SOLUZIONE

(1)

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & \alpha \\ 0 & \alpha & 3 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- $\det(A) = 3(9-\alpha^2) + 1(-3) = 27 - 3\alpha^2 - 3 = -3\alpha^2 + 24$

  A è non singolare se $\det(A) \neq 0 \rightarrow -3\alpha^2 + 24 \neq 0$

  $\rightarrow 3\alpha^2 \neq 24 \rightarrow \alpha^2 \neq 8 \rightarrow \alpha \neq \pm 2\sqrt{2}$

- Il metodo di Gauss-Seidel converge se $\rho(B_{GS}) < 1$

  dove $B_{GS} = (D-L)^{-1} \cdot U$ e

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\alpha & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\alpha \\ 0 & 0 & 0 \end{bmatrix}$$

$$D-L = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & \alpha & 3 \end{bmatrix}$$

per calcolarne l'inversa risolviamo tre sistemi $(D-L)\underline{x} = \underline{e_i}$, con $i = 1,2,3$

$$\begin{cases} 3x_1 = 1 \rightarrow x_1 = 1/3 \\ -x_1 + 3x_2 = 0 \rightarrow 3x_2 = 1/3 \rightarrow x_2 = 1/9 \\ \alpha x_2 + 3x_3 = 0 \rightarrow 3x_3 = -\frac{\alpha}{9} \rightarrow x_3 = -\frac{\alpha}{27} \end{cases}$$

$$\begin{cases} 3x_1 = 0 \rightarrow x_1 = 0 \\ -x_1 + 3x_2 = 1 \rightarrow x_2 = 1/3 \\ \alpha x_2 + 3x_3 = 0 \rightarrow 3x_3 = -\frac{\alpha}{3} \rightarrow x_3 = -\frac{\alpha}{9} \end{cases}$$

$$\begin{cases} 3x_1 = 0 \rightarrow x_1 = 0 \\ -x_1 + 3x_2 = 0 \rightarrow x_2 = 0 \\ \alpha x_2 + 3x_3 = 1 \rightarrow x_3 = 1/3 \end{cases}$$

$$(D-L)^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/9 & 1/3 & 0 \\ \frac{\alpha}{27} & -\frac{\alpha}{9} & \frac{1}{3} \end{bmatrix}$$

$$B_{GS} = (D-L)^{-1} \cdot U = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/9 & 1/3 & 0 \\ -\alpha/27 & -\alpha/9 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\alpha \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 1/9 & -\frac{\alpha}{3} \\ 0 & -\alpha/27 & +\alpha^2/9 \end{bmatrix}$$

$$\det(B_{GS} - \lambda I) = \det \begin{bmatrix} -\lambda & 1/3 & 0 \\ 0 & \frac{1}{9}-\lambda & \frac{\alpha}{3} \\ 0 & -\frac{\alpha}{27} & \frac{\alpha^2}{9}-\lambda \end{bmatrix} =$$

$$= -\lambda \left[\left(\frac{1}{9}-\lambda\right)\left(\frac{\alpha^2}{9}-\lambda\right) - \frac{\alpha^2}{81} \right] =$$

$$= -\lambda \left[\lambda^2 - \frac{1}{9}\lambda - \frac{\alpha^2}{9}\lambda + \cancel{\frac{\alpha^2}{81}} - \cancel{\frac{\alpha^2}{81}} \right] = -\lambda \left[\lambda^2 - \frac{\alpha^2+1}{9}\lambda \right] = 0$$

$$-\lambda^2 \left[\lambda - \frac{\alpha^2+1}{9} \right] = 0$$

$$-\lambda^2 = 0 \rightarrow \lambda_{1/2} = 0$$

$$\lambda - \frac{\alpha^2+1}{9} = 0 \rightarrow \lambda = \frac{\alpha^2+1}{9}$$

$$\sigma(B_{GS}) = \left\{ 0, 0, \frac{\alpha^2+1}{9} \right\} \rightarrow \rho(B_{GS}) = \frac{\alpha^2+1}{9} < 1$$

$$\alpha^2 + 1 < 9 \rightarrow \alpha^2 < 8 \rightarrow -\sqrt{8} < \alpha < \sqrt{8}$$

$$\rightarrow -2\sqrt{2} < \alpha < 2\sqrt{2}$$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

$$x^{(k+1)} = B_J x^{(k)} + \underline{f} \quad \text{con} \quad B_J = D^{-1}(L+U) \ \text{e} \ \underline{f} = D^{-1}\underline{b}.$$

$$B_J = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/3 & 0 & -2/3 \\ 0 & -2/3 & 0 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$x^{(1)} = B_J x^{(0)} + \underline{f} = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/3 & 0 & -2/3 \\ 0 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix} =$$

$$= \begin{bmatrix} 2/3 \\ 0 \\ -1/3 \end{bmatrix}$$

$$x^{(2)} = B_J x^{(1)} + \underline{f} = \begin{bmatrix} 0 & 1/3 & 0 \\ 1/3 & 0 & -2/3 \\ 0 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 2/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 4/9 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/9 \\ 1/3 \end{bmatrix}$$

