

SOLUZIONE

② @ la funzione è pari quindi $b_n = 0$.

$$a_0 = \frac{1}{2} \left\{ \int_0^1 (2-x) dx + \int_1^2 1 \cdot dx \right\} = \frac{1}{2} \left\{ \left[2x - \frac{x^2}{2} \right]_0^1 + [x]_1^2 \right\} = \\ = \frac{1}{2} \left\{ 2 - \frac{1}{2} + 2 - 1 \right\} = \frac{5}{4}$$

$$a_n = \int_0^1 (2-x) \cos(k \frac{\pi}{2} x) dx + \int_1^2 1 \cdot \cos(k \frac{\pi}{2} x) dx =$$

$$= \left[(2-x) \frac{\operatorname{sen}(k \frac{\pi}{2} x)}{k \frac{\pi}{2}} \right]_0^1 - \left[\frac{\cos(k \frac{\pi}{2} x)}{(k \frac{\pi}{2})^2} \right]_0^1 + \left[\frac{\operatorname{sen}(k \frac{\pi}{2} x)}{k \frac{\pi}{2}} \right]_1^2 =$$

$$= \frac{\operatorname{sen} k \frac{\pi}{2}}{k \frac{\pi}{2}} - \frac{\cos k \frac{\pi}{2} - 1}{(k \frac{\pi}{2})^2} - \frac{\operatorname{sen} k \frac{\pi}{2}}{k \frac{\pi}{2}} = \frac{4}{\pi^2 n^2} (1 - \cos k \frac{\pi}{2})$$

$$S_y(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos(k \frac{\pi}{2} x) + \tilde{b}_n \operatorname{sen}(k \frac{\pi}{2} x)$$

$$S_{y'}(x) = \sum_{n=1}^{\infty} -\tilde{a}_n k \frac{\pi}{2} \operatorname{sen}(k \frac{\pi}{2} x) + \tilde{b}_n k \frac{\pi}{2} \cos(k \frac{\pi}{2} x)$$

$$y' + \sqrt{2} y = f(x) \Rightarrow S_{y'}(x) + \sqrt{2} S_y(x) = S_f(x) \rightarrow$$

$$\rightarrow \sum_{n=1}^{\infty} -\tilde{a}_n k \frac{\pi}{2} \operatorname{sen}(k \frac{\pi}{2} x) + \tilde{b}_n k \frac{\pi}{2} \cos(k \frac{\pi}{2} x) +$$

$$+ \sqrt{2} \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{b}_n \cos(k \frac{\pi}{2} x) + \sqrt{2} \tilde{b}_0 \operatorname{sen}(k \frac{\pi}{2} x) =$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n \frac{\pi}{2} x) \rightarrow$$

$$\rightarrow \sqrt{2} \tilde{a}_0 + \sum_{n=1}^{\infty} (-\tilde{a}_n n \frac{\pi}{2} + \sqrt{2} \tilde{b}_n) \sin(n \frac{\pi}{2} x) +$$

$$+ (\tilde{b}_n n \frac{\pi}{2} + \sqrt{2} \tilde{a}_n) \cos(n \frac{\pi}{2} x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \frac{\pi}{2} x)$$

$$\rightarrow \sqrt{2} \tilde{a}_0 = a_0 \quad \rightarrow \tilde{a}_0 = \frac{a_0}{\sqrt{2}} = \frac{5}{2\sqrt{2}}$$

$$\rightarrow -\tilde{a}_n n \frac{\pi}{2} + \sqrt{2} \tilde{b}_n = 0$$

$$\tilde{b}_n n \frac{\pi}{2} + \sqrt{2} \tilde{a}_n = a_n$$

la matrice dei coefficienti del sistema nelle incognite \tilde{a}_n e \tilde{b}_n è

$$\begin{bmatrix} -k \frac{\pi}{2} & \sqrt{2} \\ \sqrt{2} & k \frac{\pi}{2} \end{bmatrix}$$

il cui determinante è

$$-(k \frac{\pi}{4})^2 + 2 \neq 0$$

che non si annulla mai.

