ORTHOGONAL POLYNOMIALS WITH A SKEW-HERMITIAN DIFFERENTIATION MATRIX

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We are interested in solving a PDE of the form $\partial_t u = \partial_x(a(x)\partial_x u)$, $a(x) > 0$. After discretization in the x-variable one arrives at a system $\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t)$, $\mathbf{u}(t) = [u(t,x_1), u(t,x_2), \ldots]^T \in \ell^2$, $\mathbf{u}(0) = \mathbf{u}_0$, $\mathcal{D}$ is a finite difference approximation of the partial derivative ∂_x and $\mathcal{A} = \mathrm{diag}(a(x_1), a(x_2), \ldots)$. Stability requires the system to be dissipative: $\frac{1}{2}\frac{\mathrm{d}\|\mathbf{u}\|^2}{\mathrm{d}t} = \mathbf{u}^T\mathbf{u}' = \mathbf{u}^T\mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u} = (\mathcal{D}^T\mathbf{u})^T\mathcal{A}(\mathcal{D}\mathbf{u}) < 0$. If $\mathcal{D}$ is skew symmetric, then $\mathcal{D}^T = -\mathcal{D}$ and stability is satisfied automatically. When using finite differences like $\frac{f(x+\Delta/2)-f(x-\Delta/2)}{2\Delta}$, then $\mathcal{D}$ is skew symmetric, but that is a bit of an exception. However if we want to use spectral methods, then we assume $u(t,\cdot) = \sum_n u_n(t)\varphi_n$ with $\{\varphi_n\}_n$ an orthogonal basis for $L^2(\mathbf{R})$. If the Fourier basis $\{\cos(n\xi), \sin(n\xi)\}_n$ on $[-\pi,\pi]$ or the Hermite polynomials on $\mathbf{R}$ are arranged in a column Φ, then they satisfy $\xi\hat{\Phi}(\xi) = \mathcal{J}\hat{\Phi}(\xi)$ with a symmetric Jacobi matrix $\mathcal{J}$. Take the Fourier transform with the proper weight and the result is $\Phi'(x) = \mathrm{i}\mathcal{J}\Phi(x)$, with $\mathcal{D} = \mathrm{i}\mathcal{J}$ skew Hermitian. A. Iserles and M. Webb recently used this idea and took the Fourier transforms of the Laguerre basis, which resulted in a rational basis that is essentially the rational basis found independently by F. Malmquist and S. Takenaka in 1926. A clever transformation to the unit circle allows to use the fast computation of the Fourier coefficients by FFT. In this lecture we shall explore the effect of free parameters that are still allowed in this approach.

References

[1] A. Iserles and M. Webb. A family of orthogonal rational functions and other orthogonal systems with a skew-symmetric differentiation matrix. *Journal Fourier Analysis and Applications*, 2019. To appear.