The Residue Theorem
Evaluation of Integrals and Series

RESIDUES

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z = a$ chosen as the center of C. Then, as we have seen in Chapter 6, $f(z)$ has a Laurent series about $z = a$ given by

$$
\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty} a_n(z-a)^n \\
&= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots
\end{aligned}
\tag{1}
$$

where

$$
a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz \qquad n = 0, \pm 1, \pm 2, \ldots
\tag{2}
$$

In the special case $n = -1$, we have from (2)

$$
\oint_C f(z) \, dz = 2\pi i \, a_{-1}
\tag{3}
$$

Formally we can obtain (3) from (1) by integrating term by term and using the results (Problems 21 and 22, Chapter 4)

$$
\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p = 1 \\ 0 & p = \text{integer} \neq 1 \end{cases}
\tag{4}
$$

Because of the fact that (3) involves only the coefficient a_{-1} in (1), we call a_{-1} the *residue* of $f(z)$ at $z = a$.

CALCULATION OF RESIDUES

To obtain the residue of a function $f(z)$ at $z = a$, it may appear from (1) that the Laurent expansion of $f(z)$ about $z = a$ must be obtained. However, in the case where $z = a$ is a pole of order k there is a simple formula for a_{-1} given by

$$
a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{ (z-a)^k f(z) \}
\tag{5}
$$

If $k = 1$ (simple pole) the result is especially simple and is given by

$$
a_{-1} = \lim_{z \to a} (z-a) f(z)
\tag{6}
$$

which is a special case of (5) with $k = 1$ if we define $0! = 1$.

> **Example 1:** If $f(z) = \dfrac{z}{(z-1)(z+1)^2}$, then $z = 1$ and $z = -1$ are poles of orders one and two respectively. We have, using (6) and (5) with $k = 2$,
>
> Residue at $z = 1$ is $\displaystyle \lim_{z \to 1} (z-1) \left\{ \frac{z}{(z-1)(z+1)^2} \right\} = \frac{1}{4}$
>
> Residue at $z = -1$ is $\displaystyle \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z+1)^2 \left(\frac{z}{(z-1)(z+1)^2} \right) \right\} = -\frac{1}{4}$

If $z = a$ is an essential singularity, the residue can sometimes be found by using known series expansions.

Example 2:   If $f(z) = e^{-1/z}$, then $z = 0$ is an essential singularity and from the known expansion
for e^u with $u = -1/z$ we find

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \cdots$$

from which we see that the residue at $z = 0$ is the coefficient of $1/z$ and equals -1.

THE RESIDUE THEOREM

Let $f(z)$ be single-valued and analytic inside
and on a simple closed curve C except at the
singularities $a, b, c, \ldots$ inside C which have resi-
dues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ [see Fig. 7-1].
Then the *residue theorem* states that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \quad (7)$$

i.e. the integral of $f(z)$ around C is $2\pi i$ times the
sum of the residues of $f(z)$ at the singularities
enclosed by C. Note that (7) is a generalization
of (3). Cauchy's theorem and integral formulas
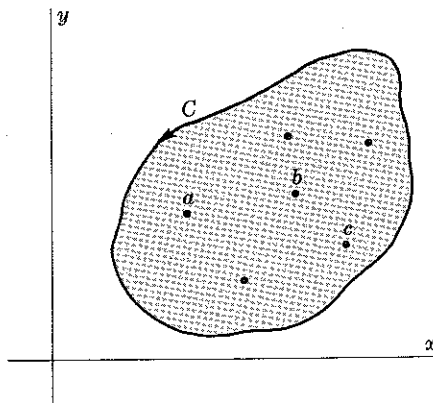are special cases of this theorem (see Problem 75).

Fig. 7-1

EVALUATION OF DEFINITE INTEGRALS

The evaluation of definite integrals is often achieved by using the residue theorem
together with a suitable function $f(z)$ and a suitable closed path or contour C, the choice
of which may require great ingenuity. The following types are most common in practice.

1. $\displaystyle\int_{-\infty}^{\infty} F(x)\,dx$, $F(x)$ is a rational function.

Consider $\displaystyle\oint_C F(z)\,dz$ along a contour C consisting of the line along the
x axis from $-R$ to $+R$ and the semicircle Γ above the x axis having this line as
diameter [Fig. 7-2]. Then let $R \to \infty$. If $F(x)$ is an even function this can be
used to evaluate $\displaystyle\int_0^{\infty} F(x)\,dx$.   See Problems 7-10.
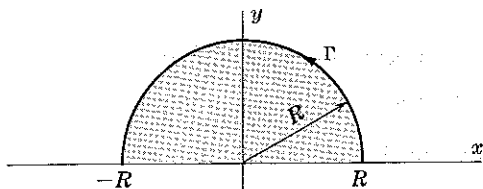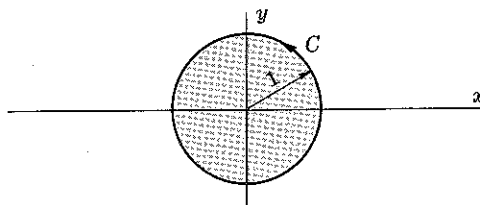

Fig. 7-2

Fig. 7-3

2. $\displaystyle\int_0^{2\pi} G(\sin\theta, \cos\theta)\,d\theta$, $G(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{z - z^{-1}}{2i}$, $\cos\theta = \dfrac{z + z^{-1}}{2}$ and $dz = ie^{i\theta}\,d\theta$ or

$d\theta = dz/iz$. The given integral is equivalent to $\displaystyle\oint_C F(z)\,dz$ where C is the unit

circle with center at the origin [Fig. 7-3]. See Problems 11-14.

3. $\displaystyle\int_{-\infty}^{\infty} F(x) \begin{Bmatrix} \cos mx \\ \sin mx \end{Bmatrix} dx$, $F(x)$ is a rational function.

Here we consider $\displaystyle\oint_C F(z)\, e^{imz}\, dz$ where C is the same contour as that in Type 1.   See Problems 15-17, and 37.

4.   Miscellaneous integrals involving particular contours.   See Problems 18-23.

SPECIAL THEOREMS USED IN EVALUATING INTEGRALS

In evaluating integrals such as those of Types 1 and 3 above, it is often necessary to show that $\displaystyle\int_{\Gamma} F(z)\, dz$ and $\displaystyle\int_{\Gamma} e^{imz} F(z)\, dz$ approach zero as $R \to \infty$.   The following theorems are fundamental.

Theorem 1. If $|F(z)| \leqq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M are constants, then if Γ is the semicircle of Fig. 7-2,

$$\lim_{R \to \infty} \int_{\Gamma} F(z)\, dz \;=\; 0$$

See Problem 7.

Theorem 2. If $|F(z)| \leqq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 0$ and M are constants, then if Γ is the semicircle of Fig. 7-2,

$$\lim_{R \to \infty} \int_{\Gamma} e^{imz} F(z)\, dz \;=\; 0$$

See Problem 15.

