Lezione 23 18/12/23

Calcola i seguenti integrali.

- $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \dfrac{2}{3}\sqrt{x^3} + c$

- $\int \dfrac{1}{4\sqrt{x}}\,dx = \dfrac{1}{4}\int x^{-\frac{1}{2}}dx = \dfrac{1}{\cancel{4}\,2}\,\dfrac{x^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + c = \dfrac{1}{2}\sqrt{x} + c$

$-\frac{1}{2}+1 = \frac{-1+2}{2} = \frac{1}{2}$

- $\int (x^2+x+10)\,dx = \dfrac{x^3}{3} + \dfrac{x^2}{2} + 10x + c$

- $\int e^{3x+1}\,dx = \dfrac{1}{3}\int 3e^{3x+1}\,dx = \dfrac{1}{3}\left[e^{3x+1}\right] + c$

$\int \dfrac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

- $\int \dfrac{1}{2x+5}\,dx = \dfrac{1}{2}\int \dfrac{2}{2x+5}\,dx = \dfrac{1}{2}\ln|2x+5| + c$ $\left(\ln\sqrt{2x+5} + c\right)$

- $\int \dfrac{\cos x + \sin x}{\sin x - \cos x}\,dx = \ln|\sin x - \cos x| + c$

$\int f'g = fg - \int fg'$

- $\int \dfrac{\ln(x^2)}{x^2}\,dx = \int \underbrace{\dfrac{1}{x^2}}_{f'}\,\underbrace{\ln(x^2)}_{g}\,dx = -\dfrac{1}{x}\ln(x^2) - \int -\dfrac{1}{\cancel{x}}\,\dfrac{2\cancel{x}}{x^2}\,dx$

$f' \quad x^{-2}$

$= -\dfrac{\ln(x^2)}{x} + 2\int \dfrac{1}{x^2}\,dx$

$f = \dfrac{x^{-1}}{-1} = -\dfrac{1}{x}$

$= -\dfrac{\ln(x^2)}{x} - \dfrac{2}{x} + c = -\dfrac{\ln(x^2)+2}{x} + c$

- $\int \dfrac{6x^2+8x-8}{x^3+2x^2-4x+3}\,dx = 2\int \dfrac{3x^2+4x-4}{x^3+2x^2-4x+3}\,dx = 2\ln|x^3+2x^2-4x+3| + c$

$\rightarrow 3x^2+4x-4$

$= \ln\left[(x^3+2x^2-4x+3)^2\right] + c$

- $\int \dfrac{5}{2x-3}\,dx = \dfrac{5}{2}\int \dfrac{2}{2x-3}\,dx = \dfrac{5}{2}\ln|2x-3| + c$

• $\int \frac{1-x}{9x^2-1}\,dx = \int \frac{1-x}{(3x-1)(3x+1)}\,dx$

$\Delta = +36 > 0$

$3x \quad 1$

$$= \int \frac{1/3}{3x-1}\,dx + \int \frac{-2/3}{3x+1}\,dx$$

$$= \frac{1}{3}\frac{1}{3}\int \frac{3}{3x-1}\,dx - \frac{2}{3}\frac{1}{3}\int \frac{3}{3x+1}\,dx$$

$$= \frac{1}{9}\ln|3x-1| - \frac{2}{9}\ln|3x+1|$$

$A, B \in \mathbb{R}$

$$\frac{A}{3x-1} + \frac{B}{3x+1} = \frac{1-x}{(3x-1)(3x+1)}$$

$$\frac{3Ax+A+3Bx-B}{(3x-1)(3x+1)}$$

$$(3A+3B)x + A - B$$

$$\begin{cases} 3A+3B = -1 \\ A-B = 1 \end{cases}$$

$$\begin{cases} 3B+3+3B = -1 \\ A = B+1 \end{cases} \rightarrow \begin{cases} 6B = -4 \\ \end{cases}$$

$$\begin{cases} B = -\frac{2}{3} \\ A = -\frac{2}{3}+1 = \frac{-2+3}{3} = \frac{1}{3} \end{cases}$$

• $\int \frac{2x^2+6x+11}{x^2+2x+10}\,dx$

$$\int 2 + \frac{2x-9}{x^2+2x+10}\,dx =$$

N, D

$$\begin{array}{r|l} 2x^2 + 6x + 11 & x^2+2x+10 \\ \hline -2x^2 - 4x - 20 & 2 \\ \hline 2x - 9 & \end{array}$$

Q, R

$$\frac{N}{D} = \frac{QD+R}{D} = Q + \frac{R}{D}$$

$$= 2x + \ln|x^2+2x+10| + -\frac{11}{3}\, \text{arctg}\left[\frac{1}{3}(x+1)\right] + C$$

- $\displaystyle\int \frac{2x-9}{x^2+2x+10}\,dx = \int \frac{2x+2-2-9}{x^2+2x+10}\,dx = \underbrace{\int \frac{2x+2}{x^2+2x+10}\,dx} + \int \frac{-11}{x^2+2x+10}\,dx$

