$

$$= \int_{-\frac{1}{2}}^{0} (-2-x)\, e^{-ik2\pi x}\,dx + \int_0^{1/2} (-2+x)\, e^{-ik2\pi x}\,dx =$$

$$= \left[(-2-x)\frac{e^{-ik2\pi x}}{-ik\pi}\right]_{-\frac{1}{2}}^{0} + \int_{-1/2}^{0} \frac{e^{-ik2\pi x}}{-ik2\pi}\,dx +$$

$$+ \left[(-2+x)\frac{e^{-ik\pi x}}{-ik\pi}\right]_0^{1/2} - \int_0^{1/2} \frac{e^{-ik\pi x}}{-ik\pi}\,dx =$$

$$= \frac{-\cancel{2\cdot 1} + \frac{3}{2}e^{+ik\pi}}{-ik\pi} + \left[\frac{e^{-ik\pi x}}{(-ik\pi)^2}\right]_{-1/2}^{0} + \frac{-\frac{3}{2}e^{-ik\pi} + \cancel{2\cdot 1}}{-ik2\pi} +$$

$$-\left[\frac{e^{-ik\pi x}}{(-ik\pi)^2}\right]_0^{1/2} =$$

$$= \frac{\frac{3}{2}\left(\cancel{e^{ik\pi}} - \cancel{e^{-ik\pi}}\right)}{-ik\pi} + \frac{1 - e^{ik\pi} - e^{-ik\pi} + 1}{(-ik\pi)^2} =$$

$$\boxed{\begin{aligned} e^{ik\pi} &= \cos(k\pi) + i\,\cancel{\mathrm{sen}(k\pi)} = (-1)^k \\ e^{-ik\pi} &= \cos(k\pi) - i\,\cancel{\mathrm{sen}(k\pi)} = (-1)^k \end{aligned}}$$

$$= \frac{2 - 2(-1)^k}{(-ik\pi)^2} = -\frac{\cancel{2}}{k^2 \cancel{4}_2 \pi^2}\left(1-(-1)^k\right) = -\frac{1}{2k^2\pi^2}\left(1-(-1)^k\right)$$

Quindi

$$Sf(x) = -\frac{7}{4} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{2k^2\pi^2}\left(1-(-1)^k\right) e^{2ik\pi x}$$