SOLUZIONE

① ⓐ $y'(x) - 4y(x) = e^{-3x} H(x)$

$\mathcal{F}\{y'(x) - 4y(x)\} = \mathcal{F}\{e^{-3x} H(x)\}$

$\mathcal{F}\{y'(x)\} - 4\mathcal{F}\{y(x)\} = F(k)$

$ik Y(k) - 4Y(k) = F(k) \rightarrow (ik-4)Y(k) = F(k)$

$\rightarrow Y(k) = \dfrac{F(k)}{ik-4} = F(k)G(k)$, con $G(k) = \dfrac{1}{ik-4}$

$y(x) = \mathcal{F}^{-1}\{Y(k)\} = \mathcal{F}^{-1}\{F(k)\cdot G(k)\} = \mathcal{F}^{-1}\{e^{-3x}H(x)\} * \mathcal{F}^{-1}\{G(k)\}$

$\mathcal{F}^{-1}\left\{\dfrac{1}{ik-4}\right\} = -\mathcal{F}^{-1}\left\{\dfrac{1}{4-ik}\right\} = -e^{4x} H(-x)$

$y(x) = \left[e^{-3x} H(x)\right] * \left[-e^{4x} H(-x)\right] =$

$= -\int_{-\infty}^{\infty} \left[e^{-3(x-y)} H(x-y)\right] \cdot \left[e^{4y} H(-y)\right] dy =$

$$\boxed{H(-y) = \begin{cases} 1 & y \le 0 \\ 0 & y > 0 \end{cases}}$$

$= -\int_{-\infty}^{0} e^{-3(x-y)} e^{4y} H(x-y)\, dy =$

$$\boxed{H(x-y) = \begin{cases} 1 & x \ge y \\ 0 & x < y \end{cases}}$$

$$= \begin{cases} -\int_{-\infty}^{x} e^{-3(x-y)} e^{4y}\, dy & x \le 0 \\ -\int_{-\infty}^{0} e^{-3(x-y)} e^{4y}\, dy & x > 0 \end{cases}$$

$$= \begin{cases} -e^{-3x}\left[\dfrac{e^{7y}}{7}\right]_{-\infty}^{x} & x \leq 0 \\ -e^{-3x}\left[\dfrac{e^{7y}}{7}\right]_{-\infty}^{0} & x > 0 \end{cases}$$

$$= \begin{cases} -\frac{1}{7}e^{-3x}\left(e^{7x} - 0\right) = -\frac{1}{7}e^{4x} & x \leq 0 \\ -\frac{1}{7}e^{-3x}(1-0) = -\frac{1}{7}e^{-3x} & x > 0 \end{cases}$$

$$= -\frac{1}{7}e^{4x}H(-x) - \frac{1}{7}e^{-3x}H(x)$$

1b) $2y'' - y = H(x+3) - H(x-4)$

$\mathcal{F}\{2y'' - y\} = \mathcal{F}\{H(x+3) - H(x-4)\}$

$2\mathcal{F}\{y''\} - \mathcal{F}\{y\} = F(k) \rightarrow 2(ik)^2 Y(k) - Y(k) = F(k)$

$\rightarrow -2k^2 Y(k) - Y(k) = F(k) \Rightarrow (-2k^2 - 1)Y(k) = F(k)$

$\rightarrow Y(k) = -\dfrac{F(k)}{2k^2+1} = -F(k)G(k)$, con $G(k) = \dfrac{1}{2k^2+1}$

$y(x) = \mathcal{F}^{-1}\{Y(k)\} = \mathcal{F}^{-1}\{-F(k)\cdot G(k)\} = -\mathcal{F}^{-1}\{F(k)\} * \mathcal{F}^{-1}\{G(k)\}$

$\mathcal{F}^{-1}\left\{\dfrac{1}{2k^2+1}\right\} = \dfrac{1}{2}\mathcal{F}^{-1}\left\{\dfrac{1}{k^2+\frac{1}{2}}\right\} = \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2}\mathcal{F}^{-1}\left\{\dfrac{\frac{2}{\sqrt{2}}}{k^2+\frac{1}{2}}\right\} = \dfrac{\sqrt{2}}{4}e^{-\frac{|x|}{\sqrt{2}}}$

