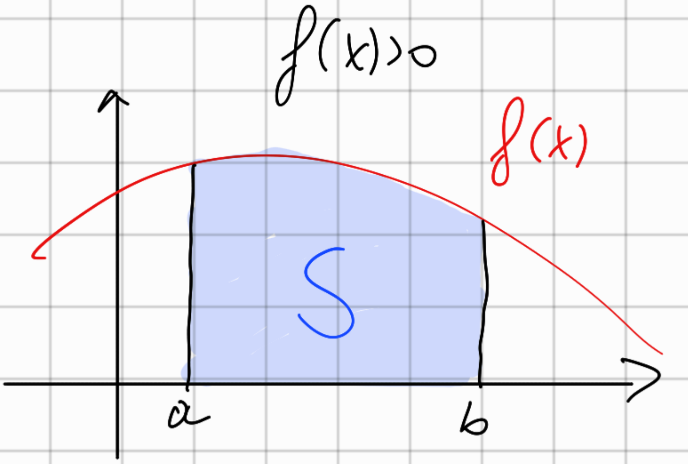
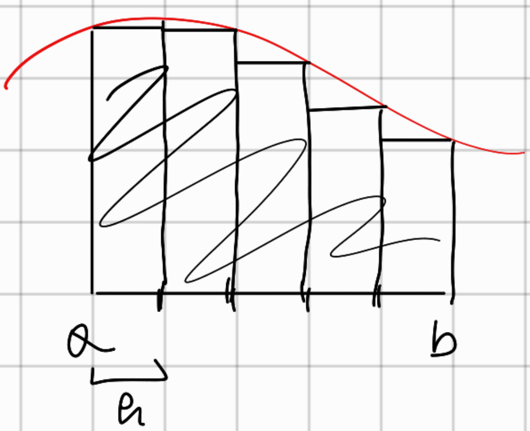
Lezione 22 15/12/23

L'INTEGRALE DEFINITO

CI PONIAMO L'OBBIETTIVO DI CALCOLARE L'AREA SOTTESA DALLA FUNZIONE $f(x)$ tra un estremo $a \in \mathbb{R}$ e $b \in \mathbb{R}$





Suddividiamo $[a,b]$ in n intervalli equidistanti

$$h = \frac{b-a}{n}$$

$x_0 = a$

$x_1 = a + h$

$x_2 = a + 2h$

$\vdots$

$x_n = a + nh = b$

CHIAMO $m_k = \min\limits_{x_{k-1} < x < x_k} \{ f(x) \}$ $\quad k = 1, 2, \dots n$

$M_k = \max\limits_{x_{k-1} < x < x_k} \{ f(x) \}$

L'AREA DEL k-esimo rettangolino è

è $\quad h \cdot m_k$ (INFERIORE)

$\quad h \cdot M_k$ (SUPERIORE)

$$S_n = h m_1 + h m_2 + \dots h m_n = h(m_1 + \dots + m_n)$$

$$= h \sum_{k=1}^{n} m_k$$

$$S_n = \dots\dots\dots = h \sum_{k=1}^{n} M_k$$

$$h \sum_{k=1}^{n} m_k = s_n \leq S \leq S_n = h \sum_{k=1}^{n} M_k$$

$$\lim_{n \to \infty} s_n \leq \lim_{n \to \infty} S \leq \lim_{n \to \infty} S_n$$

$$\lim_{n \to \infty} S = S$$

SE I DUE LIMITI SONO UGUALI

IL TEOREMA DEI CARABINIERI GARANTISCE CHE QUESTO LIMITE È S

DEFINIZIONE

Si definisce L'INTEGRALE DEFINITO da a a b di $f(x)$ in dx e si scrive

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} S_n = \boxed{S = \int_a^b f(x)\,dx}$$

← RAPPRESENTA UNA SOMMA DI INFINITI CONTRIBUTI INFINITESIMI

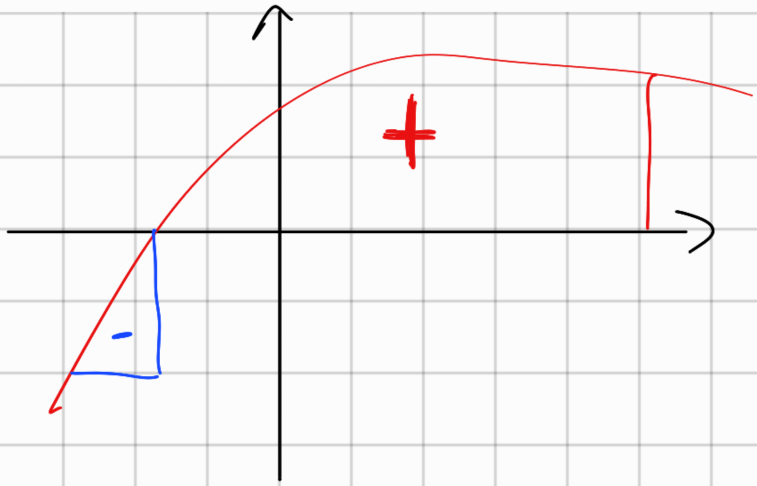
$h = \Delta x = x_{k-1} - x_k$ INCREMENTO FINITO