THE CAUCHY PRINCIPAL VALUE OF INTEGRALS

If $F(x)$ is continuous in $a \leqq x \leqq b$ except at a point x_0 such that $a < x_0 < b$, then if ϵ_1 and ϵ_2 are positive we define

$$\int_a^b F(x)\, dx \;=\; \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \left\{ \int_a^{x_0 - \epsilon_1} F(x)\, dx \;+\; \int_{x_0 + \epsilon_2}^b F(x)\, dx \right\}$$

In some cases the above limit does not exist for $\epsilon_1 \neq \epsilon_2$ but does exist if we take $\epsilon_1 = \epsilon_2 = \epsilon$. In such case we call

$$\int_a^b F(x)\, dx \;=\; \lim_{\epsilon \to 0} \left\{ \int_a^{x_0 - \epsilon} F(x)\, dx \;+\; \int_{x_0 + \epsilon}^b F(x)\, dx \right\}$$

the *Cauchy principal value* of the integral on the left.

Example: $\displaystyle\int_{-1}^{1} \frac{dx}{x^3} \;=\; \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \left\{ \int_{-1}^{-\epsilon_1} \frac{dx}{x^3} + \int_{\epsilon_2}^{1} \frac{dx}{x^3} \right\} \;=\; \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \left\{ \frac{1}{2\epsilon_2^2} - \frac{1}{2\epsilon_1^2} \right\}$

does not exist.   However, the Cauchy principal value with $\epsilon_1 = \epsilon_2 = \epsilon$ does exist and equals zero.

DIFFERENTIATION UNDER THE INTEGRAL SIGN.   LEIBNITZ'S RULE

A useful method for evaluating integrals employs *Leibnitz's rule* for differentiation under the integral sign.   This rule states that

$$\frac{d}{d\alpha} \int_a^b F(x, \alpha)\, dx \;=\; \int_a^b \frac{\partial F}{\partial \alpha}\, dx$$

The rule is valid if a and b are constants, α is a real parameter such that $\alpha_1 \leq \alpha \leq \alpha_2$ where α_1 and α_2 are constants, and $F(x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $a \leq x \leq b$, $\alpha_1 \leq \alpha \leq \alpha_2$. It can be extended to cases where the limits a and b are infinite or dependent on α.

SUMMATION OF SERIES

The residue theorem can often be used to sum various types of series. The following results are valid under very mild restrictions on $f(z)$ which are generally satisfied whenever the series converge. See Problems 24-32, and 38.

1. $\displaystyle\sum_{-\infty}^{\infty} f(n)$ $= -\{$sum of residues of $\pi \cot \pi z\, f(z)$ at all the poles of $f(z)\}$

2. $\displaystyle\sum_{-\infty}^{\infty} (-1)^n f(n)$ $= -\{$sum of residues of $\pi \csc \pi z\, f(z)$ at all the poles of $f(z)\}$

3. $\displaystyle\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right)$ $= \{$sum of residues of $\pi \tan \pi z\, f(z)$ at all the poles of $f(z)\}$

4. $\displaystyle\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right)$ $= \{$sum of residues of $\pi \sec \pi z\, f(z)$ at all the poles of $f(z)\}$

MITTAG-LEFFLER'S EXPANSION THEOREM

1. Suppose that the only singularities of $f(z)$ in the finite z plane are the simple poles $a_1, a_2, a_3, \ldots$ arranged in order of increasing absolute value.

2. Let the residues of $f(z)$ at $a_1, a_2, a_3, \ldots$ be $b_1, b_2, b_3, \ldots$.

3. Let C_N be circles of radius R_N which do not pass through any poles and on which $|f(z)| < M$, where M is independent of N and $R_N \to \infty$ as $N \to \infty$.

Then *Mittag-Leffler's expansion theorem* states that

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\}$$

SOME SPECIAL EXPANSIONS

1. $\csc z = \dfrac{1}{z} - 2z \left(\dfrac{1}{z^2 - \pi^2} - \dfrac{1}{z^2 - 4\pi^2} + \dfrac{1}{z^2 - 9\pi^2} - \cdots \right)$

2. $\sec z = \pi \left(\dfrac{1}{(\pi/2)^2 - z^2} - \dfrac{3}{(3\pi/2)^2 - z^2} + \dfrac{5}{(5\pi/2)^2 - z^2} - \cdots \right)$

3. $\tan z = 2z \left(\dfrac{1}{(\pi/2)^2 - z^2} + \dfrac{1}{(3\pi/2)^2 - z^2} + \dfrac{1}{(5\pi/2)^2 - z^2} + \cdots \right)$

4. $\cot z = \dfrac{1}{z} + 2z \left(\dfrac{1}{z^2 - \pi^2} + \dfrac{1}{z^2 - 4\pi^2} + \dfrac{1}{z^2 - 9\pi^2} + \cdots \right)$

5. $\operatorname{csch} z = \dfrac{1}{z} - 2z \left(\dfrac{1}{z^2 + \pi^2} - \dfrac{1}{z^2 + 4\pi^2} + \dfrac{1}{z^2 + 9\pi^2} - \cdots \right)$

6. $\operatorname{sech} z = \pi \left(\dfrac{1}{(\pi/2)^2 + z^2} - \dfrac{3}{(3\pi/2)^2 + z^2} + \dfrac{5}{(5\pi/2)^2 + z^2} - \cdots \right)$

7. $\tanh z = 2z \left(\dfrac{1}{z^2 + (\pi/2)^2} + \dfrac{1}{z^2 + (3\pi/2)^2} + \dfrac{1}{z^2 + (5\pi/2)^2} + \cdots \right)$

8. $\coth z = \dfrac{1}{z} + 2z \left(\dfrac{1}{z^2 + \pi^2} + \dfrac{1}{z^2 + 4\pi^2} + \dfrac{1}{z^2 + 9\pi^2} + \cdots \right)$

Solved Problems

RESIDUES AND THE RESIDUE THEOREM

1. Let $f(z)$ be analytic inside and on a simple closed curve C except at point a inside C.

   (a) Prove that

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

   i.e. $f(z)$ can be expanded into a converging Laurent series about $z = a$.

   (b) Prove that

   $$\oint_C f(z)\, dz = 2\pi i\, a_{-1}$$

   (a) This follows from Problem 25 of Chapter 6.

   (b) If we let $n = -1$ in the result of (a), we find

   $$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz, \qquad \text{i.e.} \qquad \oint_C f(z)\, dz = 2\pi i\, a_{-1}$$

   We call a_{-1} the *residue* of $f(z)$ at $z = a$.

2. Prove the *residue theorem*. If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of points $a, b, c, \ldots$ inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then

   $$\oint_C f(z)\, dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

   i.e. $2\pi i$ times the sum of the residues at all singularities enclosed by C.

Fig. 7-4

   With centers at $a, b, c, \ldots$ respectively construct circles $C_1, C_2, C_3, \ldots$ which lie entirely inside C as shown in Fig. 7-4. This can be done since $a, b, c, \ldots$ are interior points. By Theorem 5, Page 97, we have

   $$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \oint_{C_3} f(z)\, dz + \cdots \tag{1}$$

   But by Problem 1,

   $$\oint_{C_1} f(z)\, dz = 2\pi i\, a_{-1}, \qquad \oint_{C_2} f(z)\, dz = 2\pi i\, b_{-1}, \qquad \oint_{C_3} f(z)\, dz = 2\pi i\, c_{-1}, \qquad \cdots \tag{2}$$

   Then from (1) and (2) we have, as required,

   $$\oint_C f(z)\, dz = 2\pi i\, (a_{-1} + b_{-1} + c_{-1} + \cdots) = 2\pi i\, (\text{sum of residues})$$

   The proof given here establishes the residue theorem for simply-connected regions containing a finite number of singularities of $f(z)$. It can be extended to regions with infinitely many isolated singularities and to multiply-connected regions (see Problems 96 and 97).