$y(x) = -\left[H(x+3) - H(x-4)\right] * \left[\dfrac{\sqrt{2}}{4}e^{-\frac{|x|}{\sqrt{2}}}\right] =$

$$= -\frac{\sqrt{2}}{4}\int_{-\infty}^{\infty}\left[H(y+3)-H(y-4)\right]\cdot\left[e^{-\frac{|x-y|}{\sqrt{2}}}\right]dy =$$

$$\boxed{H(y+3)-H(y-4) = \begin{cases} 1 & -3\le y\le 4 \\ 0 & \text{altrove}\end{cases}}$$

$$\boxed{|x-y| = \begin{cases} x-y & x\ge y \\ y-x & x<y\end{cases}}$$

$$= -\frac{\sqrt{2}}{4}\int_{-3}^{4} e^{-\frac{|x-y|}{\sqrt{2}}}\,dy =$$

$$= \begin{cases} -\frac{\sqrt{2}}{4}\int_{-3}^{4} e^{-\frac{1}{\sqrt{2}}(y-x)}\,dy & x\le -3 \\ -\frac{\sqrt{2}}{4}\int_{-3}^{x} e^{-\frac{1}{\sqrt{2}}(x-y)}\,dy - \frac{\sqrt{2}}{4}\int_{x}^{4} e^{-\frac{1}{\sqrt{2}}(y-x)}\,dy & -3<x<4 \\ -\frac{\sqrt{2}}{4}\int_{-3}^{4} e^{-\frac{1}{\sqrt{2}}(x-y)}\,dy & x\ge 4 \end{cases}$$

$$= \begin{cases} +\frac{\sqrt{2}}{4}e^{\frac{x}{\sqrt{2}}}\left[\sqrt{2}\,e^{-\frac{y}{\sqrt{2}}}\right]_{-3}^{4} = \frac{1}{2}e^{\frac{x}{\sqrt{2}}}\left(e^{-\frac{4}{\sqrt{2}}} - e^{\frac{3}{\sqrt{2}}}\right) & x\le -3 \\ -\frac{\sqrt{2}}{4}e^{-\frac{x}{\sqrt{2}}}\left[\sqrt{2}\,e^{\frac{y}{\sqrt{2}}}\right]_{-3}^{x} + \frac{\sqrt{2}}{4}e^{\frac{x}{\sqrt{2}}}\left[\sqrt{2}\,e^{-\frac{y}{\sqrt{2}}}\right]_{x}^{4} & -3<x<4 \\ -\frac{\sqrt{2}}{4}e^{-\frac{x}{\sqrt{2}}}\left[\sqrt{2}\,e^{\frac{y}{\sqrt{2}}}\right]_{-3}^{4} = -\frac{1}{2}e^{-\frac{x}{\sqrt{2}}}\left(e^{\frac{4}{\sqrt{2}}} - e^{-\frac{3}{\sqrt{2}}}\right) & x\ge 4 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(e^{\frac{1}{\sqrt{2}}(x-4)} - e^{\frac{1}{\sqrt{2}}(x+3)}\right) & x\le -3 \\ -\frac{1}{2}e^{-\frac{x}{\sqrt{2}}}\left(e^{\frac{x}{\sqrt{2}}} - e^{-\frac{3}{\sqrt{2}}}\right) + \frac{1}{2}e^{\frac{x}{\sqrt{2}}}\left(e^{-\frac{4}{\sqrt{2}}} - e^{-\frac{x}{\sqrt{2}}}\right) & -3<x<4 \\ -\frac{1}{2}\left(e^{-\frac{1}{\sqrt{2}}(x-4)} - e^{-\frac{1}{\sqrt{2}}(x+3)}\right) & x\ge 4 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(e^{\frac{1}{\sqrt{2}}(x-4)} - e^{\frac{1}{\sqrt{2}}(x+3)}\right) & x\le -3 \\ -\frac{1}{2}\left(1 - e^{-\frac{1}{\sqrt{2}}(x+3)} - e^{\frac{1}{\sqrt{2}}(x-4)} + 1\right) = -\frac{1}{2}\left(2 - e^{-\frac{1}{\sqrt{2}}(x+3)} - e^{\frac{1}{\sqrt{2}}(x-4)}\right) & -3<x<4 \\ -\frac{1}{2}\left(e^{-\frac{1}{\sqrt{2}}(x-4)} - e^{-\frac{1}{\sqrt{2}}(x+3)}\right) & x\ge 4 \end{cases}$$

