Solution methods for ill-posed problem

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Abstract: Many problems in Science and Engineering aim to determine the cause of an observed effect. Examples include remote sensing, where one is interested in determining the cause of a measurement, and image restoration, where one is interested in determining the "true" image from an available contaminated image. The purpose of these lectures is to provide the mathematical foundation for solution methods for this kind of problems. The above problems are said to be ill-posed because

1. they may not have a solution,
2. the solution might not be unique, and
3. the solution may not depend continuously on the data provided.

Non-existence and non-unicity often can be overcome by formulating a related minimization problem. For instance, we may formulate a variational problem and determine its minimal-norm solution, which typically exists and is unique even when the original problem does not have a solution or its solution is not unique. The discontinuous dependence of the solution on the data can be handled by regularization. Regularization entails replacing the given problem by a nearby problem, whose solution depends continuously on the data. In particular, the solution of the regularized problem is less sensitive to errors in the data than the original problem. The most common regularization methods are Tikhonov regularization, truncated singular value decomposition, and truncated iteration. The latter

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regularization method is well suited for large-scale problems that are solved by an iterative method.

Many ill-posed problems may be considered underdetermined. It therefore often is beneficial to impose constraints on the computed solution that the desired solution is known to have. The constraints may include nonnegativity and sparsity. The latter refers to that in a suitable basis, the desired solution can be expressed as a linear combination of fairly few basis functions.

These lectures present techniques from linear algebra required to compute the solution of linear discrete ill-posed problems. These include direct and iterative methods. How to bridge the gap between linear discrete ill-posed problems and linear ill-posed problems also will be discussed. Application of these techniques to nonlinear problems are described. Several methods have been developed for the computation of sparse solutions. Some of these methods, as well as techniques for accelerating the computations with these methods, will be described.