②

$$A=\begin{bmatrix}\alpha & \alpha & \alpha\\ \alpha & 3 & 0\\ \alpha & 0 & 3\end{bmatrix}$$

- $\det(A)=\alpha(-3\alpha)+3(3\alpha-\alpha^2)=-3\alpha^2+9\alpha-3\alpha^2=$

$$=-6\alpha^2+9\alpha=3\alpha(3-2\alpha)\neq 0$$

$3\alpha\neq 0 \rightarrow \alpha\neq 0$

$3-2\alpha\neq 0 \rightarrow 2\alpha\neq 3 \rightarrow \alpha\neq\frac{3}{2}$

- $B_J=D^{-1}(L+U)=\begin{bmatrix}1/\alpha & 0 & 0\\ 0 & 1/3 & 0\\ 0 & 0 & 1/3\end{bmatrix}\begin{bmatrix}0 & -\alpha & -\alpha\\ -\alpha & 0 & 0\\ -\alpha & 0 & 0\end{bmatrix}=$

$$=\begin{bmatrix}0 & -1 & -1\\ -\frac{\alpha}{3} & 0 & 0\\ -\frac{\alpha}{3} & 0 & 0\end{bmatrix}$$

$\det(B_J-\lambda I)=\det\begin{bmatrix}-\lambda & -1 & -1\\ -\frac{\alpha}{3} & -\lambda & 0\\ -\frac{\alpha}{3} & 0 & -\lambda\end{bmatrix}=$

$$=-\frac{\alpha}{3}(-\lambda)-\lambda\left(\lambda^2-\frac{\alpha}{3}\right)=-\lambda\left(-\frac{\alpha}{3}+\lambda^2-\frac{\alpha}{3}\right)=$$

$$=-\lambda\left(\lambda^2-\frac{2\alpha}{3}\right)=0$$

$-\lambda=0 \rightarrow \lambda_1=0$

$\lambda^2-\frac{2\alpha}{3}=0 \rightarrow \lambda_{2/3}=\pm\sqrt{\frac{2\alpha}{3}}$

$$\sigma(B_J) = \left\{0, -\sqrt{\frac{2\alpha^2}{3}}, \sqrt{\frac{2\alpha^2}{3}}\right\} \qquad \rho(B_J) = \sqrt{\frac{2|\alpha|}{3}} < 1$$

$$\frac{2|\alpha|}{3} < 1 \to |\alpha| < \frac{3}{2} \to -\frac{3}{2} < \alpha < \frac{3}{2}$$

- $D-L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$$\begin{cases} x_1 = 1 & \to x_1 = 1 \\ x_1 + 3x_2 = 0 \to 3x_2 = -1 & \to x_2 = -\frac{1}{3} \\ x_1 + 3x_3 = 0 \to 3x_3 = -1 & \to x_3 = -\frac{1}{3} \end{cases}$$

$$\begin{cases} x_1 = 0 & \to x_1 = 0 \\ x_1 + 3x_2 = 1 & \to x_2 = 1/3 \\ x_1 + 3x_3 = 0 & \to x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 & \to x_1 = 0 \\ x_1 + 3x_2 = 0 & \to x_2 = 0 \\ x_1 + 3x_3 = 1 & \to x_3 = 1/3 \end{cases}$$

$$B_{GS} = (D-L)^{-1} \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1/3 & 0 \\ -1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}$$

$$\underline{f} = (D-L)^{-1} \cdot \underline{b} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1/3 & 0 \\ -1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{4}{3} \\ 8/3 \end{bmatrix}$$

$$\underline{x}^{(1)} = B_{GS}\underline{x}^{(0)} + \underline{f} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ -4/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ -4/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{4}{3} \\ \frac{8}{3} \end{bmatrix}$$

$$\underline{x}^{(2)} = B_{GS}\,\underline{x}^{(1)} + \underline{f} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 \\ -4/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} 8 \\ -4/3 \\ 8/3 \end{bmatrix} =$$

$$= \begin{bmatrix} -4/3 \\ 4/9 \\ 4/9 \end{bmatrix} + \begin{bmatrix} 8 \\ -4/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 20/3 \\ -8/9 \\ 28/9 \end{bmatrix}$$

③

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 1 & \alpha \end{bmatrix}$$

$$\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & \alpha-\lambda & 1 \\ 0 & 1 & \alpha-\lambda \end{bmatrix} = (2-\lambda)\left[(\alpha-\lambda)^2 - 1\right] = 0$$

$$2-\lambda = 0 \rightarrow \lambda_1 = 2$$

$$(\alpha-\lambda)^2 - 1 = 0 \rightarrow (\alpha-\lambda)^2 = 1 \rightarrow \alpha-\lambda = \pm 1$$

$$\rightarrow \lambda_{2/3} = \alpha \pm 1$$

$$\sigma(A) = \{2, \alpha-1, \alpha+1\}$$

$$\alpha - 1 > 0 \rightarrow \alpha > 1$$
$$\alpha + 1 > 0 \rightarrow \alpha > -1$$