$x^2+2x+10 \to 2x+2$

$\Delta = 4-40 = -36 < 0$

- $\displaystyle\int \frac{-11}{x^2+2x+10}\,dx = \int \frac{-11}{(x^2+2x+1)+9}\,dx$

$$\left[\operatorname{arctg}(f(x))\right]' = \frac{f'(x)}{1+f^2(x)}$$

$$\int \frac{[-11]\cdot\frac{1}{9}}{[(x+1)^2+9]\frac{1}{9}}\,dx = \int \frac{-11/9}{\underbrace{\tfrac{1}{9}(x+1)^2}+1}\,dx$$

$$= \int \frac{-11/9}{\underbrace{\left(\tfrac{1}{3}(x+1)\right)}^2+1}\,dx = -\frac{11}{\cancel{9}\,3}\cdot\cancel{3}\int \frac{1/3}{\left(\tfrac{1}{3}(x+1)\right)^2+1}\,dx$$

$f' = \frac{1}{3}$

$$= -\frac{11}{3}\operatorname{arctg}\left[\frac{1}{3}(x+1)\right] + C$$

Determina tra le primitive di $f(x)=\frac{1}{e^x+1}$ passa per $A=(0,\ln 2)$

$$\begin{cases} y'(x)=\frac{1}{e^x+1} \\ y(0)=\ln(2) \end{cases}$$

$$\int \frac{1}{e^x+1}\,dx$$

$t=e^x \rightarrow x=\ln t$

$dt=e^x\,dx$

$\frac{dt}{e^x}=dx$

$\frac{dt}{t}=dx$

$$\boxed{dx=\frac{1}{t}\,dt}$$

$$\int \frac{1}{t+1}\cdot\frac{1}{t}\,dt$$

$$\int \frac{1}{(t+1)t}\,dt \rightarrow t^2+t \quad \Delta>0$$

$$\frac{A}{t+1}+\frac{B}{t}=\frac{1}{t(t+1)}$$

$$\frac{At+B(t+1)}{t(t+1)}$$

$$= At+Bt+B$$

$$= (A+B)t+B$$

$$\begin{cases} A+B=0 \\ B=1 \end{cases} \rightarrow \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$\int -\frac{1}{t+1}\,dt+\int \frac{1}{t}\,dt=$$

$$-\ln|t+1|+\ln|t|+C$$

TOLGO VAL. ASS. PERCHÉ SO CHE $e^x>0$, $e^x+1>0$

$$-\ln(e^x+1)+\ln(e^x)+C$$

$$\begin{cases} y(x)=-\ln(e^x+1)+x+C \\ -\ln(e^0+1)+0+C=+\ln(2) \end{cases}$$

$$-\ln(2)+C=+\ln(2)$$

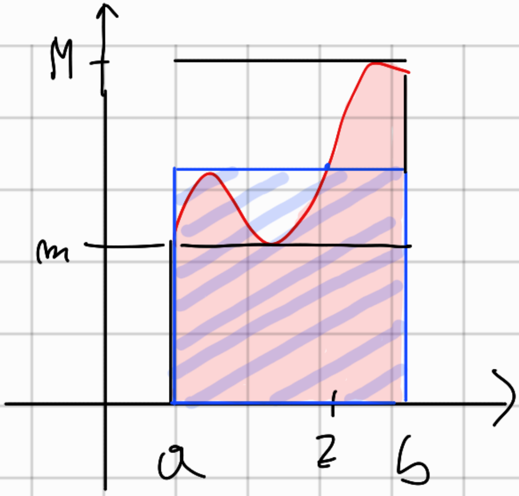
$$C=2\ln(2)$$

$$y(x)=-\ln(e^x+1)+x+2\ln(2)$$

TEOREMA DELLA MEDIA INTEGRALE

Se $f(x)$ è continua in $[a,b]$

$\Rightarrow \exists z \in [a,b]$ t.c. $\underbrace{\int_a^b f(x)\,dx}_{} = \underbrace{(b-a) f(z)}_{}$



DIMOSTRAZIONE

Se f è continua in $[a,b] \xRightarrow{\text{WEIERSTRASS}} \exists\, m, M$ rispettivamente minimo e massimo assoluti per f.

CIOÈ

$$m \leq f(x) \leq M$$

$\Downarrow$ MONOTONIA DEGLI INTEGRALI

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$\Downarrow$ AREA DEL RETTANGOLO

$$(b-a)m \leq \int_a^b f(x)\,dx \leq (b-a)M$$

$$m \leq \frac{\int_a^b f(x)\,dx}{(b-a)} \leq M$$

PER IL TEOREMA DEI VALORI INTERMEDI

$\Rightarrow \exists z \in [a,b]$ t.c. $f(z) = \dfrac{\int_a^b f(x)\,dx}{b-a}$

$$\boxed{(b-a) f(z) = \int_a^b f(x)\,dx}$$

$\square$