Lezione 17 02/12/23

ESERCIZIO : Calcolare le seguenti derivate

$f(x) = 5\ln(x) \rightarrow f'(x) = 5\,\frac{1}{x} = \frac{5}{x}$

$f(x) = e^x - 3\ln(x) \rightarrow f'(x) = e^x - 3\,\frac{1}{x} = e^x - \frac{3}{x}$

$f(x) = x\ln(x) \rightarrow f'(x) = 1\cdot\ln(x) + \cancel{x}\,\frac{1}{\cancel{x}} = \ln(x) + 1$

$f(x) = e^x \sin(x) \rightarrow f'(x) = e^x \sin(x) + e^x \cos(x) = e^x(\sin(x) + \cos(x))$

$f(x) = \underbrace{x^2\cos x}_{f}\,\underbrace{\sin x}_{g} \rightarrow f'(x) = [x^2\cos x]'\sin(x) + x^2\cos x\cos x$

$\frac{1}{2}\sin(2x) = \cos x \sin x$

$= [2x\cos x + x^2(-\sin(x))]\sin x + x^2\cos^2 x$

$= 2x\cos x\sin x - x^2\sin^2 x + x^2\cos^2 x$

$= x\sin(2x) - x^2(\underbrace{\cos^2 x - \sin^2 x}_{\cos(2x)})$

$= x\sin(2x) - x^2\cos(2x)$

$f'(x) = -\sin x$

$f(x) = \cos x$

$g(x) = \ln(x)$

$f(x) = \cos(\ln(x)) \rightarrow f'(x) = -\sin(\ln(x))\cdot\frac{1}{x} = -\frac{\sin(\ln x)}{x}$

$[\cos(3x)]^2$

$f(x) = \cos^2(3x) + 2 \rightarrow f'(x) = \underline{2\cos(3x)}\,\underline{(-\sin(3x))\cdot 3}$

$= -6\cos(3x)\sin(3x)$

$[\cos(3x)]^2$

$f(x) = x^2 \rightarrow f'(x) = 2x$

$f'(g(x)) = 2\cos(3x)$

$g'(x) = -\sin(3x)\cdot 3$

$g(x) = \cos(3x)$

$f(x) = \cos(x) \rightarrow f'(x) = -\sin(x) \rightarrow f'(g(x)) = -\sin(3x)$

$g(x) = 3x$

$g'(x) = 3$

$$f(x) = \frac{\sqrt{x}}{e^x - 1} = \frac{x^{\frac{1}{2}}}{\underbrace{e^x - 1}_{\downarrow \; e^x + 0}}$$

$$f'(x) = \frac{\frac{1}{2} \underbrace{x^{1-\frac{1}{2}}}_{= -\frac{1}{2}, \; \frac{1}{\sqrt{x}}} (e^x - 1) - \underbrace{x^{\frac{1}{2}}}_{\sqrt{x}} e^x}{(e^x - 1)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}} (e^x - 1) - \sqrt{x}\, e^x}{(e^x - 1)^2}$$

$$= \frac{1}{(e^x - 1)} \left[\frac{(e^x - 1) - 2\overset{\sqrt{x}\sqrt{x}}{x} e^x}{2\sqrt{x}} \right]$$

$$f(x) = \sqrt{x^3} = x^{\frac{3}{2}}$$

$$f'(x) = \frac{3}{2} x^{\overset{\frac{3}{2}-1}{\frac{1}{2}}} = \frac{3}{2}\sqrt{x}$$

PER CASA

- $f(x) = \dfrac{1}{\sqrt[4]{x^3}}$
- $f(x) = \dfrac{\sqrt{x^2 - 7}}{5\cos(x^3)}$