3. Let $f(z)$ be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of $f(z)$ at a is given by

   $$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

   Method 1. If $f(z)$ has a pole a of order m, then the Laurent series of $f(z)$ is

   $$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{1}$$
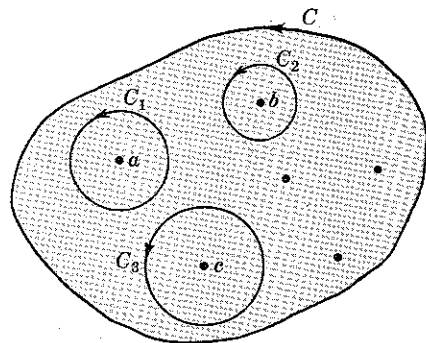
Then multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + \cdots \tag{2}$$

This represents the Taylor series about $z = a$ of the analytic function on the left. Differentiating both sides $m-1$ times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\, a_{-1} + m(m-1)\cdots 2a_0(z-a) + \cdots$$

Thus on letting $z \to a$,

$$\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\, a_{-1}$$

from which the required result follows.

Method 2. The required result also follows directly from Taylor's theorem on noting that the coefficient of $(z-a)^{m-1}$ in the expansion (2) is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\Big|_{z=a}$$

Method 3. See Problem 28, Chapter 5, Page 132.

4. Find the residues of (a) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.

(a) $f(z)$ has a double pole at $z = -1$ and simple poles at $z = \pm 2i$.

Method 1.
Residue at $z = -1$ is

$$\lim_{z \to -1} \frac{1}{1!} \frac{d}{dz}\left\{(z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} = \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Residue at $z = 2i$ is

$$\lim_{z \to 2i}\left\{(z-2i) \cdot \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

Residue at $z = -2i$ is

$$\lim_{z \to -2i}\left\{(z+2i) \cdot \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

Method 2.
Residue at $z = 2i$ is

$$\lim_{z \to 2i}\left\{\frac{(z-2i)(z^2-2z)}{(z+1)^2(z^2+4)}\right\} = \left\{\lim_{z \to 2i}\frac{z^2-2z}{(z+1)^2}\right\}\left\{\lim_{z \to 2i}\frac{z-2i}{z^2+4}\right\}$$

$$= \frac{-4-4i}{(2i+1)^2} \cdot \lim_{z \to 2i}\frac{1}{2z} = \frac{-4-4i}{(2i+1)^2} \cdot \frac{1}{4i} = \frac{7+i}{25}$$

using L'Hospital's rule. In a similar manner, or by replacing i by $-i$ in the result, we can obtain the residue at $z = -2i$.

(b) $f(z) = e^z \csc^2 z = \dfrac{e^z}{\sin^2 z}$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$, i.e. $z = m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$.

Method 1.
Residue at $z = m\pi$ is

$$\lim_{z \to m\pi} \frac{1}{1!} \frac{d}{dz}\left\{(z-m\pi)^2 \frac{e^z}{\sin^2 z}\right\}$$

$$= \lim_{z \to m\pi} \frac{e^z[(z-m\pi)^2 \sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2 \cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u \to 0} e^{u + m\pi} \left\{ \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u} \right\}$$

$$= e^{m\pi} \left\{ \lim_{u \to 0} \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u} \right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note

that $\lim_{u \to 0} \dfrac{u^3}{\sin^3 u} = \lim_{u \to 0} \left(\dfrac{u}{\sin u} \right)^3 = 1$ and thus write the limit as

$$e^{m\pi} \lim_{u \to 0} \left(\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} \cdot \frac{u^3}{\sin^3 u} \right)$$

$$= e^{m\pi} \lim_{u \to 0} \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} = e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit we can instead use the series
expansions $\sin u = u - u^3/3! + \cdots$, $\cos u = 1 - u^2/2! + \cdots$.

Method 2 (using Laurent's series).

In this method we expand $f(z) = e^z \csc^2 z$ in a Laurent series about $z = m\pi$ and obtain the
coefficient of $1/(z - m\pi)$ as the required residue. To make the calculation easier let $z = u + m\pi$.
Then the function to be expanded in a Laurent series about $u = 0$ is $e^{m\pi + u} \csc^2(m\pi + u) = e^{m\pi} e^u \csc^2 u$. Using the Maclaurin expansions for e^u and $\sin u$, we find using long division

$$e^{m\pi} e^u \csc^2 u = \frac{e^{m\pi}\left(1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \cdots\right)}{\left(u - \dfrac{u^3}{3!} + \dfrac{u^5}{5!} - \cdots\right)^2} = \frac{e^{m\pi}\left(1 + u + \dfrac{u^2}{2} + \cdots\right)}{u^2\left(1 - \dfrac{u^2}{6} + \dfrac{u^4}{120} - \cdots\right)^2}$$

$$= \frac{e^{m\pi}\left(1 + u + \dfrac{u^2}{2!} + \cdots\right)}{u^2\left(1 - \dfrac{u^2}{3} + \dfrac{2u^4}{45} + \cdots\right)} = e^{m\pi}\left(\frac{1}{u^2} + \frac{1}{u} + \frac{5}{6} + \frac{u}{3} + \cdots\right)$$

and so the residue is $e^{m\pi}$.

5. Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

We have as in Method 2 of Problem 4(b),

$$F(z) = \frac{\cos z \cosh z}{z^3 \sin z \sinh z} = \frac{\left(1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots\right)\left(1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots\right)}{z^3\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right)\left(z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots\right)}$$

$$= \frac{\left(1 - \dfrac{z^4}{6} + \cdots\right)}{z^5\left(1 - \dfrac{z^4}{90} + \cdots\right)} = \frac{1}{z^5}\left(1 - \frac{7z^4}{45} + \cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another method. The result can also be obtained by finding

$$\lim_{z \to 0} \frac{1}{4!} \frac{d^4}{dz^4} \left\{ z^5 \frac{\cos z \cosh z}{z^3 \sin z \sinh z} \right\}$$

but this method is much more laborious than that given above.

6. Evaluate $\dfrac{1}{2\pi i} \displaystyle\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\, dz$ around the circle C with equation $|z| = 3$.

The integrand $\dfrac{e^{zt}}{z^2(z^2 + 2z + 2)}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$
[roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left\{ z^2 \frac{e^{zt}}{z^2(z^2 + 2z + 2)} \right\} = \lim_{z \to 0} \frac{(z^2 + 2z + 2)(te^{zt}) - (e^{zt})(2z + 2)}{(z^2 + 2z + 2)^2} = \frac{t - 1}{2}$$

Residue at $z = -1 + i$ is

$$\lim_{z \to -1 + i} \left\{ [z - (-1 + i)] \frac{e^{zt}}{z^2(z^2 + 2z + 2)} \right\} = \lim_{z \to -1 + i} \left\{ \frac{e^{zt}}{z^2} \right\} \lim_{z \to -1 + i} \left\{ \frac{z + 1 - i}{z^2 + 2z + 2} \right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2} \cdot \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1 - i$ is

$$\lim_{z \to -1 - i} \left\{ [z - (-1 - i)] \frac{e^{zt}}{z^2(z^2 + 2z + 2)} \right\} = \frac{e^{(-1-i)t}}{4}$$