Dalla prima equazione ottieniamo

$$\sqrt{2} \tilde{b}_n = \tilde{a}_n n \frac{\pi}{2} \rightarrow \tilde{b}_n = \frac{k n}{2\sqrt{2}} \tilde{a}_n$$

che sostituito nella seconda

$$\left(\frac{k n}{2\sqrt{2}} \tilde{a}_n \right) k \frac{\pi}{2} + \sqrt{2} \tilde{a}_n = a_k \rightarrow \left(\frac{k^2 n^2}{4\sqrt{2}} + \sqrt{2} \right) \tilde{a}_n = a_k$$

$$\rightarrow \tilde{a}_n = \frac{a_n}{\left(\frac{k^2 n^2}{4\sqrt{2}} + \sqrt{2} \right)} \rightarrow \tilde{a}_n = \frac{a_n}{\left(\frac{k^2 n^2 + 8}{4\sqrt{2}} \right)} \rightarrow$$

$$\rightarrow \tilde{a}_n = \frac{4\sqrt{2}}{\kappa^2 n^2 + 8} \quad a_n = \frac{4\sqrt{2}}{\kappa^2 n^2 + 8} \left(\frac{4}{\kappa^2 n^2} (1 - \cos \kappa \frac{n\pi}{2}) \right) = \\ = \frac{16\sqrt{2}}{\kappa^2 n^2 (\kappa^2 n^2 + 8)} (1 - \cos \kappa \frac{n\pi}{2})$$

$$\tilde{b}_n = \frac{\kappa n}{2\sqrt{2}} \quad \tilde{a}_n = \frac{\kappa n}{2\sqrt{2}} \left(\frac{16\sqrt{2}}{\kappa^2 n^2 (\kappa^2 n^2 + 8)} (1 - \cos \kappa \frac{n\pi}{2}) \right) = \\ = \frac{8}{\kappa n (\kappa^2 n^2 + 8)} (1 - \cos \kappa \frac{n\pi}{2})$$

$$S(x) = \frac{5}{2\sqrt{2}} + \sum_{n=1}^{\infty} \frac{16\sqrt{2}}{\kappa^2 n^2 (\kappa^2 n^2 + 8)} (1 - \cos \kappa \frac{n\pi}{2}) \cos \left(\kappa \frac{n\pi}{2} x \right) + \\ + \frac{8}{\kappa n (\kappa^2 n^2 + 8)} (1 - \cos \kappa \frac{n\pi}{2}) \operatorname{sen} \left(\kappa \frac{n\pi}{2} x \right)$$

$$\textcircled{B} \quad a_0 = \frac{1}{2} \int_{-1}^1 (2x+1) dx = \frac{1}{2} \left[x^2 + x \right]_{-1}^1 = \frac{1}{2} \left\{ 1 + 1 - 1 + 1 \right\} = \textcircled{A}$$

$$a_n = \int_{-1}^1 (2x+1) \cos(\kappa n x) dx = \left[(2x+1) \frac{\operatorname{sen}(\kappa n x)}{\kappa n} \right]_{-1}^1 + 2 \left[\frac{\cos(\kappa n x)}{(\kappa n)^2} \right]_{-1}^1$$

$$= \frac{2}{\kappa^2 n^2} (\cos(\kappa n) - \cos(-\kappa n)) = 0$$

$$b_n = \int_{-1}^1 (2x+1) \operatorname{sen}(\kappa n x) dx = \left[(2x+1) \left(-\frac{\cos(\kappa n x)}{\kappa n} \right) \right]_{-1}^1 + 2 \left[\frac{\operatorname{sen}(\kappa n x)}{(\kappa n)^2} \right]_{-1}^1$$

$$= -\frac{1}{k\pi} \left(3(-1)^k + (-1)^k \right) = -\frac{4}{k\pi} (-1)^k$$

$$Sy(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos(n\pi x) + \tilde{b}_n \sin(n\pi x)$$

$$Sy''(x) = \sum_{n=1}^{\infty} -\tilde{a}_n (n\pi)^2 \cos(n\pi x) - \tilde{b}_n (n\pi)^2 \sin(n\pi x)$$

$$2y'' + y = f(x) \quad \rightarrow 2Sy''(x) + Sy(x) = Sf(x)$$

$$\rightarrow \sum_{n=1}^{\infty} -2\tilde{a}_n (n\pi)^2 \cos(n\pi x) - 2\tilde{b}_n (n\pi)^2 \sin(n\pi x) +$$