PUNTI DI NON DERIVABILITÀ

Ricordiamo che f è DERIVABILE SE $\exists$ FINITI $f'_+(x_0) = f'_-(x_0)$

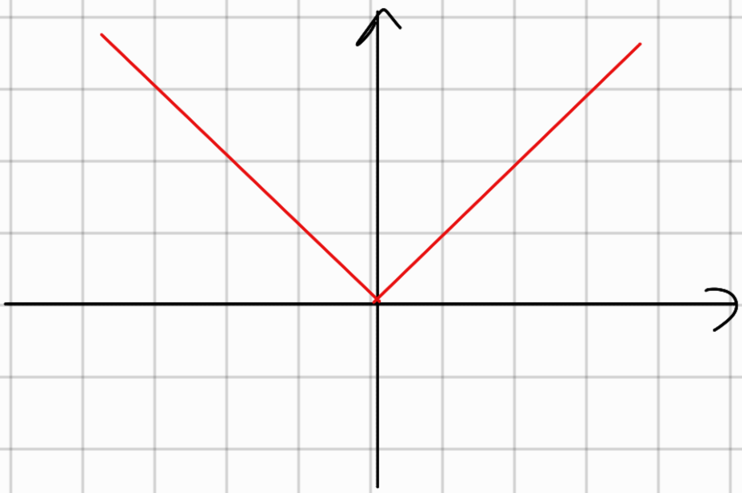
$$\left(\Leftrightarrow \exists \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \right)$$

CASO 1 PUNTO ANGOLOSO

ALMENO UNO DEI DUE LIMITI È FINITO

$f'_+(x_0) = \ell \in \mathbb{R}$

$f'_-(x) \in \mathbb{R} + \{+\infty, -\infty\}$



ESEMPIO

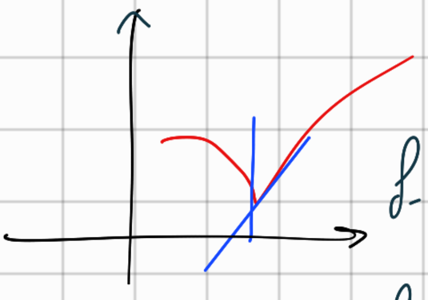
$|x|$ in $x_0 = 0$

$f'_+(0) = 1$

$f'_-(0) = -1$

$$f(x) = \begin{cases} e^x & x < 0 \\ x^2+1 & x \geq 0 \end{cases}$$

PER CASA VERIFICARE
SE 0 È UN PUNTO ANGOLOSO



$f'_-(x_0) = -\infty$

$f'_+(x_0) = \ell$

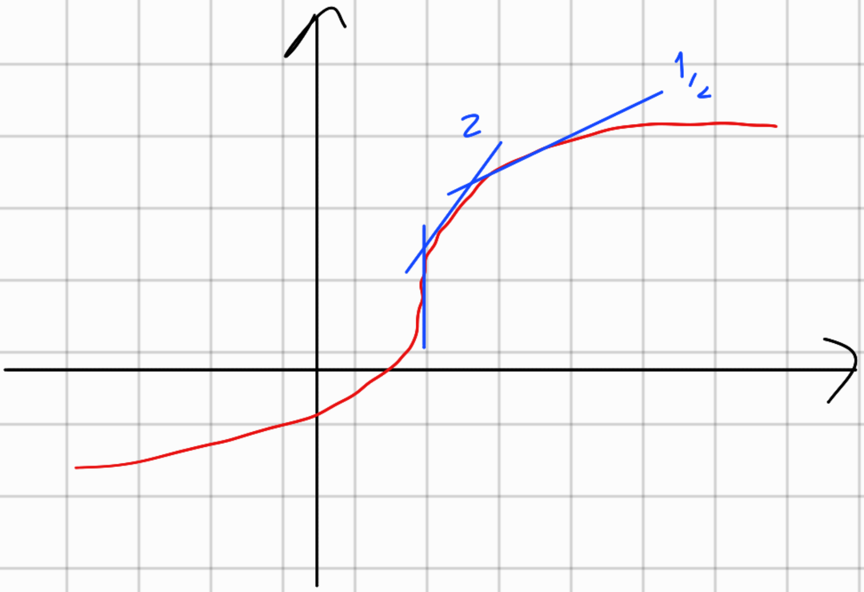
$\lim_{x \to 0^-} e^x$

$\lim_{x \to 0^+} x^2+1$

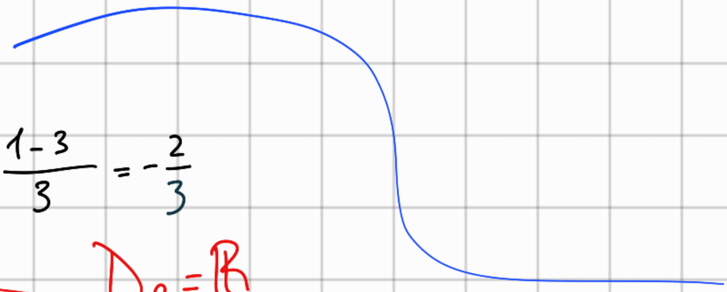
SONO UGUALI O NO?

