COMPUTING THE ROOTS OF MANDELBROT POLYNOMIALS: AN EXPERIMENTAL ANALYSIS

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The celebrated Mandelbrot set is formed by the complex numbers c such that the sequence $x_0 = 0$, $x_{i+1} = x_i^2 + c$, does not diverge to infinity. Relevant points of this set are the numbers c for which the sequence $\{x_i\}$ provides a cycle of finite length k. These values are the roots of the polynomial $p_k(z)$ of degree $n = 2^k - 1$ defined by the recurrence $p_1(z) = z + 1$, $p_{i+1} = zp_i(z)^2 + 1$, $i = 1, \dots, k-1$. Efforts to compute these roots have been done by several authors [2], [4]. In this talk we provide an algorithm based on the Ehrlich-Aberth iterations [1] complemented by the Fast Multipole Method of [3], and by the fast search of near neighbors of a set of complex numbers, that have a cost of roughly $O(n)$ arithmetic operations per step. In our experiments, the number of iterations needed to arrive at numerical convergence is practically constant. This allows to compute the roots of $p_k(x)$ up to degree $n = 2^{24} - 1$ in a few minutes on a laptop with 16 GB RAM and an Intel I3 processor. Larger degrees can be treated on platforms with a higher amount of RAM.

References

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