$$+ \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos(n\pi x) + \tilde{b}_n \sin(n\pi x) =$$

$$= a_0 + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \rightarrow$$

$$\rightarrow \begin{cases} \tilde{a}_0 = a_0 \\ (-2\tilde{a}_n k^2 \pi^2 + \tilde{a}_n) \cos(n\pi x) = 0 \end{cases} \rightarrow \tilde{a}_0 = ④$$

$$\rightarrow \begin{cases} (-2\tilde{b}_n k^2 \pi^2 + \tilde{b}_n) \sin(n\pi x) = b_n \sin(n\pi x) \end{cases}$$

Le due equazioni sono disaccoppiate.

Dalla prima

$$(-2k^2\pi^2 + 1)\tilde{a}_n = 0 \rightarrow \tilde{a}_n = 0$$

Dalla seconda

$$(-2k^2\pi^2 + 1)b_n = b_n \rightarrow b_n = \frac{b_n}{-2k^2\pi^2 + 1} = -\frac{4(-1)^k}{k\pi(1-2k^2\pi^2)}$$

$$S_y(x) = 1 + \sum_{n=1}^{\infty} \left(-\frac{4(-1)^n}{n\pi(1-n^2\alpha^2)} \right) \sin(n\pi x)$$

④ La funzione è pari quindi $b_n = 0$.

$$a_0 = \int_0^1 x \, dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}$$

$$a_n = 2 \int_0^1 x \cos(n\pi x) \, dx = 2 \left\{ x \frac{\sin(n\pi x)}{n\pi} \right\}_0^1 + \left[\frac{\cos(n\pi x)}{(n\pi)^2} \right]_0^1$$

$$= \frac{2}{n^2\pi^2} ((-1)^n - 1)$$

$$S_y(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos(n\pi x) + \tilde{b}_n \sin(n\pi x)$$

$$S_y'(x) = \sum_{n=1}^{\infty} -\tilde{a}_n (n\pi)^2 \cos(n\pi x) - \tilde{b}_n (n\pi)^2 \sin(n\pi x)$$

$$3y'' + 2y = f(x) \rightarrow 3S_y''(x) + 2S_y(x) = S_f(x) \rightarrow$$

$$\rightarrow \sum_{n=1}^{\infty} -3\tilde{a}_n n^2\pi^2 \cos(n\pi x) - 3\tilde{b}_n n^2\pi^2 \sin(n\pi x) +$$

$$+ 2\tilde{a}_0 + \sum_{n=1}^{\infty} 2\tilde{a}_n \cos(n\pi x) + 2\tilde{b}_n \sin(n\pi x) =$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$\rightarrow \begin{cases} 2\tilde{a}_0 = a_0 \\ (-3\tilde{a}_n n^2\pi^2 + 2\tilde{a}_n) \cos(n\pi x) = a_n \cos(n\pi x) \\ (-3\tilde{b}_n n^2\pi^2 + 2\tilde{b}_n) \sin(n\pi x) = 0 \end{cases} \rightarrow \tilde{a}_0 = \frac{a_0}{2} = \frac{1}{4}$$

Le due equazioni sono disaccoppiate.

Dalla prima:

$$(-3u^2\alpha^2 + 2)\tilde{a}_n = a_n \rightarrow \tilde{a}_n = \frac{a_n}{2 - 3u^2\alpha^2} =$$
$$= \frac{2((-1)^n - 1)}{u^2\alpha^2(2 - 3u^2\alpha^2)}$$

Dalla seconda:

$$(-3u^2\alpha^2 + 2)\tilde{b}_n = 0 \rightarrow \tilde{b}_n = 0$$

$$Sy(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{u^2\alpha^2(2 - 3u^2\alpha^2)} \cos(n\alpha x)$$

②

$$\mathcal{F}\left\{ \frac{\sin 8x}{x^2+3} \right\} = \frac{\pi}{2i\sqrt{3}} \begin{bmatrix} -\sqrt{3}|-(\kappa-8)| & -\sqrt{3}|-(\kappa+8)| \\ e^{-\sqrt{3}|\kappa-8|} & e^{-\sqrt{3}|\kappa+8|} \end{bmatrix}$$