CASO 2 FLESSO VERTICALE

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = +\infty \quad (-\infty)$$



I LIMITI SONO UGUALI
MA ENTRAMBI INFINITI



$$\frac{1}{3} - 1 = \frac{1-3}{3} = -\frac{2}{3}$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \leftarrow D_f = \mathbb{R}$$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \leftarrow D_{f'} = \mathbb{R} \setminus \{0\}$$

$$\lim_{x\to 0^+} \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{0^+} = +\infty$$

← 0 È UN P.TO DI FLESSO VERTICALE

$0 \in D_f \quad 0 \notin D_{f'}$

e $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = +\infty$

NOTA: Se la retta tangente in un p.to è VERTICALE

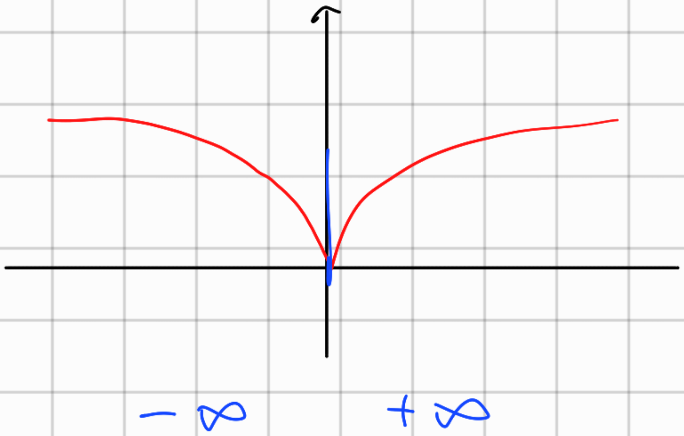
$\Rightarrow f'(x) \longrightarrow \infty$

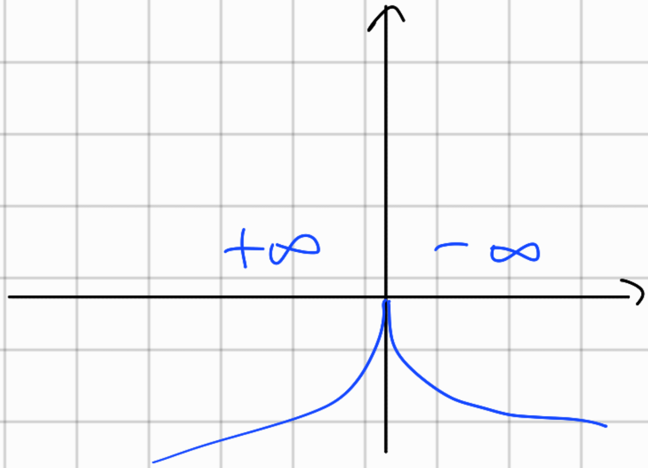
$\boxed{\text{CASO 3}}$ PUNTI DI CUSPIDE

$$f'_+(x_0) = +\infty$$

$$f'_-(x_0) = -\infty$$

e viceversa





ESEMPIO

$$\frac{2-3}{3} = -\frac{1}{3}$$

$$f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}} \qquad D_f = \mathbb{R}$$

$$f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}} \qquad D_{f'} = \mathbb{R} - \{0\}$$

