SOLUZIONE

1

$$A=\begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[\text{1-3}]{\text{scambio}} \begin{array}{c} \\ \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{array}\begin{bmatrix} 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[\text{2-4}]{\text{scambio}} \begin{array}{c} \\ \\ 0 \\ \frac{1}{2} \end{array}\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} \\ \\ \\ \frac{1}{2} \end{array}\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} = U$$

$$L=\begin{bmatrix} 1 \\ \cancel{1/3} \\ \cancel{2/3} \\ 0 \end{bmatrix} \rightarrow L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1 & 0 \\ 1/3 & 1/2 & 1/2 & 1 \end{bmatrix}$$

$$I=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{1-3}]{} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{2-4}]{} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = P$$

- $\det(A) = (-1)^{\#\text{scambi}} \prod_{i=1}^{4} u_{ii} = (-1)^2 \cdot \left[\cancel{3} \cdot 2 \cdot \frac{\cancel{4}^{2}}{\cancel{3}} \cdot \left(-\frac{1}{2}\right)\right] = 1 \cdot (-4) = -4$

$$\begin{cases} L\underline{y} = P\underline{e}_i \\ U\underline{x} = \underline{y} \end{cases}$$

- $P\underline{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$L\underline{y} = P\underline{e}_2 \Rightarrow \begin{cases} y_1 = 0 \rightarrow y_1 = 0 \\ y_2 = 0 \rightarrow y_2 = 0 \\ \frac{2}{3}y_1 + y_3 = 0 \rightarrow y_3 = 0 \\ \frac{1}{3}y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3 + y_4 = 1 \rightarrow y_4 = 1 \end{cases}$$

$$U\underline{x} = \underline{y} \Rightarrow \begin{cases} 3x_1 + x_3 = 0 \rightarrow x_1 = 0 \\ 2x_2 + x_4 = 0 \rightarrow 2x_2 = 2 \rightarrow x_2 = 1 \\ \frac{4}{3}x_3 = 0 \rightarrow x_3 = 0 \\ -\frac{1}{2}x_4 = 1 \rightarrow x_4 = -2 \end{cases}$$

$$P\underline{e}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow L\underline{y} = P\underline{e}_3 \rightarrow \begin{cases} y_1 = 1 \rightarrow y_1 = 1 \\ y_2 = 0 \rightarrow y_2 = 0 \\ \frac{2}{3}y_1 + y_3 = 0 \rightarrow y_3 = -\frac{2}{3} \\ \frac{1}{3}y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3 + y_4 = 0 \end{cases}$$

$$\rightarrow y_4 = -\frac{1}{3} + \frac{1}{3} = 0$$

$$U\underline{x} = \underline{y} \rightarrow \begin{cases} 3x_1 + x_3 = 1 \rightarrow 3x_1 = 1 + \frac{1}{2} = \frac{3}{2} \rightarrow x_1 = \frac{1}{2} \\ 2x_2 + x_4 = 0 \rightarrow x_2 = 0 \\ \frac{4}{3}x_3 = -\frac{2}{3} \rightarrow x_3 = -1/2 \\ -\frac{1}{2}x_4 = 0 \rightarrow x_4 = 0 \end{cases}$$

②

$$A = \begin{array}{c c} \begin{bmatrix} 2 & -2 & 2 & 2 \\ 1 & 0 & 0 & 3 \\ 4 & 0 & 3 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix} & \begin{matrix} \\ 1/2 \\ 2 \\ 0 \end{matrix} \end{array} \rightarrow \begin{array}{c c} \begin{bmatrix} 2 & -2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & -1 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} & \begin{matrix} \\ \\ 4 \\ 0 \end{matrix} \end{array}$$

$$\rightarrow \begin{array}{c c} \begin{bmatrix} 2 & -2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & -11 \\ 0 & 0 & 3 & 2 \end{bmatrix} & \begin{matrix} \\ \\ \\ 1 \end{matrix} \end{array} \rightarrow \begin{bmatrix} 2 & -2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & -11 \\ 0 & 0 & 0 & 13 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- $\det(A) = \prod_{i=1}^{4} u_{ii} = 2 \cdot 1 \cdot 3 \cdot 13 = 78$

