A CRITICAL VIEW ON THE GLT THEORY: PAST, PRESENT, PERSPECTIVES, AND BEYOND

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The idea of Generalized Locally Toeplitz (GLT) sequences has been introduced as a generalization both of classical Toeplitz sequences and of variable coefficient differential operators and, for every sequence of the class, it has been demonstrated that it is possible to give a rigorous description of the asymptotic spectrum in terms of a function (the symbol) that can be easily identified.

For every $r, d \geq 1$ the r-block d-level GLT class has nice $*$-algebra features and indeed it has been proven that it is stable under linear combinations, products, and inversion when the sequence which is inverted shows a sparsely vanishing symbol (sparsely vanishing symbol = a symbol whose minimal singular value vanishes at most in a set of zero Lebesgue measure). Furthermore, the GLT $*$-algebras virtually include any approximation of partial differential equations (PDEs), fractional differential equations (FDEs), integro-differential equations (IDEs) by local methods (Finite Difference, Finite Element, Isogeometric Analysis etc) and, based on this, we demonstrate that our results on GLT sequences can be used in a PDE/FDE/IDE setting in various directions, including preconditioning, multigrid, spectral detection of branches, fast 'matrix-less' computation of eigenvalues, stability issues, asymptotic low-rank structures, and challenges such as the GLT use in tensors, stochastic, machine learning algorithms. We will discuss also the impact and the further potential of the theory with special attention to new tools and to new directions as those based on symmetrization tricks, on the extra-dimensional approach, and on blocking structures/operations.

References

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