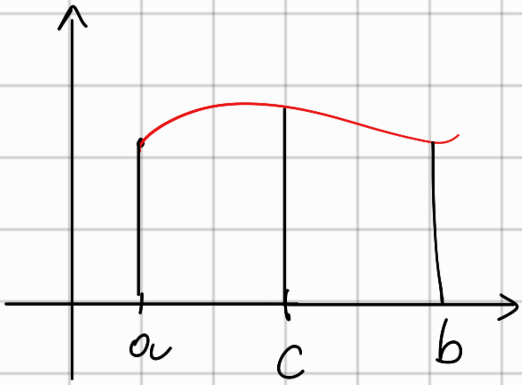
Lezione 23

26/11/24

PROPRIETÀ



- $$\int_a^a f(x)\,dx = 0$$

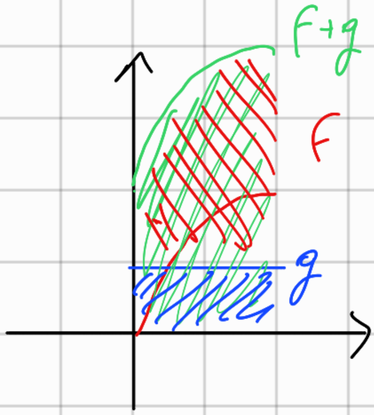
- $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$a < b$

- $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$a < c < b$

LINEARITÀ



- $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

- $$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

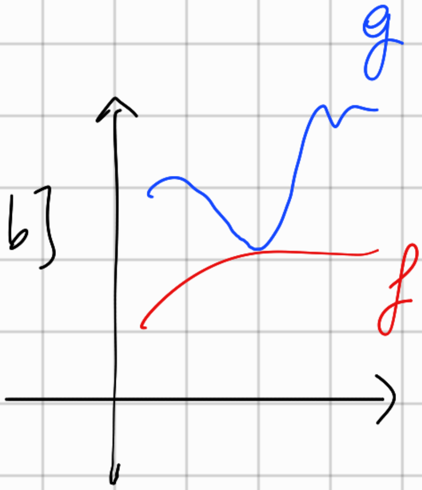
$\alpha \in \mathbb{R}$

MONOTONIA

$$f(x) \leq g(x) \quad \forall x \in [a,b]$$

$$\Downarrow$$

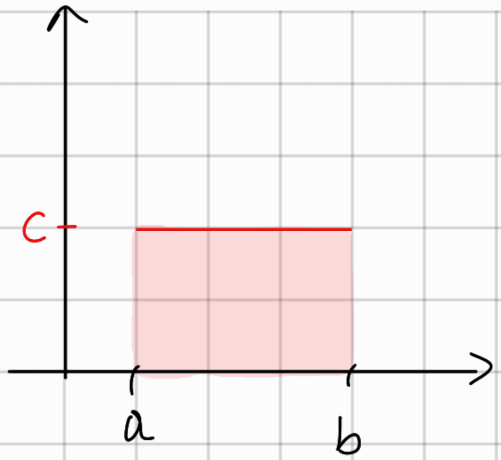
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$



ESEMPIO

INTEGRALE DELLA FUNZIONE COSTANTE $c \in \mathbb{R}$

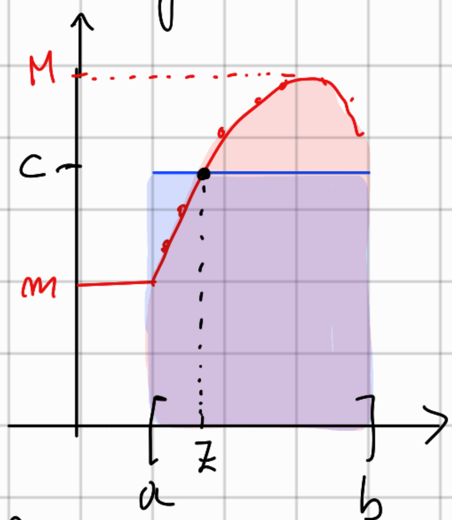
$$\int_a^b c\,dx = (b-a)c$$



TEOREMA DELLA MEDIA INTEGRALE

Se $f(x)$ CONTINUA in $[a,b]$ allora $\exists\, z \in [a,b]$ tale che

$$\int_a^b f(x)\,dx = (b-a)f(z)$$



DIM.