↓ propn ⑦ $\kappa_0 = 8$

$$\mathcal{F}\left\{ \frac{1}{x^2+3} \right\} = 2\pi \left(\frac{1}{2\sqrt{3}} e^{-\sqrt{3}|\kappa|} \right)$$

↓ propn ⑤

$$\mathcal{F}^{-1}\left\{ \frac{1}{\kappa^2+3} \right\} = \frac{1}{2\sqrt{3}} \mathcal{F}^{-1}\left\{ \frac{2\sqrt{3}}{\kappa^2+3} \right\} = \frac{1}{2\sqrt{3}} e^{-\sqrt{3}|\kappa|}$$



$$\mathcal{F}^{-1} \left\{ \frac{e^{-3ik}}{5+i(6-2k)} \right\} = \frac{1}{2} e^{\frac{i}{2}(x-3)} H(3-x) e^{-3i(x-3)} =$$

\uparrow

$$= \frac{1}{2} e^{\frac{i}{2}(5-3i)(x-3)} H(3-x)$$

↓ propn ② $x_0=3$

$$\mathcal{F}^{-1} \left\{ \frac{1}{5+i(6-2k)} \right\} = \frac{1}{2} \mathcal{F}^{-1} \left\{ \frac{1}{\frac{5}{2}+i(3-k)} \right\} = \frac{1}{2} e^{\frac{i}{2}x} H(-x) e^{-3ix}$$

\uparrow

↓ propn. ③ $k_0=-3$

$$\mathcal{F}^{-1} \left\{ \frac{1}{\frac{5}{2}-ik} \right\} = e^{\frac{i}{2}x} H(-x)$$

$$\mathcal{F} \left\{ \frac{e^{-2ix}}{2x^2+1} \right\} = \frac{\sqrt{2}}{2}\pi e^{-\frac{|-(k+2)|}{\sqrt{2}}} = \frac{\sqrt{2}}{2}\pi e^{-\frac{1}{\sqrt{2}}|k+2|}$$

\uparrow

↓ propn ③ $k_0=-2$

$$\mathcal{F} \left\{ \frac{1}{2x^2+1} \right\}$$

↓ propn. ⑤

$$= \frac{1}{2}\pi \frac{\sqrt{2}}{\sqrt{2}} e^{-\frac{|k|}{\sqrt{2}}}$$

\uparrow

$$\mathcal{F}^{-1} \left\{ \frac{1}{2k^2+1} \right\} = \frac{1}{2} \mathcal{F}^{-1} \left\{ \frac{1}{k^2+\frac{1}{2}} \right\} = \frac{1}{2} \frac{\sqrt{2}}{2} \mathcal{F}^{-1} \left\{ \frac{\frac{2}{\sqrt{2}}}{k^2+\frac{1}{2}} \right\} = \frac{\sqrt{2}}{4} e^{-\frac{|x|}{\sqrt{2}}}$$

H

$$\mathcal{F}^{-1} \left\{ \frac{i(u-z)}{9+(u-z)^2} e^{-3iu} \right\} = -\frac{1}{2} e^{2ix} [e^{-3(x-3)} H(x-3) - e^{3(x-3)} H(3-x)]$$

↓ mom. ① $x_0=3$



$$\mathcal{F}^{-1} \left\{ \frac{i(u-z)}{9+(u-z)^2} \right\} = -\frac{1}{2} e^{2ix} [e^{-3x} H(x) - e^{3x} H(-x)]$$

↓ mom. ③ $k_0=2$



$$\mathcal{F}^{-1} \left\{ \frac{ik}{9+u^2} \right\} = -\frac{1}{2} \mathcal{F}^{-1} \left\{ \frac{2ik}{9+u^2} \right\} = -\frac{1}{2} [e^{-3x} H(x) - e^{3x} H(-x)]$$

\mapsto

$$\mathcal{F} \left\{ \frac{H(-x) \cos 3x}{e^{-4x}} \right\} = \frac{1}{2} \left[\frac{1}{4-i(u-3)} + \frac{1}{4-i(u+3)} \right]$$

↓ mom. ⑥ $k_0=3$



$$\mathcal{F} \left\{ H(-x) e^{4x} \right\} = \frac{1}{4-ik}$$