Then by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)} \, dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i \left\{ \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4} \right\}$$

$$= 2\pi i \left\{ \frac{t-1}{2} + \frac{1}{2} e^{-t} \cos t \right\}$$

i.e.,

$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)} \, dz = \frac{t-1}{2} + \frac{1}{2} e^{-t} \cos t$$

DEFINITE INTEGRALS OF THE TYPE $\displaystyle\int_{-\infty}^{\infty} F(x) \, dx$

7. If $|F(z)| \leqq M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, prove that $\displaystyle\lim_{R \to \infty} \int_{\Gamma} F(z) \, dz = 0$ where Γ is the semicircular arc of radius R shown in Fig. 7-5.
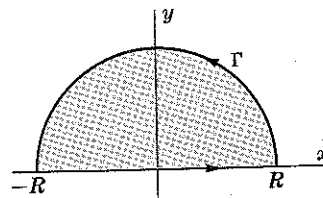
Fig. 7-5

By Property 5, Page 93, we have

$$\left| \int_{\Gamma} F(z) \, dz \right| \leqq \frac{M}{R^k} \cdot \pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$.   Then

$$\lim_{R \to \infty} \left| \int_{\Gamma} F(z) \, dz \right| = 0 \qquad \text{and so} \qquad \lim_{R \to \infty} \int_{\Gamma} F(z) \, dz = 0$$

8. Show that for $z = Re^{i\theta}$, $|f(z)| \leqq \dfrac{M}{R^k}$, $k > 1$ if $f(z) = \dfrac{1}{z^6 + 1}$.

If $z = Re^{i\theta}$, $|f(z)| = \left| \dfrac{1}{R^6 e^{6i\theta} + 1} \right| \leqq \dfrac{1}{|R^6 e^{6i\theta}| - 1} = \dfrac{1}{R^6 - 1} \leqq \dfrac{2}{R^6}$ if R is large enough (say $R > 2$, for example) so that $M = 2$, $k = 6$.

Note that we have made use of the inequality $|z_1 + z_2| \geqq |z_1| - |z_2|$ with $z_1 = R^6 e^{6i\theta}$ and $z_2 = 1$.

9. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{x^6 + 1}$.

Consider $\displaystyle\oint_C \frac{dz}{z^6 + 1}$, where C is the closed contour of Fig. 7-5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6 + 1 = 0$ when $z = e^{\pi i/6},\ e^{3\pi i/6},\ e^{5\pi i/6},\ e^{7\pi i/6},\ e^{9\pi i/6},\ e^{11\pi i/6}$, these are simple poles of $1/(z^6 + 1)$. Only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$ and $e^{5\pi i/6}$ lie within C. Then using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z \to e^{\pi i/6}} \left\{ (z - e^{\pi i/6}) \frac{1}{z^6 + 1} \right\} = \lim_{z \to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z \to e^{3\pi i/6}} \left\{ (z - e^{3\pi i/6}) \frac{1}{z^6 + 1} \right\} = \lim_{z \to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z \to e^{5\pi i/6}} \left\{ (z - e^{5\pi i/6}) \frac{1}{z^6 + 1} \right\} = \lim_{z \to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-25\pi i/6}$$

Thus
$$\oint_C \frac{dz}{z^6 + 1} = 2\pi i \left\{ \tfrac{1}{6} e^{-5\pi i/6} + \tfrac{1}{6} e^{-5\pi i/2} + \tfrac{1}{6} e^{-25\pi i/6} \right\} = \frac{2\pi}{3}$$

i.e.,
$$\int_{-R}^{R} \frac{dx}{x^6 + 1} + \int_{\Gamma} \frac{dz}{z^6 + 1} = \frac{2\pi}{3} \qquad (1)$$

Taking the limit of both sides of (1) as $R \to \infty$ and using Problems 7 and 8, we have

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^6 + 1} = \int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = \frac{2\pi}{3} \qquad (2)$$

Since $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = 2 \int_{0}^{\infty} \frac{dx}{x^6 + 1}$, the required integral has the value $\pi/3$.

10. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2 + 1)^2 (x^2 + 2x + 2)} = \frac{7\pi}{50}$.

The poles of $\dfrac{z^2}{(z^2 + 1)^2 (z^2 + 2z + 2)}$ enclosed by the contour C of Fig. 7-5 are $z = i$ of order 2 and $z = -1 + i$ of order 1.

Residue at $z = i$ is
$$\lim_{z \to i} \frac{d}{dz} \left\{ (z - i)^2 \frac{z^2}{(z + i)^2 (z - i)^2 (z^2 + 2z + 2)} \right\} = \frac{9i - 12}{100}.$$

Residue at $z = -1 + i$ is $\displaystyle\lim_{z \to -1 + i} (z + 1 - i) \frac{z^2}{(z^2 + 1)^2 (z + 1 - i)(z + 1 + i)} = \frac{3 - 4i}{25}$.

Then
$$\oint_C \frac{z^2\, dz}{(z^2 + 1)^2 (z^2 + 2z + 2)} = 2\pi i \left\{ \frac{9i - 12}{100} + \frac{3 - 4i}{25} \right\} = \frac{7\pi}{50}$$

or
$$\int_{-R}^{R} \frac{x^2\, dx}{(x^2 + 1)^2 (x^2 + 2x + 2)} + \int_{\Gamma} \frac{z^2\, dz}{(z^2 + 1)^2 (z^2 + 2z + 2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem 7, we obtain the required result.

DEFINITE INTEGRALS OF THE TYPE $\displaystyle\int_{0}^{2\pi} G(\sin\theta, \cos\theta)\, d\theta$

11. Evaluate $\displaystyle\int_{0}^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i}$, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + z^{-1}}{2}$, $dz = iz\, d\theta$ so that

$$\int_{0}^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint_C \frac{dz/iz}{3 - 2(z + z^{-1})/2 + (z - z^{-1})/2i} = \oint_C \frac{2\, dz}{(1 - 2i)z^2 + 6iz - 1 - 2i}$$

where C is the circle of unit radius with center at the origin (Fig. 7-6).

The poles of $\dfrac{2}{(1-2i)z^2 + 6iz - 1 - 2i}$ are the simple poles

Fig. 7-6

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)}$$

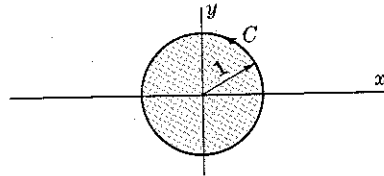$$= \frac{-6i \pm 4i}{2(1-2i)} = 2 - i, \ (2-i)/5$$

Only $(2-i)/5$ lies inside C.

Residue at $(2-i)/5 = \lim\limits_{z \to (2-i)/5} \{z - (2-i)/5\} \left\{ \dfrac{2}{(1-2i)z^2 + 6iz - 1 - 2i} \right\}$

$$= \lim\limits_{z \to (2-i)/5} \frac{2}{2(1-2i)z + 6i} = \frac{1}{2i} \quad \text{by L'Hospital's rule.}$$

Then $\displaystyle\oint_C \frac{2\, dz}{(1-2i)z^2 + 6iz - 1 - 2i} = 2\pi i \left(\frac{1}{2i}\right) = \pi,$ the required value.

12. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b \sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ if $a > |b|$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i}$, $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \oint_C \frac{dz/iz}{a + b(z - z^{-1})/2i} = \oint_C \frac{2\, dz}{bz^2 + 2aiz - b}$$

where C is the circle of unit radius with center at the origin, as shown in Fig. 7-6.