f è CONTINUA IN $[a,b]$

$\Downarrow$ TEO. WEIERSTRASS

$\exists\, M, m$ massimo e minimo della funzione

$\Downarrow$

$$m \leq f(x) \leq M \qquad \forall x \in [a,b]$$

$\downarrow$ MONOTONIA

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$$

$\downarrow$ DIVIDO PER $(b-a)$

$$m \leq \boxed{\frac{\int_a^b f(x)\,dx}{b-a}} \leq M$$

è UN VALORE COMPRESO TRA MASSIMO E MINIMO

SICCOME f CONTINUA IN $[a,b]$

IL TEOREMA DEI VALORI INTERMEDI CI GARANTISCE CHE

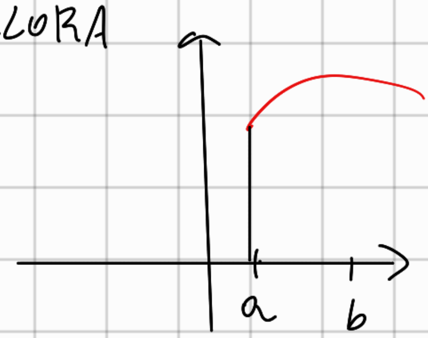
$$\exists\, z \in [a,b] \text{ t.c. } f(z) = \frac{\int_a^b f(x)\,dx}{b-a}$$

$$(b-a)\, f(z) = \int_a^b f(x)\,dx$$

□

TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

Se f è CONTINUA IN $[a,b]$ ALLORA



$$F(x) = \int_a^x f(t)\,dt$$

È DERIVABILE ED È UNA PRIMITIVA DI $f(x)$

CIOÈ $F'(x) = f(x)$

↓ COROLLARIO

- SE $F(x)$ è UNA PRIMITIVA DI f

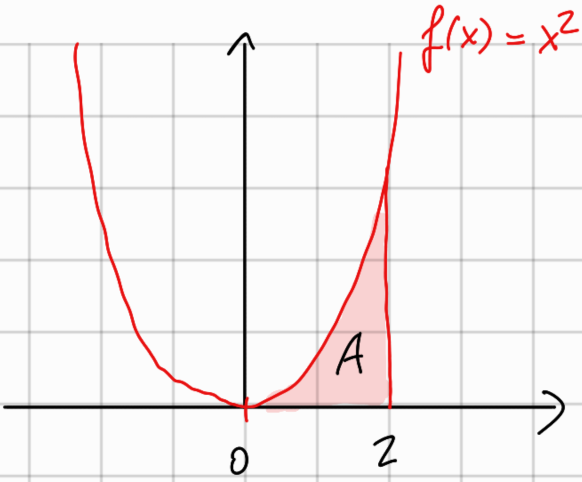
$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

NO DIM

ESEMPIO

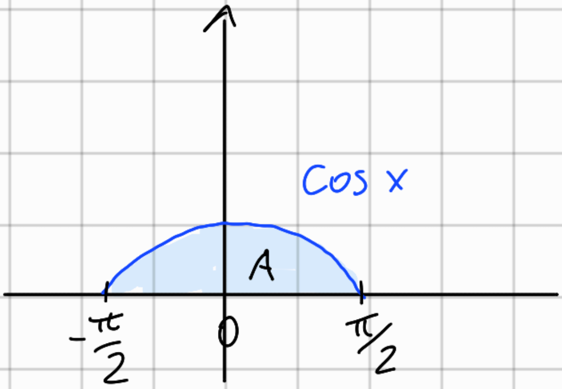
$$A = \int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = $$

$$= \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

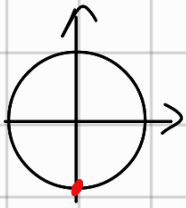


ESEMPIO

$$A = \int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin x\Big|_{-\pi/2}^{\pi/2} =$$



$$= \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) =$$



$$= 1 - (-1) = 1 + 1 = 2$$

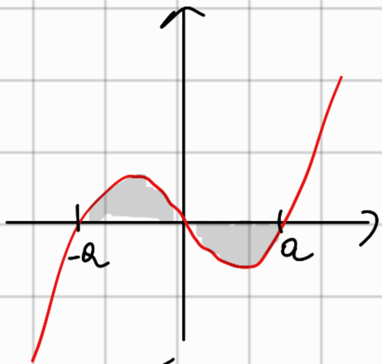
PROPOSIZIONE

SE f PARI (cioe' $f(x) = f(-x)$)

