Optimally conditioned Vandermonde-like matrices

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Vandermonde matrices arise frequently in computational mathematics in problems that require polynomial approximation, differentiation, or integration. These matrices are defined by a set of n distinct nodes $x_1, x_2, \ldots, x_n$ and a monomial basis. A difficulty with Vandermonde matrices is that they typically are quite ill-conditioned when the nodes are real and n is not very small. The ill-conditioning often can be reduced significantly by using a basis of orthonormal polynomials $p_0, p_1, \ldots, p_{n-1}$, with $\deg(p_j) = j$. This was first observed by Gautschi. The matrices so obtained are commonly referred to as Vandermonde-like and are of the form $V_{n,n} = [p_{i-1}(x_j)]_{i,j=1}^{n} \in R^{n\times n}$. Gautschi analyzed optimally conditioned and optimally scaled square Vandermonde and Vandermonde-like matrices with real nodes. We extend Gautschi's analysis to rectangular Vandermonde-like matrices with real nodes, as well as to Vandermonde-like matrices with nodes on the unit circle in the complex plane. Additionally, we investigate existence and uniqueness of optimally conditioned Vandermonde-like matrices. Finally, we discuss properties of rectangular Vandermonde and Vandermonde-like matrices $V_{N,n}$ of order $N \times n$, $N \neq n$, with Chebyshev nodes or with equidistant nodes on the unit circle in the complex plane, and show that the condition number of these matrices can be bounded independently of the number of nodes.

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