- $\begin{cases} L\underline{y} = \underline{b} \\ U\underline{x} = \underline{y} \end{cases}$

$$L\underline{y} = \underline{b} \rightarrow \begin{cases} y_1 = -4 & \rightarrow y_1 = -4 \\ \frac{1}{2}y_1 + y_2 = -4 & \rightarrow \rightarrow y_2 = -4+2 = -2 \\ 2y_1 + 4y_2 + y_3 = -8 & \rightarrow y_3 = \cancel{-8} \cancel{+8} + 8 = 8 \\ y_3 + y_4 = -5 & \rightarrow y_4 = -5-8 = -13 \end{cases}$$

$$U\underline{x}=\underline{y} \to \begin{cases} 2x_1-2x_2+2x_3+2x_4=-4 \quad (A) \\ x_2-x_3+2x_4=-2 \to x_2=-\cancel{2}-1+\cancel{2} \to x_2=-1 \\ 3x_3-11x_4=8 \to 3x_3=8-11 \to x_3=-1 \\ 13x_4=-13 \to x_4=-1 \end{cases}$$

$$(A)\quad 2x_1=-4-\cancel{2}+\cancel{2}+2=-2 \to x_1=-1$$

(3)

$$A=\begin{bmatrix} 4 & 1 & 0 & 8 \\ 6 & 2 & -1 & 1 \\ 6 & 3 & -4 & 1 \\ 2 & 4 & 8 & 2 \end{bmatrix} \xrightarrow[\text{1-2}]{\text{scambio}} \begin{array}{c} \\ \frac{2}{3} \\ 1 \\ \frac{1}{3} \end{array}\begin{bmatrix} 6 & 2 & -1 & 1 \\ 4 & 1 & 0 & 8 \\ 6 & 3 & -4 & 1 \\ 2 & 4 & 8 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 6 & 2 & -1 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{22}{3} \\ 0 & 1 & -3 & 0 \\ 0 & \frac{10}{3} & \frac{25}{3} & \frac{5}{3} \end{bmatrix} \xrightarrow[\text{2-4}]{\text{scambio}} \begin{array}{c} \\ \\ \frac{3}{10} \\ -\frac{1}{10} \end{array}\begin{bmatrix} 6 & 2 & -1 & 1 \\ 0 & \frac{10}{3} & \frac{25}{3} & \frac{5}{3} \\ 0 & 1 & -3 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

$$\to \begin{array}{c} \\ \\ \\ -\frac{3}{11} \end{array}\begin{bmatrix} 6 & 2 & -1 & 1 \\ 0 & \frac{10}{3} & \frac{25}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{11}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} & \frac{45}{6} \end{bmatrix} \to \begin{bmatrix} 6 & 2 & -1 & 1 \\ 0 & \frac{10}{3} & \frac{25}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{11}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{81}{11} \end{bmatrix} = U$$

- $$\det(A)=(-1)^{\#\text{scambi}}\prod_{i=1}^{4} u_{ii} = (-1)^2\left[6\cdot\frac{\overset{5}{\cancel{10}}}{\cancel{3}}\cdot\left(-\frac{\cancel{11}}{\underset{1}{\cancel{2}}}\right)\left(\frac{\overset{27}{\cancel{81}}}{\cancel{11}}\right)\right]=-810$$

$$L=\begin{bmatrix}1\\2/3\\1\\1/3\end{bmatrix} \rightarrow L=\begin{bmatrix}1&0&0&0\\1/3&1&0&0\\1&3/10&1&0\\2/3&-1/10&-3/11&1\end{bmatrix}$$

$$I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow[1-2]{}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow[2-4]{}\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{bmatrix}=P$$

- $\begin{cases}L\underline{y}=P\underline{b}\\ U\underline{x}=\underline{y}\end{cases}$ $\qquad \underline{b}=\begin{bmatrix}-5\\2\\-2\\4\end{bmatrix}\xrightarrow[1-2]{}\begin{bmatrix}2\\-5\\-2\\4\end{bmatrix}\xrightarrow[2-4]{}\begin{bmatrix}2\\4\\-2\\-5\end{bmatrix}$

$$L\underline{y}=P\underline{b}\rightarrow\begin{cases}y_1=2 \rightarrow y_1=2\\ \frac{1}{3}y_1+y_2=4 \rightarrow y_2=4-\frac{2}{3}=\frac{10}{3}\\ y_1+\frac{3}{10}y_2+y_3=-2 \rightarrow y_3=-2-2-1=-5\\ \frac{2}{3}y_1-\frac{1}{10}y_2-\frac{3}{11}y_3+y_4=-5 \rightarrow y_4=-5-\frac{4}{3}+\frac{1}{3}-\frac{15}{11}=\\ \qquad =-\frac{81}{11}\end{cases}$$

$$U\underline{x}=\underline{y}\rightarrow\begin{cases}6x_1+2x_2-x_3+x_4=2 \rightarrow 6x_1=2+2+1+1=6 \rightarrow x_1=1\\ \frac{10}{3}x_2+\frac{25}{3}x_3+\frac{5}{3}x_4=\frac{10}{3} \rightarrow \frac{10}{3}x_2=\frac{10}{3}-\frac{25}{3}+\frac{5}{3} \rightarrow x_2=-1\\ -\frac{11}{2}x_3-\frac{1}{2}x_4=-5 \rightarrow -\frac{11}{2}x_3=-5-\frac{1}{2} \rightarrow x_3=1\\ \frac{81}{11}x_4=-\frac{81}{11} \rightarrow x_4=-1\end{cases}$$