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

SE f e' DISPARI ($f(x) = -f(-x)$)

$$\int_{-a}^{a} f(x)\,dx = 0$$

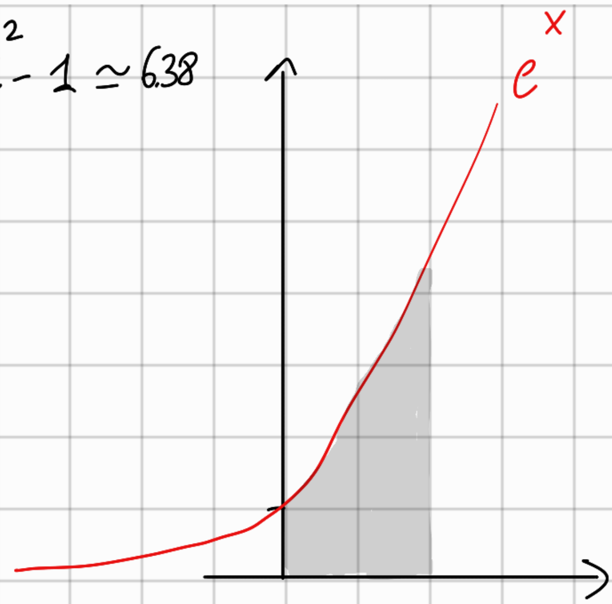


Siccome $\cos(x)$ e' PARI $\Rightarrow$

$$A = 2\int_0^{\pi/2} \cos x\,dx = 2\left[\sin x\right]_0^{\pi/2} =$$

$$= 2\left[1 - 0\right] = 2$$

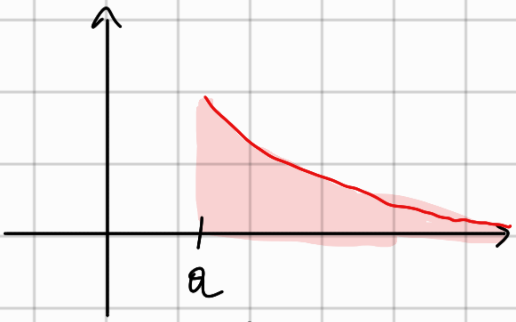
$$\int_0^2 e^x\,dx = e^x\Big|_0^2 = e^2 - e^0 = e^2 - 1 \simeq 6.38$$



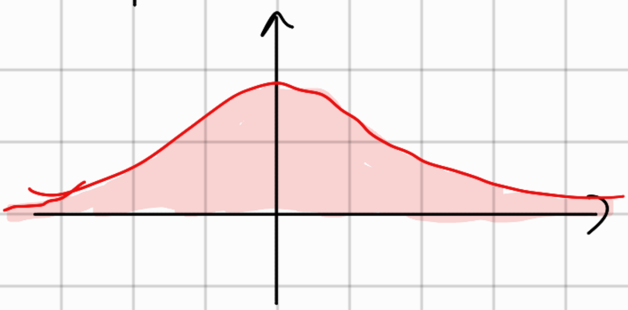
SI PUÒ ESTENDERE LA DEFINIZIONE

INTEGRALI IMPROPRI

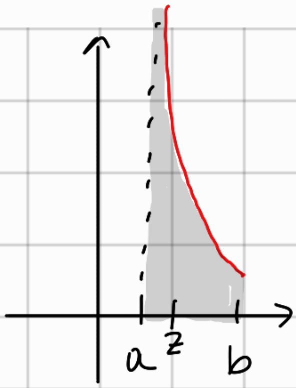
$$= \int_{-\infty}^{b} f(x)\,dx = \lim_{z\to-\infty} \int_{z}^{b} f(x)\,dx$$



$$\int_{a}^{+\infty} f(x)\,dx = \lim_{z\to+\infty} \int_{a}^{z} f(x)\,dx$$

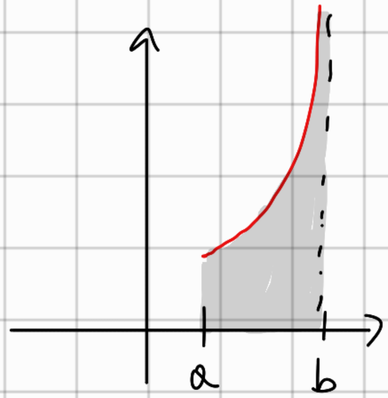


$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{z\to+\infty} \int_{-z}^{z} f(x)\,dx$$