1c) $y'' + 6y' + 5y = \delta(x-3)$

$$\mathcal{F}\{y''\} + 6\mathcal{F}\{y'\} + 5\mathcal{F}\{y\} = \mathcal{F}\{\delta(x-3)\}$$

$$(ik)^2 Y(k) + 6ik\,Y(k) + 5\,Y(k) = F(k)$$

$$F(k) = \mathcal{F}\{\delta(x-3)\} = e^{-3ik} \cdot 1 = e^{-3ik}$$

$$\left((ik)^2 + 6ik + 5\right) Y(k) = F(k)$$

$$\rightarrow Y(k) = \frac{e^{-3ik}}{(ik)^2 + 6ik + 5} = \frac{e^{-3ik}}{(ik+5)(ik+1)}$$

$$\mathcal{F}^{-1}\{Y(k)\} = \mathcal{F}^{-1}\left\{\frac{e^{-3ik}}{(ik+5)(ik+1)}\right\} \underset{\substack{②\\ x_0=3}}{\rightarrow} \mathcal{F}^{-1}\left\{\frac{1}{ik+5} \cdot \frac{1}{ik+1}\right\} =$$

$$= \mathcal{F}^{-1}\left\{\frac{1}{ik+5}\right\} * \mathcal{F}^{-1}\left\{\frac{1}{ik+1}\right\} = \left[e^{-5x} H(x)\right] * \left[e^{-x} H(x)\right] =$$

$$= \int_{-\infty}^{\infty} \left[e^{-5(x-y)} H(x-y)\right] \cdot \left[e^{-y} H(y)\right] dy =$$

$$\boxed{H(y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}}$$

$$\boxed{H(x-y) = \begin{cases} 1 & x \geq y \\ 0 & x < y \end{cases}}$$

$$= \int_{0}^{\infty} e^{-5(x-y)} e^{-y} H(x-y)\, dy =$$

$$= \begin{cases} 0 & x < 0 \\ \displaystyle\int_0^x e^{-5(x-y)} e^{-y}\, dy & x \geq 0 \end{cases}$$

$$=\begin{cases} 0 & x<0 \\ e^{-5x}\left[\dfrac{e^{4y}}{4}\right]_0^x = \dfrac{1}{4}e^{-5x}\left(e^{4x}-1\right) = \dfrac{1}{4}\left(e^{-x}-e^{-5x}\right) & x\geq 0 \end{cases}$$

$$=\frac{1}{4}\left(e^{-x}-e^{-5x}\right)H(x)$$

prop ② con $x_0=3$

$$y(x)=\frac{1}{4}\left(e^{-(x-3)}-e^{-5(x-3)}\right)H(x-3)$$

②ⓐ $\mathcal{F}\left\{3(x-2)e^{-2|x-2|}\right\} = -24e^{-2ik}\dfrac{ik}{(4+k^2)^2}$

$\downarrow$ ② $x_0=2$ $\uparrow$

$$3\mathcal{F}\left\{xe^{-2|x|}\right\} = i\left(\frac{12}{4+k^2}\right)' = 12i\,\frac{0-2k}{(4+k^2)^2} = -24\frac{ik}{(4+k^2)^2}$$

$\downarrow$ ⑨ $\uparrow$

$$3\mathcal{F}\left\{e^{-2|x|}\right\} = 3\cdot\frac{4}{4+k^2} = \frac{12}{4+k^2}$$

⑤ $\mathcal{F}^{-1}\left\{9e^{-\frac{(k+4)^2}{4}}\right\} = \dfrac{9}{\sqrt{\pi}}e^{-4ix}e^{-x^2} = \dfrac{9}{\sqrt{\pi}}e^{-x(4i+x)}$

$\downarrow$ ③ $k_0=-4$ $\uparrow$

$$9\mathcal{F}^{-1}\left\{e^{-\frac{k^2}{4}}\right\} = \frac{9}{\sqrt{\pi}}\mathcal{F}^{-1}\left\{\sqrt{\pi}e^{-\frac{k^2}{4}}\right\} = \frac{9}{\sqrt{\pi}}e^{-x^2}$$

