Let $\{\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2, \vec{\boldsymbol{v}}_3, \vec{\boldsymbol{v}}_4\}$ be a basis of a four dimensional vector space V. To construct an orthonormal basis of V, we proceed as follows:

$$\vec{\boldsymbol{w}}_1 = \vec{\boldsymbol{v}}_1, \tag{1a}$$

$$\vec{\boldsymbol{u}}_1 = \frac{\vec{\boldsymbol{w}}_1}{\|\vec{\boldsymbol{w}}_1\|}, \tag{1b}$$

$$\vec{\boldsymbol{w}}_2 = \vec{\boldsymbol{v}}_2 - (\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1, \tag{1c}$$

$$\vec{\boldsymbol{u}}_2 = \frac{\vec{\boldsymbol{w}}_2}{\|\vec{\boldsymbol{w}}_2\|}, \tag{1d}$$

$$\vec{\boldsymbol{w}}_3 = \vec{\boldsymbol{v}}_3 - (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 - (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_2)\vec{\boldsymbol{u}}_2, \tag{1e}$$

$$\vec{\boldsymbol{u}}_3 = \frac{\vec{\boldsymbol{w}}_3}{\|\vec{\boldsymbol{w}}_3\|}, \tag{1f}$$

$$\vec{\boldsymbol{w}}_4 = \vec{\boldsymbol{v}}_4 - (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 - (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_2)\vec{\boldsymbol{u}}_2 - (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_3)\vec{\boldsymbol{u}}_3, \tag{1g}$$

$$\vec{\boldsymbol{u}}_4 = \frac{\vec{\boldsymbol{w}}_4}{\|\vec{\boldsymbol{w}}_4\|}. \tag{1h}$$

Then $\{\vec{\boldsymbol{u}}_1, \vec{\boldsymbol{u}}_2, \vec{\boldsymbol{u}}_3, \vec{\boldsymbol{u}}_4\}$ is an orthonormal basis of V.

Let us now use (1a) and (1b) to write

$$\vec{\boldsymbol{v}}_1 = \|\vec{\boldsymbol{w}}_1\|\vec{\boldsymbol{u}}_1. \tag{2a}$$

Then we use (1c) and (1d) to write

$$\vec{\boldsymbol{v}}_2 = \|\vec{\boldsymbol{w}}_2\|\vec{\boldsymbol{u}}_2 + (\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1. \tag{2b}$$

Next we use (1e) and (1f) to write

$$\vec{\boldsymbol{v}}_3 = \|\vec{\boldsymbol{w}}_3\|\vec{\boldsymbol{u}}_3 + (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 + (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_2)\vec{\boldsymbol{u}}_2. \tag{2c}$$

Finally, we use (1g) and (1h) to write

$$\vec{\boldsymbol{v}}_4 = \|\vec{\boldsymbol{w}}_4\|\vec{\boldsymbol{u}}_4 + (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_1)\vec{\boldsymbol{u}}_1 + (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_2)\vec{\boldsymbol{u}}_2 + (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_3)\vec{\boldsymbol{u}}_3. \tag{2d}$$

Let us now write (2a)-(2d) in matrix form by lining up the vectors in either basis as columns of a matrix. We get

$$\underbrace{\left(\vec{\boldsymbol{v}}_1 \,\vdots\, \vec{\boldsymbol{v}}_2 \,\vdots\, \vec{\boldsymbol{v}}_3 \,\vdots\, \vec{\boldsymbol{v}}_4\right)}_{\text{given matrix } M} = \underbrace{\left(\vec{\boldsymbol{u}}_1 \,\vdots\, \vec{\boldsymbol{u}}_2 \,\vdots\, \vec{\boldsymbol{u}}_3 \,\vdots\, \vec{\boldsymbol{u}}_4\right)}_{\text{orthogonal matrix } Q} \underbrace{\begin{pmatrix} \|\vec{\boldsymbol{w}}_1\| & (\vec{\boldsymbol{v}}_2, \vec{\boldsymbol{u}}_1) & (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_1) & (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_1) \\ 0 & \|\vec{\boldsymbol{w}}_2\| & (\vec{\boldsymbol{v}}_3, \vec{\boldsymbol{u}}_2) & (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_2) \\ 0 & 0 & \|\vec{\boldsymbol{w}}_3\| & (\vec{\boldsymbol{v}}_4, \vec{\boldsymbol{u}}_3) \\ 0 & 0 & 0 & \|\vec{\boldsymbol{w}}_4\| \end{pmatrix}}_{\text{upper triangular matrix } R}. \tag{3}$$

In other words, the matrix M constructed from the given basis vectors has been factorized in the form

$$M = QR,$$

where Q is an orthogonal matrix and R is an upper triangular matrix with positive diagonal elements. Such a factorization is called a *QR-factorization* of M. Once Q is known (by lining up the orthonormal basis vectors as columns), the upper triangular factor R is easily constructed:

$$R = Q^T M,$$

where Q^T stands for the transpose of Q.[1] QR-factorizations play an important role in solving linear systems numerically, because they allow one to reduce an arbitrary linear system to a linear system with an orthogonal matrix followed by an upper triangular system.

[1]Note that $Q^T Q$ is the identity matrix, because the columns of Q form an orthonormal system