SOLUZIONE

① Calcoliamo gli autovalori:

$$\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & a & 0 \\ a & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} =$$

$$= (2-\lambda)\left[(1-\lambda)(2-\lambda)-1\right] - a\left[a(2-\lambda)\right] =$$

$$= (2-\lambda)\left[(1-\lambda)(2-\lambda)-1-a^2\right] = (2-\lambda)\left[2-3\lambda+\lambda^2-1-a^2\right] =$$

$$= (2-\lambda)\left[\lambda^2-3\lambda+1-a^2\right] = 0$$

$$(2-\lambda) = 0 \rightarrow \lambda_1 = 2$$

$$\lambda^2 - 3\lambda + 1 - a^2 = 0 \rightarrow \lambda_{2/3} = \frac{3 \pm \sqrt{9-4(1-a^2)}}{2} =$$

$$= \frac{3 \pm \sqrt{9-4+4a^2}}{2} = \frac{3 \pm \sqrt{5+4a^2}}{2}$$

$$\sigma(A) = \left\{2, \frac{3-\sqrt{5+4a^2}}{2}, \frac{3+\sqrt{5+4a^2}}{2}\right\}$$

Gli autovalori 2 e $\frac{3+\sqrt{5+4a^2}}{2}$ sono sicuramente positivi. Verifichiamo per quali valori di a lo è anche $\frac{3-\sqrt{5+4a^2}}{2}$.

$$\frac{3-\sqrt{5+4a^2}}{2}>0 \rightarrow 3-\sqrt{5+4a^2}>0 \rightarrow 3>\sqrt{5+4a^2}$$

$$\rightarrow 9>5+4a^2 \rightarrow 4>4a^2 \rightarrow a^2<1 \rightarrow -1<a<1$$

A è definita positiva per $-1<a<1$.

② A non è simmetrica, quindi dobbiamo calcolare A^TA:

$$A^TA=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

e calcolarne gli autovalori:

$$\det(A^TA-\lambda I)=\det\begin{bmatrix}2-\lambda&1&1\\1&2-\lambda&1\\1&1&2-\lambda\end{bmatrix}=$$

$$=(2-\lambda)\left[(2-\lambda)^2-1\right]-\left[(2-\lambda)-1\right]+\left[1-(2-\lambda)\right]=$$

$$=(2-\lambda)\left[(2-\lambda-1)(2-\lambda+1)\right]-\left[2-\lambda-1\right]-\left[2-\lambda-1\right]=$$

$$=\left[2-\lambda-1\right]\left[(2-\lambda)(2-\lambda+1)-1-1\right]=$$

$$=\left[1-\lambda\right]\left[\lambda^2-5\lambda+6-2\right]=(1-\lambda)(\lambda^2-5\lambda+4)=0$$

$(1-\lambda)=0 \rightarrow \lambda_1 = 1$

$$(\lambda^2-5\lambda+4)=0 \rightarrow \lambda_{2/3} = \frac{5 \pm \sqrt{25-16}}{2} \begin{cases} \rightarrow \lambda_2 = 4 \\ \rightarrow \lambda_3 = 1 \end{cases}$$

$\sigma(A^TA) = \{1,1,4\} \rightarrow \rho(A^TA) = 4 \rightarrow \|A\|_2 = 2$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow 2 \\ \rightarrow 2 \\ \rightarrow 2 \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ 2 & 2 & 2 \end{matrix}$$

$\|A\|_\infty = \max\{2,2,2\} = 2$

$\|A\|_1 = \max\{2,2,2\} = 2$

o—o

B è simmetrica quindi possiamo calcolarne gli autovalori:

$$\det(B-\lambda I) = \det \begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} =$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 4\right] = (1-\lambda)\left[\lambda^2 - 2\lambda - 3\right] = 0$$

$(1-\lambda)=0 \rightarrow \lambda_1 = 1$

$$(\lambda^2-2\lambda-3)=0 \rightarrow \lambda_{2/3} = \frac{2 \pm \sqrt{4+12}}{2} \begin{cases} \rightarrow \lambda_2 = 3 \\ \rightarrow \lambda_3 = -1 \end{cases}$$

$$\sigma(B)=\{1,-1,3\} \rightarrow \rho(B)=3 \rightarrow \|B\|_2=3$$

$$B=\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow 3 \\ \rightarrow 3 \\ \rightarrow 1 \end{matrix} \qquad \|B\|_\infty=\|B\|_1=\max\{3,3,1\}=3$$

③

$$C=\begin{bmatrix} 2 & 0 \\ 1 & \beta \end{bmatrix} \begin{matrix} \rightarrow 2 \\ \rightarrow 1+|\beta| \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ 3 & |\beta| \end{matrix}$$

$$\|C\|_\infty=\max\{2,1+|\beta|\}=\begin{cases} 2 & \text{se } 2\geq 1+|\beta| \rightarrow -1<\beta<1 \\ 1+|\beta| & \text{se } 2<1+|\beta| \rightarrow \text{altrove} \end{cases}$$

$$\|C\|_1=\max\{3,|\beta|\}=\begin{cases} 3 & \text{se } 3\geq|\beta| \rightarrow -3<\beta<3 \\ |\beta| & \text{se } 3<|\beta| \rightarrow \text{altrove} \end{cases}$$

④ D è simmetrica quindi possiamo calcolarne gli autovalori.

$$\det(D-\lambda I)=\det\begin{bmatrix} 5-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & \beta-\lambda \end{bmatrix}=$$

$$=(\beta-\lambda)\left[(5-\lambda)(1-\lambda)-4\right]=(\beta-\lambda)\left[\lambda^2-6\lambda+1\right]=0$$

$$(\beta-\lambda)=0 \to \lambda_1=\beta$$

$$(\lambda^2-6\lambda+1)=0 \to \lambda_{2/3}=\frac{6\pm\sqrt{36-4}}{2} \to \lambda_2=3-2\sqrt{2}, \quad \lambda_3=3+2\sqrt{2}$$

$$\sigma(D)=\{\beta,\ 3-2\sqrt{2},\ 3+2\sqrt{2}\}$$

$$\rho(D)=\begin{cases} |\beta| & \text{se} \quad |\beta|\geq 3+2\sqrt{2} \to \beta\leq -3-2\sqrt{2} \wedge \beta\geq 3+2\sqrt{2} \\ 3+2\sqrt{2} & \text{se} \quad |\beta|<3+2\sqrt{2} \to -3-2\sqrt{2}<\beta<3+2\sqrt{2} \end{cases}$$

$$D=\begin{bmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{matrix} \to 7 \\ \to 3 \\ \to |\beta| \end{matrix}$$

$$\|D\|_\infty=\|D\|_1=\max\{7,3,|\beta|\}=\begin{cases} 7 & \text{se} \quad 7>|\beta| \to -7<\beta<7 \\ |\beta| & \text{se} \quad |\beta|>7 \to \text{altrimenti} \end{cases}$$