dx INCREMENTO INFINITESIMO

QUALSIASI

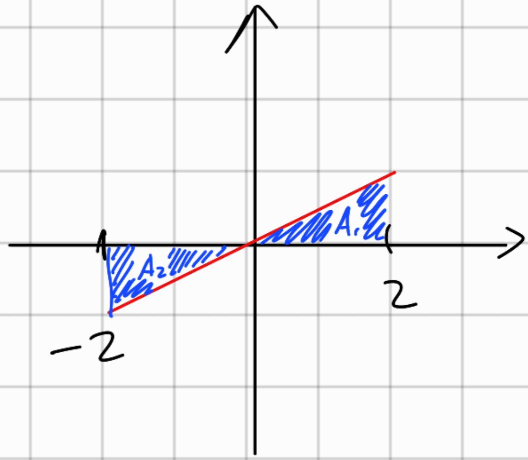
SE $f(x) \gtrless 0$

L'AREA SOTTO L'ASSE X VA CONTATA COL SEGNO -



ESEMPIO

$$\int_{-2}^{2} \frac{x}{2}\,dx = A_1 - A_2 = 0$$



PROPRIETÀ

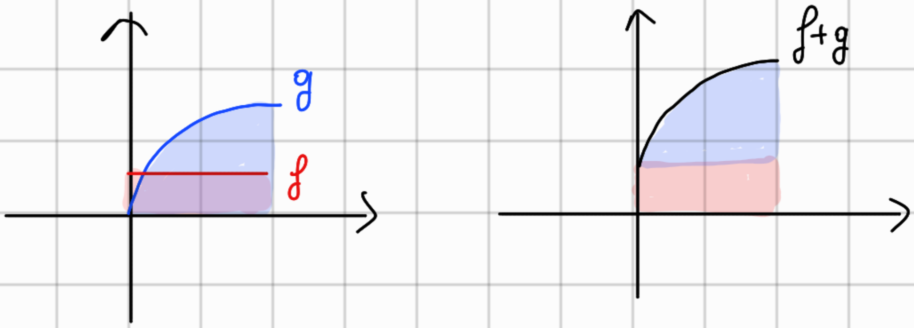
$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad a<b<c$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad \text{se } b>a$$

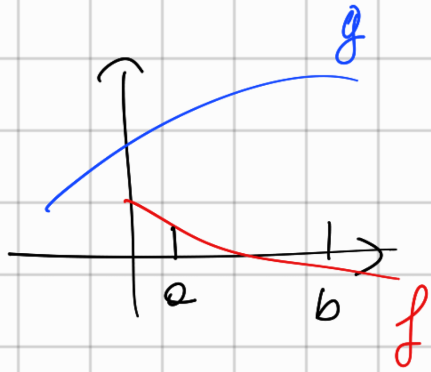
LINEARITÀ

- $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx \quad \int_a^b f(x)\,dx$
- $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$



MONOTONIA SE $f(x) < g(x) \quad \forall x \in [a,b]$

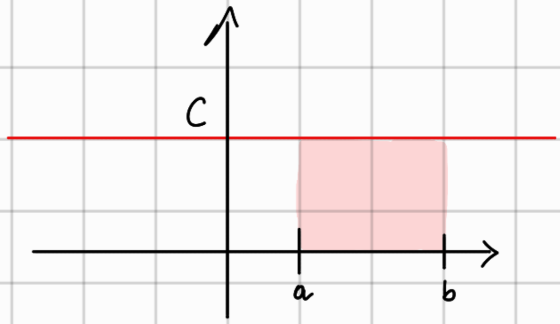
ALLORA $\int_a^b f(x)dx < \int_a^b g(x)dx$