The poles of $\dfrac{2}{bz^2 + 2aiz - b}$ are obtained by solving $bz^2 + 2aiz - b = 0$ and are given by

$$z = \frac{-2ai \pm \sqrt{-4a^2 + 4b^2}}{2b} = \frac{-ai \pm \sqrt{a^2 - b^2}\, i}{b}$$

$$= \left\{ \frac{-a + \sqrt{a^2 - b^2}}{b} \right\} i, \ \left\{ \frac{-a - \sqrt{a^2 - b^2}}{b} \right\} i$$

Only $\dfrac{-a + \sqrt{a^2 - b^2}}{b} i$ lies inside C, since

$$\left| \frac{-a + \sqrt{a^2 - b^2}}{b} i \right| = \left| \frac{\sqrt{a^2 - b^2} - a}{b} \cdot \frac{\sqrt{a^2 - b^2} + a}{\sqrt{a^2 - b^2} + a} \right| = \left| \frac{b}{(\sqrt{a^2 - b^2} + a)} \right| < 1 \quad \text{if } a > |b|$$

Residue at $z_1 = \dfrac{-a + \sqrt{a^2 - b^2}}{b} i = \lim\limits_{z \to z_1} (z - z_1) \dfrac{2}{bz^2 + 2aiz - b}$

$$= \lim\limits_{z \to z_1} \frac{2}{2bz + 2ai} = \frac{1}{bz_1 + ai} = \frac{1}{\sqrt{a^2 - b^2}\, i}$$

by L'Hospital's rule.

Then $\displaystyle\oint_C \frac{2\, dz}{bz^2 + 2aiz - b} = 2\pi i \left(\frac{1}{\sqrt{a^2 - b^2}\, i} \right) = \frac{2\pi}{\sqrt{a^2 - b^2}},$ the required value.

13. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \frac{\pi}{12}.$

If $z = e^{i\theta}$, then $\cos\theta = \dfrac{z + z^{-1}}{2}$, $\cos 3\theta = \dfrac{e^{3i\theta} + e^{-3i\theta}}{2} = \dfrac{z^3 + z^{-3}}{2}$, $dz = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \oint_C \frac{(z^3 + z^{-3})/2}{5 - 4(z + z^{-1})/2} \frac{dz}{iz} = -\frac{1}{2i} \oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\, dz$$

where C is the contour of Fig. 7-6.

The integrand has a pole of order 3 at $z = 0$ and a simple pole $z = \frac{1}{2}$ inside C.

Residue at $z = 0$ is $\lim\limits_{z \to 0} \dfrac{1}{2!} \dfrac{d^2}{dz^2} \left\{ z^3 \cdot \dfrac{z^6 + 1}{z^3(2z-1)(z-2)} \right\} = \dfrac{21}{8}$.

Residue at $z = \frac{1}{2}$ is $\lim\limits_{z \to 1/2} \left\{ \left(z - \dfrac{1}{2} \right) \cdot \dfrac{z^6 + 1}{z^3(2z-1)(z-2)} \right\} = -\dfrac{65}{24}$.

Then $\quad -\dfrac{1}{2i} \oint_C \dfrac{z^6 + 1}{z^3(2z-1)(z-2)} \, dz = -\dfrac{1}{2i}(2\pi i) \left\{ \dfrac{21}{8} - \dfrac{65}{24} \right\} = \dfrac{\pi}{12} \quad$ as required.

14. Show that $\quad \displaystyle\int_0^{2\pi} \dfrac{d\theta}{(5 - 3\sin\theta)^2} = \dfrac{5\pi}{32}$.

Letting $z = e^{i\theta}$, we have $\sin\theta = (z - z^{-1})/2i$, $dz = ie^{i\theta} d\theta = iz \, d\theta$ and so

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2} = \oint_C \frac{dz/iz}{\{5 - 3(z - z^{-1})/2i\}^2} = -\frac{4}{i} \oint_C \frac{z \, dz}{(3z^2 - 10iz - 3)^2}$$

where C is the contour of Fig. 7-6.

The integrand has poles of order 2 at $\quad z = \dfrac{10i \pm \sqrt{-100 + 36}}{6} = \dfrac{10i \pm 8i}{6} = 3i, \ i/3$. Only the pole $i/3$ lies inside C.

Residue at $z = i/3 \quad = \quad \lim\limits_{z \to i/3} \dfrac{d}{dz} \left\{ (z - i/3)^2 \cdot \dfrac{z}{(3z^2 - 10iz - 3)^2} \right\}$

$\qquad\qquad\qquad = \quad \lim\limits_{z \to i/3} \dfrac{d}{dz} \left\{ (z - i/3)^2 \cdot \dfrac{z}{(3z - i)^2 (z - 3i)^2} \right\} = -\dfrac{5}{256}$.

Then $\qquad\qquad -\dfrac{4}{i} \oint_C \dfrac{z \, dz}{(3z^2 - 10iz - 3)^2} = -\dfrac{4}{i}(2\pi i)\left(\dfrac{-5}{256} \right) = \dfrac{5\pi}{32}$

Another method.

From Problem 12, we have for $a > |b|$,

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Then by differentiating both sides with respect to a (considering b as constant) using Leibnitz's rule, we have

$$\frac{d}{da} \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \int_0^{2\pi} \frac{\partial}{\partial a} \left(\frac{1}{a + b\sin\theta} \right) d\theta = -\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2}$$

$$= \frac{d}{da} \left(\frac{2\pi}{\sqrt{a^2 - b^2}} \right) = \frac{-2\pi a}{(a^2 - b^2)^{3/2}}$$

i.e., $$\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

Letting $a = 5$ and $b = -3$, we have

$$\int_0^{2\pi} \frac{d\theta}{(5 - 3\sin\theta)^2} = \frac{2\pi(5)}{(5^2 - 3^2)^{3/2}} = \frac{5\pi}{32}$$

DEFINITE INTEGRALS OF THE TYPE $\displaystyle\int_{-\infty}^{\infty} F(x) \begin{Bmatrix} \cos mx \\ \sin mx \end{Bmatrix} dx$

15. If $|F(z)| \leq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 0$ and M are constants, prove that

$$\lim_{R \to \infty} \int_\Gamma e^{imz} F(z) \, dz = 0$$

where Γ is the semicircular arc of Fig. 7-5 and m is a positive constant.

If $z = Re^{i\theta}$, $\displaystyle\int_\Gamma e^{imz} F(z) \, dz = \int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta}) \, iRe^{i\theta} \, d\theta$. Then

$$\left| \int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta}\, d\theta \right| \;\leqq\; \int_0^\pi \left| e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta} \right| d\theta$$

$$= \int_0^\pi \left| e^{imR\cos\theta \,-\, mR\sin\theta}\, F(Re^{i\theta})\, iRe^{i\theta} \right| d\theta$$

$$= \int_0^\pi e^{-mR\sin\theta}\, |F(Re^{i\theta})|\, R\, d\theta$$

$$\leqq \frac{M}{R^{k-1}} \int_0^\pi e^{-mR\sin\theta}\, d\theta \;=\; \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

Now $\sin\theta \geqq 2\theta/\pi$ for $0 \leqq \theta \leqq \pi/2$, as can be seen geometrically from Fig. 7-7 or analytically from Prob. 99.