A è definita positiva per $\alpha > 1$.

$$D-L=\begin{bmatrix}2&0&0\\0&\alpha&0\\0&1&\alpha\end{bmatrix}$$

$$\begin{cases}2x_1=1 & \to x_1=1/2\\ \alpha x_2=0 & \to x_2=0\\ x_2+\alpha x_3=0 & \to x_3=0\end{cases}$$

$$\begin{cases}2x_1=0 & \to x_1=0\\ \alpha x_2=1 & \to x_2=1/\alpha\\ x_2+\alpha x_3=0 & \to \alpha x_3=-1/\alpha \to x_3=-1/\alpha^2\end{cases}$$

$$\begin{cases}2x_1=0 & \to x_1=0\\ \alpha x_2=0 & \to x_2=0\\ x_2+\alpha x_3=1 & \to x_3=1/\alpha\end{cases}$$

$$B_{GS}=(D-L)^{-1}\cdot U=\begin{bmatrix}1/2&0&0\\0&1/\alpha&0\\0&-1/\alpha^2&1/\alpha\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-1\\0&0&0\end{bmatrix}=$$

$$=\begin{bmatrix}0&0&0\\0&0&-1/\alpha\\0&0&1/\alpha^2\end{bmatrix}\qquad \sigma(B_{GS})=\left\{0,0,\frac{1}{\alpha^2}\right\}$$

$$\rho(B_{GS})=\frac{1}{\alpha^2}<1 \to \alpha^2>1 \to \alpha< \quad \wedge \ \alpha>1$$

$$B_J=D^{-1}(L+U)=\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&-1/2\\0&-1/2&0\end{bmatrix}$$

$$\underline{f}=D^{-1}\cdot\underline{b}=\begin{bmatrix}1/2&0&0\\0&1/2&0\\0&0&1/2\end{bmatrix}\begin{bmatrix}4\\2\\1\end{bmatrix}=\begin{bmatrix}2\\1\\1/2\end{bmatrix}$$

$$\underline{x}^{(1)} = B_J\underline{x}^{(0)} + \underline{f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\underline{x}^{(2)} = B_J\underline{x}^{(1)} + \underline{f} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ -1/4 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/4 \\ 1/4 \end{bmatrix}$$

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$$A = \begin{bmatrix} \alpha & 0 & \alpha \\ 0 & 2 & 0 \\ \alpha & 0 & 2 \end{bmatrix} \qquad \det(A) = 2(2\alpha - \alpha^2) = 2\alpha(2-\alpha) \neq 0$$

$$2\alpha \neq 0 \to \alpha \neq 0$$

$$2-\alpha \neq 0 \to \alpha \neq 2$$

$$B_J = D^{-1}(L+U) = \begin{bmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -\frac{\alpha}{2} & 0 & 0 \end{bmatrix}$$

$$\det(B_J - \lambda I) = -\lambda\left(\lambda^2 - \frac{\alpha}{2}\right) = 0$$

$$\lambda_1 = 0$$

$$\lambda_{2/3} = \pm\sqrt{\frac{\alpha}{2}}$$

$$\sigma(B_J) = \left\{ 0, -\sqrt{\frac{\alpha}{2}}, \sqrt{\frac{\alpha}{2}} \right\} \rightarrow \rho(B_J) = \sqrt{\frac{|\alpha|}{2}} < 1$$

$$\frac{|\alpha|}{2} < 1 \rightarrow |\alpha| < 2 \rightarrow -2 < \alpha < 2$$

$$D - L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\begin{cases} x_1 = 1 & \rightarrow x_1 = 1 \\ 2x_2 = 0 & \rightarrow x_2 = 0 \\ x_1 + 2x_3 = 0 & \rightarrow 2x_3 = -1 \rightarrow x_3 = -1/2 \end{cases}$$

$$\begin{cases} x_1 = 0 & \rightarrow x_1 = 0 \\ 2x_2 = 1 & \rightarrow x_2 = 1/2 \\ x_1 + 2x_3 = 0 & \rightarrow 2x_3 = 0 \rightarrow x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 & \rightarrow x_1 = 0 \\ 2x_2 = 0 & \rightarrow x_2 = 0 \\ x_1 + 2x_3 = 1 & \rightarrow x_3 = 1/2 \end{cases}$$

$$B_{GS} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\underline{f} = (D-L)^{-1}\underline{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\underline{x}^{(1)} = B_{GS}\underline{x}^{(0)} + \underline{f} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\underline{x}^{(2)} = B_{GS}\underline{x}^{(1)} + \underline{f} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

Il metodo è consistente poiché

$$x^{(1)} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} \Rightarrow x^{(2)} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix}$$

ed è convergente ($\rho(B_{GS}) = \frac{1}{2} < 1$) quindi la soluzione è $\underline{x} = [1, \frac{1}{2}, 0]^T$.