ESEMPIO

INTEGRALE DELLA FUNZIONE COSTANTE $f(x) = C \in \mathbb{R}$

$$\int_a^b C\,dx = \underbrace{(b-a)C}_{\text{AREA DEL RETTANGOLO}}$$

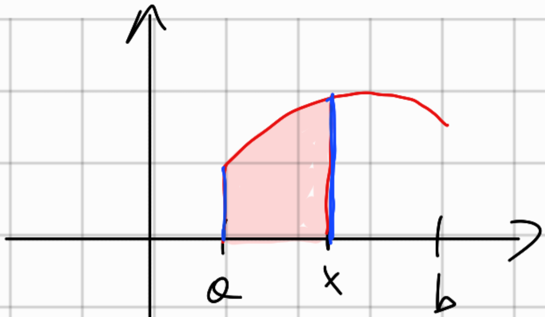


TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

Se f è CONTINUA in $[a,b]$ allora

$F(x) = \int_a^x f(t)dt$ è DERIVABILE ed è UNA PRIMITIVA DI f

CIOÈ $\boxed{F'(x) = f(x)}$



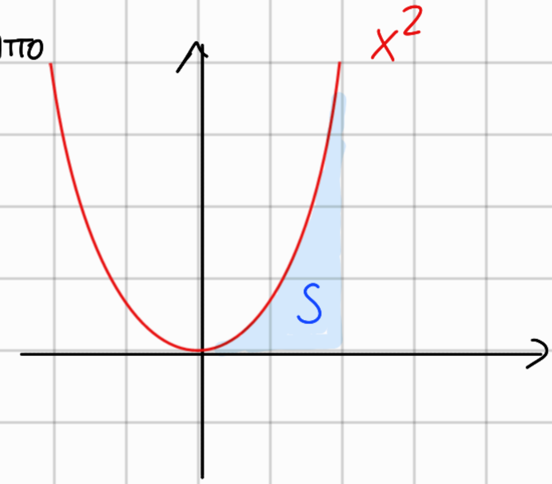
Da questo teorema si può ricavare la seguente formula

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

DOVE F È UNA QUALSIASI PRIMITIVA DI f

CIOÈ CALCOLO F CON UN INTEGRALE INDEFINITO $F(x) = \int f(x)\,dx$

QUESTA FORMULA È MOLTO IMPORTANTE PERCHÉ CI FORNISCE UN MODO PER CALCOLARE L'AREA IN MODO SEMPLICE ED ESATTO CIOÈ NON APPROSSIMATO

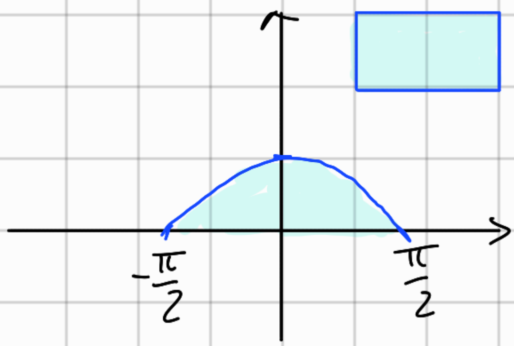


ESEMPIO

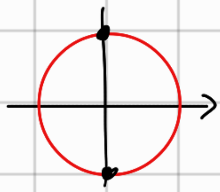
$$S = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

ESEMPIO



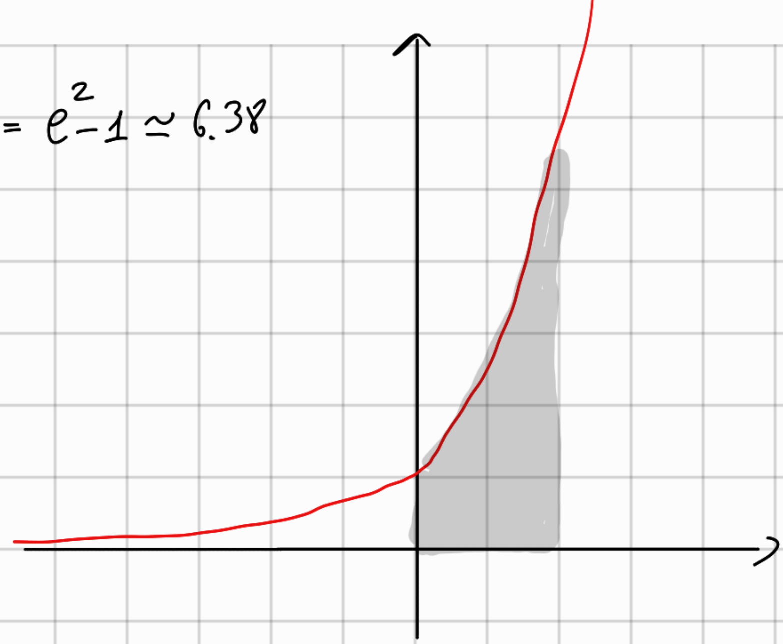
$$\int_{-\pi/2}^{\pi/2} \cos(x)\,dx = \left[\sin x\right]\Big|_{-\pi/2}^{\pi/2}$$