Then the last integral is less than or equal to

$$\frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta \;=\; \frac{\pi M}{mR^k}(1 - e^{-mR})$$

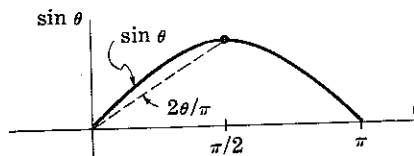As $R \to \infty$ this approaches zero, since m and k are positive, and the required result is proved.

Fig. 7-7

16. Show that $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+1}\, dx \;=\; \frac{\pi}{2} e^{-m}, \quad m > 0.$

Consider $\displaystyle\oint_C \frac{e^{imz}}{z^2+1}\, dz$ where C is the contour of Fig. 7-5. The integrand has simple poles at $z = \pm i$, but only $z = i$ lies inside C.

Residue at $z = i$ is $\displaystyle\lim_{z \to i}\left\{ (z-i)\frac{e^{imz}}{(z-i)(z+i)} \right\} = \frac{e^{-m}}{2i}.$ Then

$$\oint_C \frac{e^{imz}}{z^2+1}\, dz \;=\; 2\pi i\left(\frac{e^{-m}}{2i}\right) \;=\; \pi e^{-m}$$

or

$$\int_{-R}^R \frac{e^{imx}}{x^2+1}\, dx \;+\; \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz \;=\; \pi e^{-m}$$

i.e.,

$$\int_{-R}^R \frac{\cos mx}{x^2+1}\, dx \;+\; i\int_{-R}^R \frac{\sin mx}{x^2+1}\, dx \;+\; \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz \;=\; \pi e^{-m}$$

and so

$$2\int_0^R \frac{\cos mx}{x^2+1}\, dx \;+\; \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz \;=\; \pi e^{-m}$$

Taking the limit as $R \to \infty$ and using Problem 15 to show that the integral around Γ approaches zero, we obtain the required result.

17. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{x\sin \pi x}{x^2+2x+5}\, dx.$

Consider $\displaystyle\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\, dz$ where C is the contour of Fig. 7-5. The integrand has simple poles at $z = -1 \pm 2i$, but only $z = -1+2i$ lies inside C.

Residue at $z = -1+2i$ is $\displaystyle\lim_{z \to -1+2i}\left\{ (z+1-2i)\cdot\frac{ze^{i\pi z}}{z^2+2z+5} \right\} = (-1+2i)\frac{e^{-i\pi - 2\pi}}{4i}.$ Then

$$\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\, dz \;=\; 2\pi i(-1+2i)\left(\frac{e^{-i\pi-2\pi}}{4i}\right) \;=\; \frac{\pi}{2}(1-2i)e^{-2\pi}$$

or

$$\int_{-R}^R \frac{xe^{i\pi x}}{x^2+2x+5}\, dx \;+\; \int_\Gamma \frac{ze^{i\pi z}}{z^2+2z+5}\, dz \;=\; \frac{\pi}{2}(1-2i)e^{-2\pi}$$

i.e.,

$$\int_{-R}^R \frac{x\cos \pi x}{x^2+2x+5}\, dx \;+\; i\int_{-R}^R \frac{x\sin \pi x}{x^2+2x+5}\, dx \;+\; \int_\Gamma \frac{ze^{i\pi z}}{z^2+2z+5}\, dz \;=\; \frac{\pi}{2}(1-2i)e^{-2\pi}$$

Taking the limit as $R \to \infty$ and using Problem 15 to show that the integral around Γ approaches zero, this becomes

$$\int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2 + 2x + 5} \, dx \; + \; i \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} \, dx \;\; = \;\; \frac{\pi}{2} e^{-2\pi} \; - \; i\pi e^{-2\pi}$$

Equating real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2 + 2x + 5} \, dx \;\; = \;\; \frac{\pi}{2} e^{-2\pi}, \qquad \int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} \, dx \;\; = \;\; -\pi e^{-2\pi}$$

Thus we have obtained the value of another integral in addition to the required one.

MISCELLANEOUS DEFINITE INTEGRALS

18. Show that $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}$.

The method of Problem 16 leads us to consider the integral of e^{iz}/z around the contour of Fig. 7-5. However, since $z = 0$ lies on this path of integration and since we cannot integrate through a singularity, we modify that contour by indenting the path at $z = 0$, as shown in Fig. 7-8, which we call contour C' or $ABDEFGHJA$.

Since $z = 0$ is outside C', we have

Fig. 7-8

$$\oint_{C'} \frac{e^{iz}}{z} \, dz \;\; = \;\; 0$$

or

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} \, dx + \int_{HJA} \frac{e^{iz}}{z} \, dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x} \, dx + \int_{BDEFG} \frac{e^{iz}}{z} \, dz \;\; = \;\; 0$$

Replacing x by $-x$ in the first integral and combining with the third integral, we find

$$\int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x} \, dx + \int_{HJA} \frac{e^{iz}}{z} \, dz + \int_{BDEFG} \frac{e^{iz}}{z} \, dz \;\; = \;\; 0$$

or

$$2i \int_{\epsilon}^{R} \frac{\sin x}{x} \, dx \;\; = \;\; - \int_{HJA} \frac{e^{iz}}{z} \, dz - \int_{BDEFG} \frac{e^{iz}}{z} \, dz$$

Let $\epsilon \to 0$ and $R \to \infty$. By Problem 15, the second integral on the right approaches zero. Letting $z = \epsilon e^{i\theta}$ in the first integral on the right, we see that it approaches

$$-\lim_{\epsilon \to 0} \int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i \epsilon e^{i\theta} \, d\theta \;\; = \;\; -\lim_{\epsilon \to 0} \int_{\pi}^{0} i e^{i\epsilon e^{i\theta}} \, d\theta \;\; = \;\; \pi i$$

since the limit can be taken under the integral sign.

Then we have

$$\lim_{\substack{R \to \infty \\ \epsilon \to 0}} 2i \int_{\epsilon}^{R} \frac{\sin x}{x} \, dx \;\; = \;\; \pi i \qquad \text{or} \qquad \int_{0}^{\infty} \frac{\sin x}{x} \, dx \;\; = \;\; \frac{\pi}{2}$$

19. Prove that

$$\int_{0}^{\infty} \sin x^2 \, dx \;\; = \;\; \int_{0}^{\infty} \cos x^2 \, dx \;\; = \;\; \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

Let C be the contour indicated in Fig. 7-9, where AB is the arc of a circle with center at O and radius R. By Cauchy's theorem,
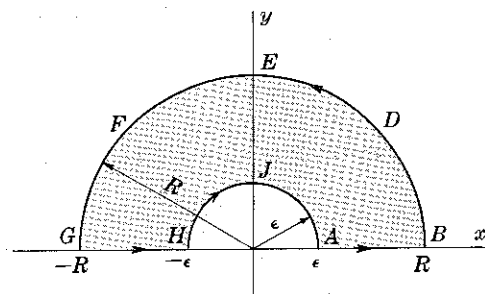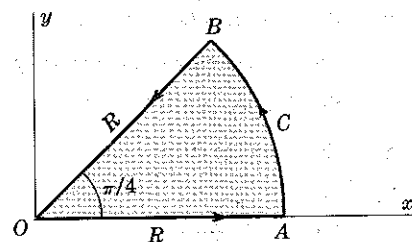