(c) $\mathcal{F}\left\{\left(e^{-2x}H(x)\sin 4x\right) * \left(\frac{e^{4ix}}{3x^2+6}\right)\right\}$

$$\Big\downarrow 10$$

$$\mathcal{F}\left\{e^{-2x}H(x)\sin 4x\right\} \cdot \mathcal{F}\left\{\frac{e^{4ix}}{3x^2+6}\right\} = (**)$$

$$\mathcal{F}\left\{e^{-2x}H(x)\sin 4x\right\} = \frac{1}{2i}\left[\frac{1}{2+i(k-4)} - \frac{1}{2+i(k+4)}\right]$$

$$\Big\downarrow (7) \quad k_0 = 4$$

$$\mathcal{F}\left\{e^{-2x}H(x)\right\} = \frac{1}{2+ik}$$

$$\mathcal{F}\left\{\frac{e^{4ix}}{3x^2+6}\right\} = \frac{\pi}{3\sqrt{2}}\, e^{-\sqrt{2}\,|-(k-4)|} = \frac{\pi}{3\sqrt{2}}\, e^{-\sqrt{2}\,|k-4|}$$

$$\Big\downarrow (3) \quad k_0 = 4$$

$$\mathcal{F}\left\{\frac{1}{3x^2+6}\right\} = \frac{2\pi}{3\cancel{6}\sqrt{2}}\, e^{-\sqrt{2}\,|-k|}$$

$$\Big\downarrow (5)$$

$$\mathcal{F}^{-1}\left\{\frac{1}{3k^2+6}\right\} = \frac{1}{3}\mathcal{F}^{-1}\left\{\frac{1}{k^2+2}\right\} = \frac{1}{6\sqrt{2}}\mathcal{F}^{-1}\left\{\frac{2\sqrt{2}}{k^2+2}\right\} = \frac{1}{6\sqrt{2}}\, e^{-\sqrt{2}\,|x|}$$

$$(**) = \frac{\pi}{6\sqrt{2}\, i}\, e^{-\sqrt{2}\,|k-4|}\left[\frac{1}{2+i(k-4)} - \frac{1}{2+i(k+4)}\right]$$

(d) $\left[H(x+3) - H(x-4)\right] * e^{-|x|} =$

$$= \int_{-\infty}^{\infty} \left[H(y+3) - H(y-4)\right] \cdot e^{-|x-y|}\, dy =$$

$$= \int_{-3}^{4} e^{-|x-y|}\, dy =$$

$$= \begin{cases} \displaystyle\int_{-3}^{4} e^{-(y-x)}\, dy & x \leq -3 \\ \displaystyle\int_{-3}^{x} e^{-(x-y)}\, dy + \int_{x}^{4} e^{-(y-x)}\, dy & -3 < x < 4 \\ \displaystyle\int_{-3}^{4} e^{-(x-y)}\, dy & x \geq 4 \end{cases}$$

$$= \begin{cases} e^{x}\left[-e^{-y}\right]_{-3}^{4} & x \leq -3 \\ e^{-x}\left[e^{y}\right]_{-3}^{x} + e^{x}\left[-e^{-y}\right]_{x}^{4} & 3 < x < 4 \\ e^{-x}\left[e^{y}\right]_{-3}^{4} & x \geq 4 \end{cases}$$

$$= \begin{cases} e^{x}(e^{-4} - e^{3}) = -e^{x-4} + e^{x+3} & x \leq -3 \\ e^{-x}(e^{x} - e^{-3}) - e^{x}(e^{-4} - e^{-x}) & -3 < x < 4 \\ e^{-x}(e^{4} - e^{-3}) = e^{-(x-4)} - e^{-(x+3)} & x \geq 4 \end{cases}$$

$$= \begin{cases} -e^{x-4} + e^{x+3} & x \leq -3 \\ 1 - e^{-(x+3)} - e^{x-4} + 1 = 2 - e^{-(x+3)} - e^{x-4} & -3 < x < 4 \\ e^{-(x-4)} - e^{-(x+3)} & x \geq 4 \end{cases}$$

(e) $$\mathcal{F}^{-1}\left\{\frac{e^{-2ik}}{k^2-4k+7}\right\} = \frac{1}{2\sqrt{3}}\, e^{2i(x-2)}\, e^{-\sqrt{3}\,|x-2|}$$

$\downarrow$ ② $x_0 = 2$ $\qquad\qquad\qquad \uparrow$

$$\mathcal{F}^{-1}\left\{\frac{1}{k^2-4k+7}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{(k-2)^2+3}\right\} = \frac{1}{2\sqrt{3}}\, e^{2ix}\, e^{-\sqrt{3}|x|}$$

$\downarrow$ ③ $k_0 = 2$ $\qquad\qquad \uparrow$

$$\mathcal{F}^{-1}\left\{\frac{1}{k^2+3}\right\} = \frac{1}{2\sqrt{3}}\,\mathcal{F}^{-1}\left\{\frac{2\sqrt{3}}{k^2+3}\right\} = \frac{1}{2\sqrt{3}}\, e^{-\sqrt{3}\,|x|}$$