$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) =$$

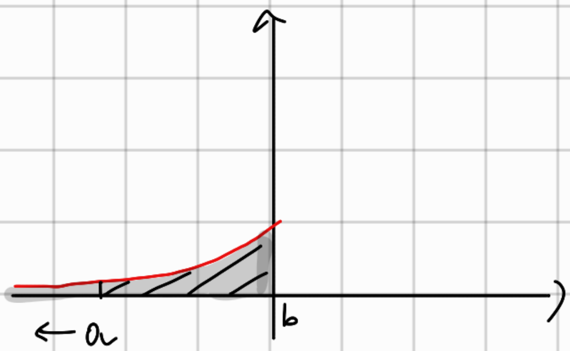
$$= 1 - (-1) = 2$$

$$\int_0^2 e^x\,dx = \left[e^x\right]_0^2 = e^2 - e^0 = e^2 - 1 \simeq 6.38$$

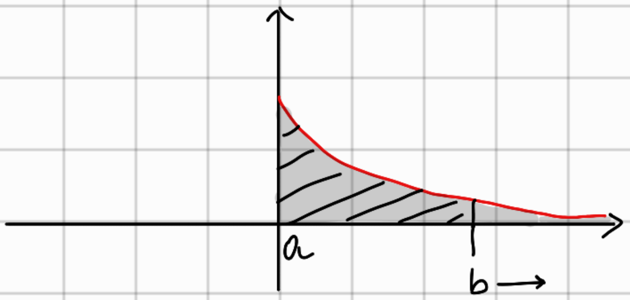


INTEGRALI GENERALIZZATI

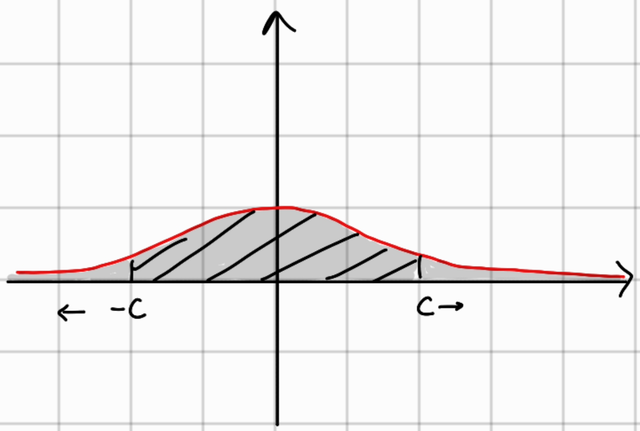
- Se vogliamo calcolare lungo un intervallo illimitato.



$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$



$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

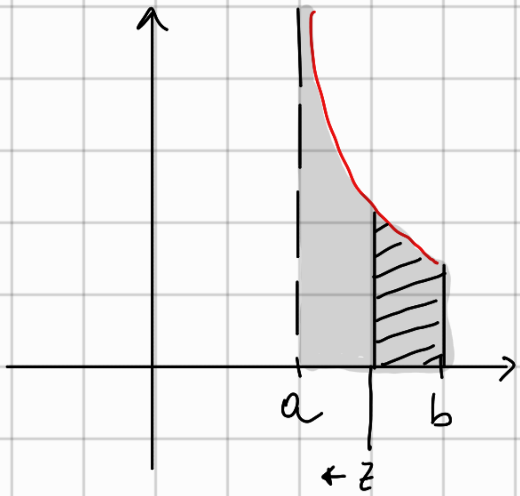


$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{c \to +\infty} \int_{-c}^{c} f(x)\,dx$$

Se la funzione presenta un asintoto verticale ....