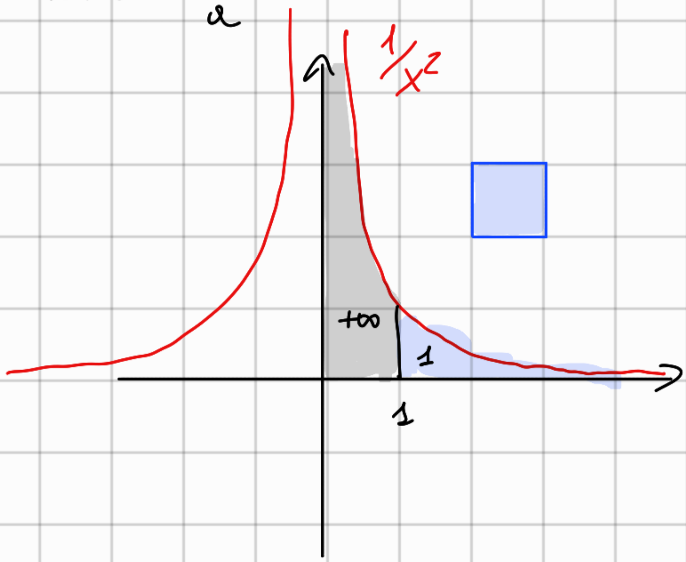
Se $\lim_{x \to a^+} f(x) = \infty$ A.V.

$$\int_a^b f(x) = \lim_{z \to a^+} \int_z^b f(x)\,dx$$



Se $\lim_{x \to b^-} f(x) = \infty$

$$\int_a^b f(x) = \lim_{z \to b^-} \int_a^z f(x)\,dx$$



$$f(x) = \frac{1}{x^2}$$

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{x^2}\,dx$$

$$= \lim_{z \to 0^+} \int_z^1 x^{-2}\,dx = \lim_{z \to 0^+} \left[-\frac{1}{x}\right]_z^1 =$$

$$= \lim_{z \to 0^+} \left[-\frac{1}{1} + \frac{1}{z}\right] =$$

$$= \lim_{z \to 0^+} \left(\frac{1}{z} - 1\right) = +\infty$$

$$\frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} x^{-2}\,dx = \lim_{z\to+\infty} \int_1^{z} x^{-2}\,dx =$$

$$= \lim_{z\to+\infty} \left(-\frac{1}{x}\right)\Big|_1^{z} = \lim_{z\to+\infty} \left[-\frac{1}{z}+\frac{1}{1}\right] =$$

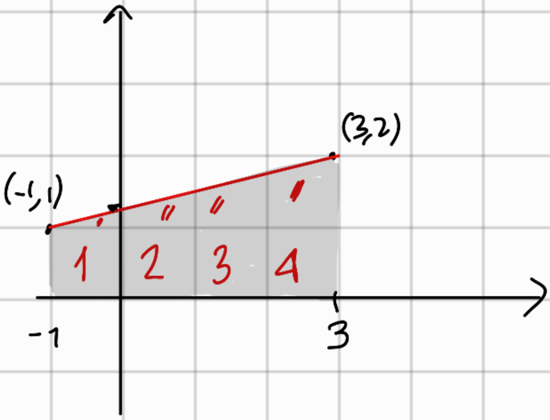
$$= \lim_{z\to+\infty} \left[1-\underbrace{\frac{1}{z}}_{0}\right] = 1$$

PER CASA

- $\displaystyle\int_{-\infty}^{0} e^{x}\,dx$
- $\displaystyle\int_0^1 \frac{1}{x}\,dx$, $\displaystyle\int_1^{+\infty} \frac{1}{x}\,dx$
- $\displaystyle\int_1^{+\infty} \frac{2}{\sqrt{x}}\,dx$