Fig. 7-9

$$\oint_{C} e^{iz^2} \, dz \;\; = \;\; 0$$

or

$$\int_{OA} e^{iz^2}\, dz \;+\; \int_{AB} e^{iz^2}\, dz \;+\; \int_{BO} e^{iz^2}\, dz \;=\; 0 \tag{1}$$

Now on OA, $z = x$ (from $x = 0$ to $x = R$); on AB, $z = Re^{i\theta}$ (from $\theta = 0$ to $\theta = \pi/4$); on BO, $z = re^{\pi i/4}$ (from $r = R$ to $r = 0$). Hence from (1),

$$\int_0^R e^{ix^2}\, dx \;+\; \int_0^{\pi/4} e^{iR^2 e^{2i\theta}}\, iRe^{i\theta}\, d\theta \;+\; \int_R^0 e^{ir^2 e^{\pi i/2}}\, e^{\pi i/4}\, dr \;=\; 0 \tag{2}$$

i.e.,

$$\int_0^R (\cos x^2 + i \sin x^2)\, dx \;=\; e^{\pi i/4} \int_0^R e^{-r^2}\, dr \;-\; \int_0^{\pi/4} e^{iR^2 \cos 2\theta - R^2 \sin 2\theta}\, iRe^{i\theta}\, d\theta \tag{3}$$

We consider the limit of (3) as $R \to \infty$. The first integral on the right becomes [see Problem 14, Chapter 10]

$$e^{\pi i/4} \int_0^\infty e^{-r^2}\, dr \;=\; \frac{\sqrt{\pi}}{2}\, e^{\pi i/4} \;=\; \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}} \tag{4}$$

The absolute value of the second integral on the right of (3) is

$$\left| \int_0^{\pi/4} e^{iR^2 \cos 2\theta - R^2 \sin 2\theta}\, iRe^{i\theta}\, d\theta \right| \;\leqq\; \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\, R\, d\theta$$

$$= \;\frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin \phi}\, d\phi$$

$$\leqq\; \frac{R}{2} \int_0^{\pi/2} e^{-2R^2 \phi/\pi}\, d\phi$$

$$= \;\frac{\pi}{4R}(1 - e^{-R^2})$$

where we have used the transformation $2\theta = \phi$ and the inequality $\sin \phi \geqq 2\phi/\pi$, $0 \leqq \phi \leqq \pi/2$ (see Problem 15). This shows that as $R \to \infty$ the second integral on the right of (3) approaches zero. Then (3) becomes

$$\int_0^\infty (\cos x^2 + i \sin x^2)\, dx \;=\; \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}$$

and so equating real and imaginary parts we have, as required,

$$\int_0^\infty \cos x^2\, dx \;=\; \int_0^\infty \sin x^2\, dx \;=\; \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

20. Show that $\displaystyle \int_0^\infty \frac{x^{p-1}}{1+x}\, dx \;=\; \frac{\pi}{\sin p\pi}$, $\quad 0 < p < 1$.

Consider $\displaystyle \oint_C \frac{z^{p-1}}{1+z}\, dz$. Since $z = 0$ is a branch point, choose C as the contour of Fig. 7-10 where the positive real axis is the branch line and where AB and GH are actually coincident with the x axis but are shown separated for visual purposes.

The integrand has the simple pole $z = -1$ inside C.

Residue at $z = -1 = e^{\pi i}$ is

$$\lim_{z \to -1} (z+1) \frac{z^{p-1}}{1+z} \;=\; (e^{\pi i})^{p-1} \;=\; e^{(p-1)\pi i}$$

Then $\displaystyle \oint_C \frac{z^{p-1}}{1+z}\, dz \;=\; 2\pi i e^{(p-1)\pi i}$ or, omitting the integrand,
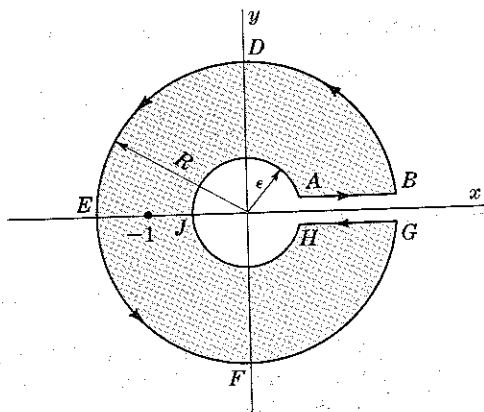

Fig. 7-10

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)\pi i}$$

We thus have

$$\int_\epsilon^R \frac{x^{p-1}}{1+x}\, dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}\, iRe^{i\theta}\, d\theta}{1 + Re^{i\theta}} + \int_R^\epsilon \frac{(xe^{2\pi i})^{p-1}}{1 + xe^{2\pi i}}\, dx$$

$$+ \int_{2\pi}^0 \frac{(\epsilon e^{i\theta})^{p-1}\, i\epsilon e^{i\theta}\, d\theta}{1 + \epsilon e^{i\theta}} = 2\pi i e^{(p-1)\pi i}$$

where we have used $z = xe^{2\pi i}$ for the integral along GH, since the argument of z is increased by 2π in going around the circle $BDEFG$.

Taking the limit as $\epsilon \to 0$ and $R \to \infty$ and noting that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^{p-1}}{1+x}\, dx + \int_\infty^0 \frac{e^{2\pi i(p-1)}\, x^{p-1}}{1+x}\, dx = 2\pi e^{(p-1)\pi i}$$

or

$$(1 - e^{2\pi i(p-1)}) \int_0^\infty \frac{x^{p-1}}{1+x}\, dx = 2\pi i e^{(p-1)\pi i}$$

so that

$$\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{2\pi i e^{(p-1)\pi i}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}} = \frac{\pi}{\sin p\pi}$$

21. Prove that $\quad \displaystyle\int_0^\infty \frac{\cosh ax}{\cosh x}\, dx = \frac{\pi}{2\cos(\pi a/2)} \quad$ where $|a| < 1$.

Consider $\displaystyle\oint_C \frac{e^{az}}{\cosh z}\, dz$ where C is a rectangle having vertices at $-R$, R, $R + \pi i$, $-R + \pi i$ (see Fig. 7-11).

The poles of $e^{az}/\cosh z$ are simple and occur where $\cosh z = 0$, i.e. $z = (n + \frac{1}{2})\pi i$, $n = 0, \pm 1, \pm 2, \ldots$. The only pole enclosed by C is $\pi i/2$.