$$\lim_{x \to 0^-} \frac{2}{3\sqrt[3]{x}} = \frac{2}{0^-} = -\infty$$

$$\lim_{x \to 0^+} \frac{2}{3\sqrt[3]{x}} = \frac{2}{0^+} = +\infty$$

CASO 4

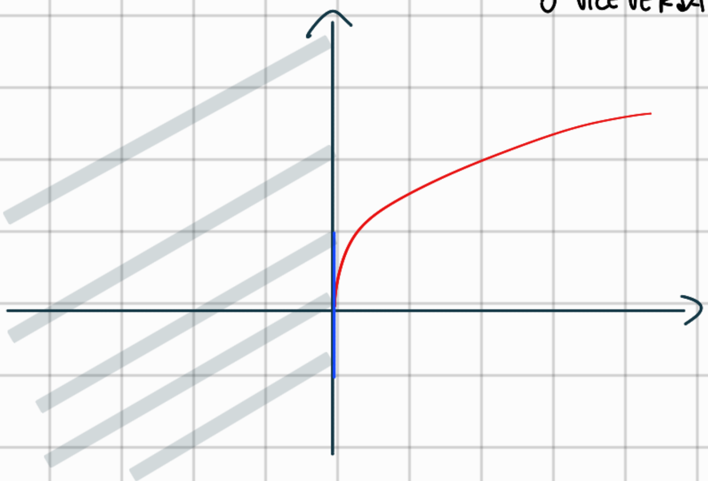
SE LA FUNZIONE NON ESISTE DA UN LATO $\exists f'_+(x_0)$ $\nexists f'_-(x_0)$

O VICEVERSA

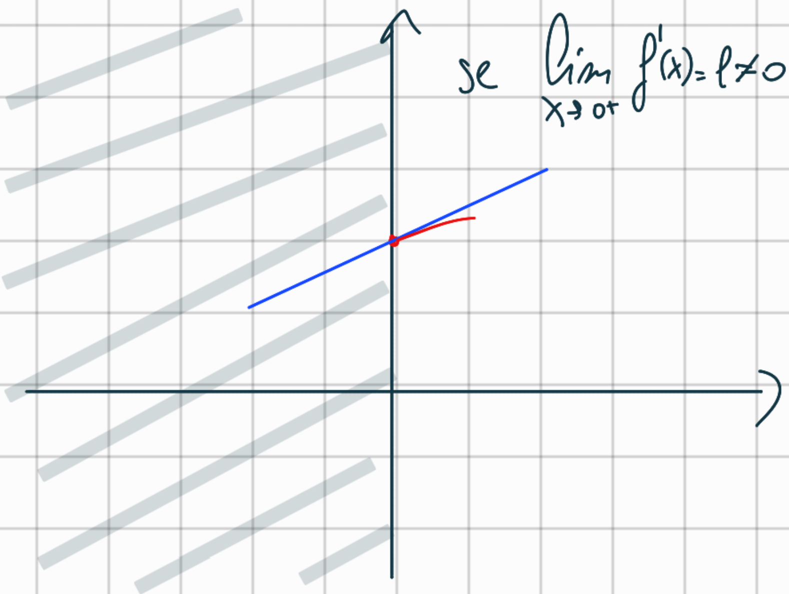
PUNTO A TANGENTE VERTICALE

$f(x) = \sqrt{x}$

$\lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty$



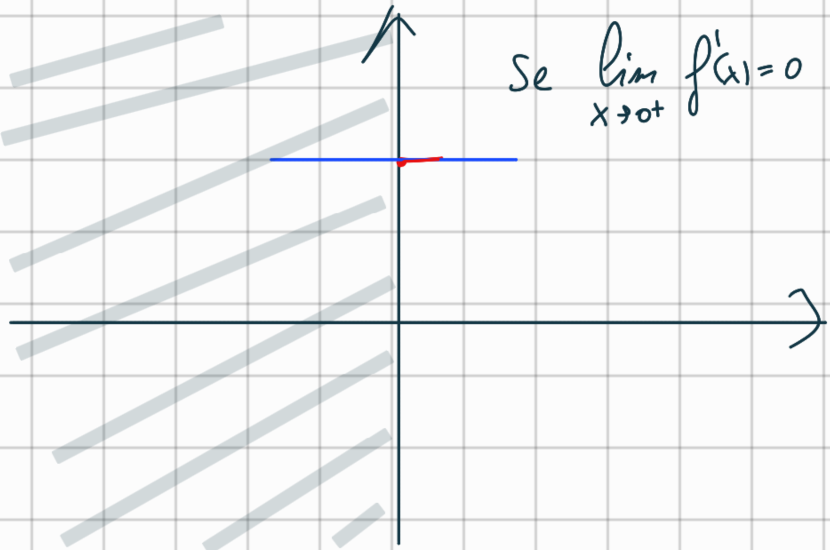
PUNTO A TANGENTE OBLIQUA



se $\lim_{x \to 0^+} f'(x) = l \neq 0$

PUNTO A TANGENTE ORIZZONTALE

$f(x) = \sqrt{x^3} \rightarrow$



Se $\lim_{x \to 0^+} f'(x) = 0$

TEOREMA DI DE L'HOSPITAL (NO DIM)