ESERCIZIO

Scrivi l'integrale che calcola l'area rappresentata in figura



$$\int_{-1}^{3} f(x)\,dx$$

$$y = mx + q$$

$$\begin{cases} 1 = m(-1) + q \\ 2 = m(3) + q \end{cases} \rightarrow \begin{cases} -m + q = 1 \\ 3m + q = 2 \end{cases} \rightarrow \begin{cases} q = m + 1 \\ 3m + m + 1 = 1 \end{cases}$$

$$\rightarrow \begin{cases} \text{"} \\ 4m = 1 \end{cases} \rightarrow \begin{cases} q = 5/4 \\ m = 1/4 \end{cases}$$

$$\int_{-1}^{3} \left(\frac{1}{4}x + \frac{5}{4}\right) dx = \frac{1}{4} \int_{-1}^{3} (x + 5)\, dx = \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_{-1}^{3} =$$

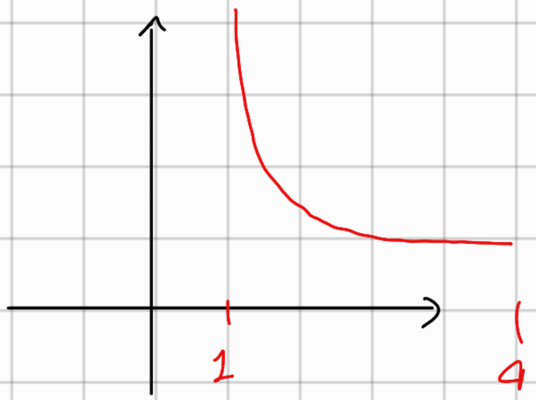
$$= \frac{1}{4}\Big[\underbrace{\frac{9}{2} + 15}_{F(b)} \underbrace{- \frac{1}{2} + 5}_{-F(a)}\Big] = \frac{1}{4}\left[4 + 20\right] = 6$$

ESERCIZIO

DIRE SE LE SEGUENTI FUNZIONI SODDISFANO LE IPOTESI DEL TEOREMA DELLA MEDIA INTEGRALE

$y = \dfrac{4}{x-1}$ $\quad [1, 4]$ → NON È CONTINUA IN 1 → NO

$x \neq -1$



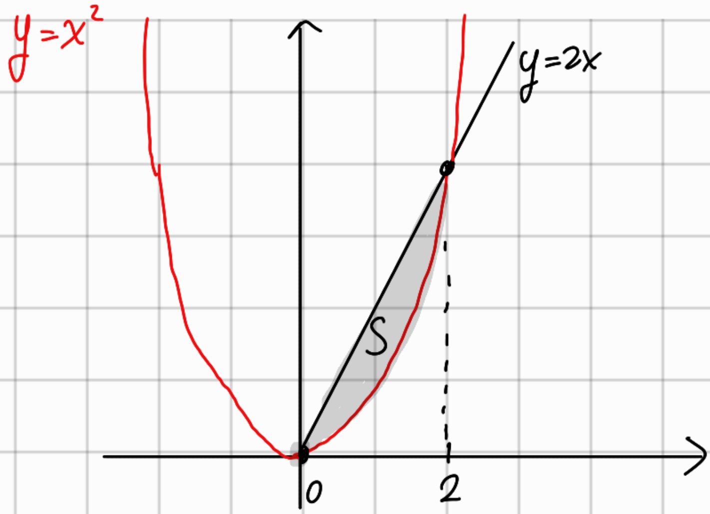
$y = \sqrt{x}$ $\quad [0, 6]$ $\quad$ SÌ

$$\Rightarrow \exists\, z \in [0, 6] \text{ t.c. } \int_0^6 \sqrt{x}\, dx = f(z)$$

$y = \ln(x+3)$ $\quad (-1, 0]$ $\quad$ NO

$$\begin{pmatrix} x + 3 > 0 \\ x > -3 \end{pmatrix}$$

Calcola l'area in figura



$$\begin{cases} y = x^2 \\ y = 2x \end{cases}$$

$\downarrow$

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0 \quad \rightarrow \quad \begin{matrix} x = 0 \\ x = 2 \end{matrix}$$

$$S = \int_0^2 2x\,dx - \int_0^2 x^2\,dx = \quad \left(\text{OPPURE} \int_0^2 (2x - x^2)\,dx \right)$$

$$= x^2 \Big|_0^2 - \frac{x^3}{3}\Big|_0^2 = 4 - 0 - \left[\frac{8}{3} - 0\right] = 4 - \frac{8}{3} = \frac{12-8}{3} = \frac{4}{3}$$

PER CASA

$$\int_2^4 \frac{2x^2+x+1}{2x-1}\,dx$$