ON THE SPECTRAL DISTRIBUTION AND MAXIMAL RESULTS FOR GEOMETRIC MEANS OF HPD GLT MATRIX SEQUENCES

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In this work, we extend our previous analysis [1] on the spectral distribution of the geometric mean of matrix-sequences formed by Hermitian Positive Definite (HPD) matrices under the framework of Generalized Locally Toeplitz (GLT) $*$-algebra. Building upon our earlier results, we address the necessity of the invertibility assumption on the GLT symbols for ensuring that the geometric mean admits the expected spectral distribution. Motivated by the fact that inversion is required mainly due to non-commutativity, we consider the case where the GLT symbols commute and rigorously prove that the invertibility assumption can be relaxed, thereby resolving a longstanding conjecture in the field.

Furthermore, we extend the spectral analysis to the Karcher mean for more than two HPD GLT matrix-sequences, showing that the resulting mean remains within the GLT class, with the symbol given by the geometric mean of the individual symbols. Numerical experiments validate our theoretical findings and illustrate the extremal spectral behavior and the emergence of GLT momentary symbols in the case of non-commuting, degenerate symbols, where the standard distribution formula fails or becomes ill-posed.

Finally, our results are extended to the multilevel block case (for $r, d \geq 1$), offering a broader generalization and deeper numerical validation for the spectral theory of geometric means of structured matrix-sequences.

References

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