ABBR. DH

Date due funzioni $f(x)$ e $g(x)$ definita in $I(x_0)$

SE $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ oppure $\frac{\infty}{\infty}$

- $f(x), g(x)$ SONO DERIVABILI in $I(x_0) \setminus \{x_0\}$
- $g'(x) \neq 0$ IN $I(x_0) \setminus \{x_0\}$

SE $\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \Rightarrow \lim \frac{f(x)}{g(x)} = l$

ESEMPI DI APPLICAZIONI

$$\lim_{x \to +\infty} \frac{4x^2+4}{\ln(x)} = +\infty$$

VERIFICA

$$\lim_{x \to +\infty} \frac{8x}{1/x} = \lim_{x \to +\infty} 8x^2 = +\infty$$

$$\lim_{x \to 1^+} \frac{3x^3-9x+6}{5x^5+2x^4-33x+26} = \frac{3-9+6}{5+2-33+26} = \frac{0}{0}$$

$$= \lim_{\substack{x \to 1^+ \\ DH}} \frac{9x^2-9}{25x^4+8x^3-33} \overset{=}{DH} \lim_{x \to 1^+} \frac{18\cancel{x}}{100x^{\cancel{3}\,2}+24x^{\cancel{2}}} = \lim_{x \to 1} \frac{18}{100x^2+24x} =$$

$$= \frac{\cancel{18}^{\,9}}{\cancel{124}_{\,62}} = \frac{9}{62}$$

CON L'ALTRO MODO

$$\lim_{x \to 1^+} \frac{(3x+6)\cancel{(x-1)^2}}{(5x^3+12x^2+19x+26)\cancel{(x-1)^2}} = \frac{3+6}{5+12+19+26} = \frac{9}{62}$$

SCOMPOSIZIONE DEL DENOMINATORE

	5	2	0	0	-33	26
1		5	7	7	7	-26
	5	7	7	7	-26	/
1		5	12	19	26	
	5	12	19	26	/	

SCOMPOSIZIONE DEL NUMERATORE

	3	0	-9	6
1		3	3	-6
	3	3	-6	
1		3	6	
	3	6		

ESERCIZIO

$$\lim_{x \to -2^+} \underbrace{\ln(x+2)}_{\downarrow \, -\infty} \underbrace{\sqrt{x+2}}_{\downarrow \, 0}$$

$$D = \begin{cases} x+2 > 0 \\ x+2 \geq 0 \end{cases} \;\Rightarrow\; x > -2$$

$$\lim_{x \to -2^+} \frac{\ln(x+2) \to -\infty}{\left(\frac{1}{\sqrt{x+2}}\right) \to +\infty} \overset{DH}{=} \lim_{x \to -2^+} \frac{\frac{1}{x+2}}{-\frac{1}{2}(x+2)^{-\frac{3}{2}}} = \lim_{x \to -2^+} (x+2)^{-1}(x+2)^{+\frac{3}{2}}(-2) =$$

$$= \lim_{x \to -2^+} -2(x+2)^{\frac{1}{2}} = \lim_{x \to -2^+} -2\sqrt{x+2} = 0$$

ATTENZIONE

NON CONFONDERSI CON LA DERIVATA DEL RAPPORTO

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \neq \frac{f'(x)}{g'(x)}$$

Per casa

Risolvere con DH i seguenti limiti

- $\lim\limits_{x \to 1} \dfrac{3x^3 - 9x + 6}{5x^5 + 2x^4 - 33x + 26}$

- $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin x - 1}{\cos^2 x}$

- $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

- $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x}$

- $\lim\limits_{x \to -\infty} \dfrac{x^2 - 3x + 2}{1 - x^3}$

- $\lim\limits_{x \to +\infty} \dfrac{3x + \ln x}{7x - 2}$

- $\lim\limits_{x \to 3^+} \dfrac{\ln(x-3)}{\ln(x^2 - 9)}$

- $\lim\limits_{x \to +\infty} \dfrac{e^{x+3}}{x}$