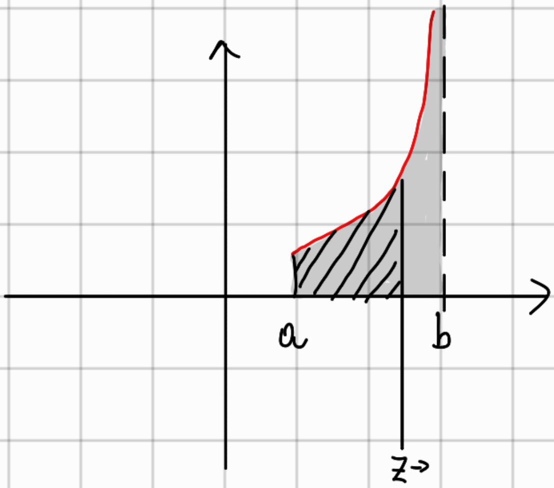
- ... nel primo estremo

$$\int_a^b f(x)\,dx = \lim_{z\to a^+} \int_z^b f(x)\,dx$$



- ... nel secondo estremo

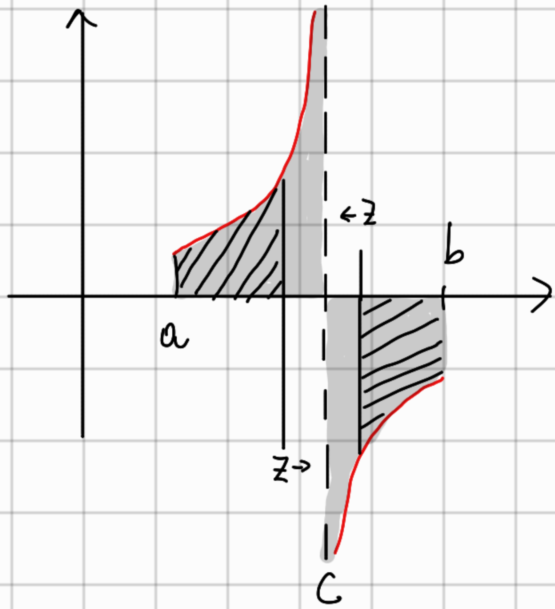
$$\int_a^b f(x)\,dx = \lim_{z\to b^-} \int_a^z f(x)\,dx$$



- ... in un punto $c \in (a,b)$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx =$$

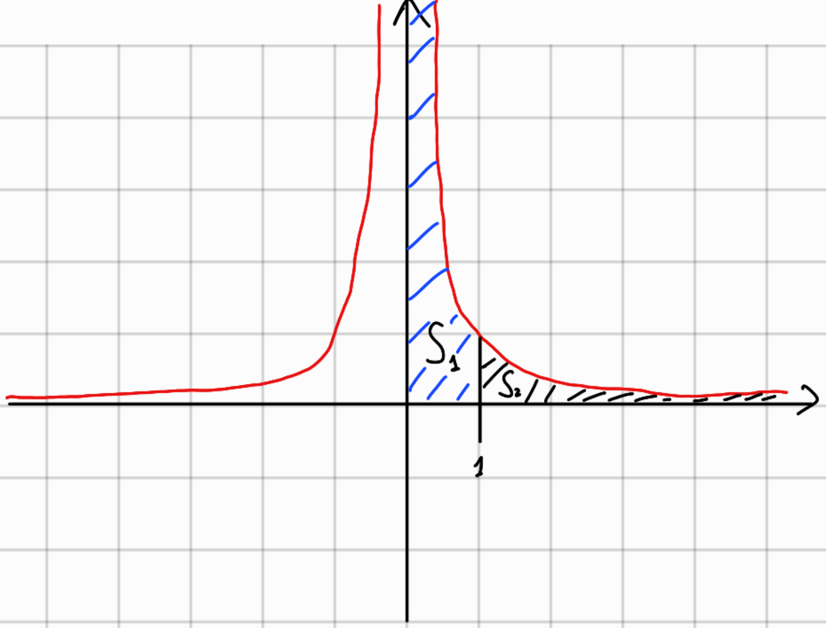
$$= \lim_{z\to c^-} \int_a^z f(x)\,dx + \lim_{z\to c^+} \int_z^b f(x)\,dx$$



ESEMPIO

A.V $x=0$

$$S_1 = \int_0^1 \frac{1}{x^2}\,dx =$$



$$= \lim_{z\to 0^+} \int_z^1 x^{-2}\,dx = \lim_{z\to 0^+} \left[-\frac{1}{x}\right]_z^1 = \lim_{z\to 0^+} \left[-\frac{1}{1} - \left(-\frac{1}{z}\right)\right] =$$

$$\lim_{z\to 0^+} \left[-1 + \frac{1}{z}\right] = +\infty$$

$$\boxed{\frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}}$$

PER CASA $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}$

- $$S_2 = \int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b x^{-2}\,dx = \lim_{b\to+\infty} \Big[\underbrace{-\frac{1}{x}}_{F(x)}\Big]_1^b =$$

$$= \lim_{b\to+\infty} \overbrace{\left[-\frac{1}{b} - \left(-\frac{1}{1}\right)\right]}^{F(b)-F(a)} = \lim_{b\to+\infty}\left[-\frac{1}{b}+1\right] = 1$$