Fig. 7-11

Residue of $\dfrac{e^{az}}{\cosh z}$ at $z = \pi i/2$ is

$$\lim_{z \to \pi i/2} (z - \pi i/2) \frac{e^{az}}{\cosh z} = \frac{e^{a\pi i/2}}{\sinh(\pi i/2)} = \frac{e^{a\pi i/2}}{i\sin(\pi/2)} = -ie^{a\pi i/2}$$

Then by the residue theorem,

$$\oint_C \frac{e^{az}}{\cosh z}\, dz = 2\pi i(-ie^{a\pi i/2}) = 2\pi e^{a\pi i/2}$$

This can be written

$$\int_{-R}^R \frac{e^{ax}}{\cosh x}\, dx + \int_0^\pi \frac{e^{a(R+iy)}}{\cosh(R+iy)}\, i\, dy + \int_R^{-R} \frac{e^{a(x+\pi i)}}{\cosh(x+\pi i)}\, dx$$

$$+ \int_\pi^0 \frac{e^{a(-R+iy)}}{\cosh(-R+iy)}\, i\, dy = 2\pi e^{a\pi i/2} \qquad (1)$$

As $R \to \infty$ the second and fourth integrals on the left approach zero. To show this let us consider the second integral. Since

$$|\cosh(R+iy)| = \left| \frac{e^{R+iy} + e^{-R-iy}}{2} \right| \geqq \tfrac{1}{2}\{|e^{R+iy}| - |e^{-R-iy}|\} = \tfrac{1}{2}(e^R - e^{-R}) \geqq \tfrac{1}{4}e^R$$

we have

$$\left| \int_0^\pi \frac{e^{a(R+iy)}}{\cosh(R+iy)}\, i\, dy \right| \leqq \int_0^\pi \frac{e^{aR}}{\frac{1}{4}e^R}\, dy = 4\pi e^{(a-1)R}$$

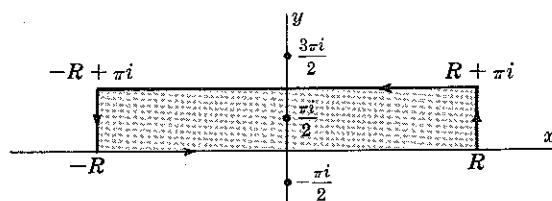and the result follows on noting that the right side approaches zero as $R \to \infty$ since $|a| < 1$. In a similar

manner we can show that the fourth integral on the left of (1) approaches zero as $R \to \infty$. Hence (1) becomes

$$\lim_{R \to \infty} \left\{ \int_{-R}^{R} \frac{e^{ax}}{\cosh x} \, dx \;+\; e^{a\pi i} \int_{-R}^{R} \frac{e^{ax}}{\cosh x} \, dx \right\} \;=\; 2\pi e^{a\pi i/2}$$

since $\cosh(x + \pi i) = -\cosh x$.   Thus

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ax}}{\cosh x} \, dx \;=\; \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x} \, dx \;=\; \frac{2\pi e^{a\pi i/2}}{1 + e^{a\pi i}} \;=\; \frac{2\pi}{e^{a\pi i/2} + e^{-a\pi i/2}} \;=\; \frac{\pi}{\cos(\pi a/2)}$$

Now
$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh x} \, dx \;+\; \int_{0}^{\infty} \frac{e^{ax}}{\cosh x} \, dx \;=\; \frac{\pi}{\cos(\pi a/2)}$$

Then replacing x by $-x$ in the first integral, we have

$$\int_{0}^{\infty} \frac{e^{-ax}}{\cosh x} \, dx \;+\; \int_{0}^{\infty} \frac{e^{ax}}{\cosh x} \, dx \;=\; 2 \int_{0}^{\infty} \frac{\cosh ax}{\cosh x} \, dx \;=\; \frac{\pi}{\cos(\pi a/2)}$$

from which the required result follows.

22. Prove that $\displaystyle \int_{0}^{\infty} \frac{\ln(x^2+1)}{x^2+1} \, dx \;=\; \pi \ln 2.$

Consider $\displaystyle \oint_{C} \frac{\ln(z+i)}{z^2+1} \, dz$ around the contour C consisting of the real axis from $-R$ to R and the semicircle Γ of radius R (see Fig. 7-12).
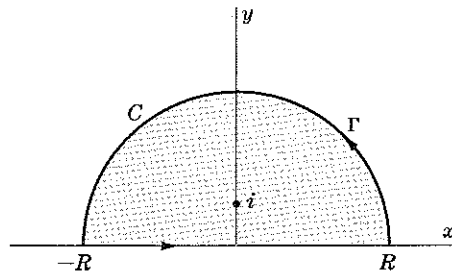
The only pole of $\ln(z+i)/(z^2+1)$ inside C is the simple pole $z = i$, and the residue is

$$\lim_{z \to i} (z-i) \frac{\ln(z+i)}{(z-i)(z+i)} \;=\; \frac{\ln(2i)}{2i}$$

Fig. 7-12

Hence by the residue theorem,

$$\oint_{C} \frac{\ln(z+i)}{z^2+1} \, dz \;=\; 2\pi i \left\{ \frac{\ln(2i)}{2i} \right\} \;=\; \pi \ln(2i) \;=\; \pi \ln 2 + \tfrac{1}{2}\pi^2 i \tag{1}$$

on writing $\ln(2i) = \ln 2 + \ln i = \ln 2 + \ln e^{\pi i/2} = \ln 2 + \pi i/2$ using principal values of the logarithm. The result can be written

$$\int_{-R}^{R} \frac{\ln(x+i)}{x^2+1} \, dx \;+\; \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} \, dz \;=\; \pi \ln 2 + \tfrac{1}{2}\pi^2 i$$

or

$$\int_{-R}^{0} \frac{\ln(x+i)}{x^2+1} \, dx \;+\; \int_{0}^{R} \frac{\ln(x+i)}{x^2+1} \, dx \;+\; \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} \, dz \;=\; \pi \ln 2 + \tfrac{1}{2}\pi^2 i$$

Replacing x by $-x$ in the first integral, this can be written

$$\int_{0}^{R} \frac{\ln(i-x)}{x^2+1} \, dx \;+\; \int_{0}^{R} \frac{\ln(i+x)}{x^2+1} \, dx \;+\; \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} \, dz \;=\; \pi \ln 2 + \tfrac{1}{2}\pi^2 i$$

or, since $\ln(i-x) + \ln(i+x) = \ln(i^2 - x^2) = \ln(x^2+1) + \pi i$,

$$\int_{0}^{R} \frac{\ln(x^2+1)}{x^2+1} \, dx \;+\; \int_{0}^{R} \frac{\pi i}{x^2+1} \, dx \;+\; \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} \, dz \;=\; \pi \ln 2 + \tfrac{1}{2}\pi^2 i \tag{2}$$

As $R \to \infty$ we can show that the integral around Γ approaches zero (see Problem 101). Hence on taking real parts we find, as required,

$$\lim_{R \to \infty} \int_{0}^{R} \frac{\ln(x^2+1)}{x^2+1} \, dx \;=\; \int_{0}^{\infty} \frac{\ln(x^2+1)}{x^2+1} \, dx \;=\; \pi \ln 2$$

23. Prove that $\displaystyle\int_0^{\pi/2} \ln \sin x \, dx \;=\; \int_0^{\pi/2} \ln \cos x \, dx \;=\; -\tfrac{1}{2}\pi \ln 2$

Letting $x = \tan\theta$ in the result of Problem 22, we find

$$\int_0^{\pi/2} \frac{\ln(\tan^2\theta + 1)}{\tan^2\theta + 1} \sec^2\theta \, d\theta \;=\; -2\int_0^{\pi/2} \ln\cos\theta \, d\theta \;=\; \pi \ln 2$$

from which

$$\int_0^{\pi/2} \ln\cos\theta \, d\theta \;=\; -\tfrac{1}{2}\pi \ln 2 \tag{1}$$

which establishes part of the required result. Letting $\theta = \pi/2 - \phi$ in (1), we find

$$\int_0^{\pi/2} \ln\sin\phi \, d\phi \;=\; -\tfrac{1}{2}\pi \ln 2$$

SUMMATION OF SERIES

24. Let C_N be a square with vertices at

$(N+\tfrac{1}{2})(1+i), \quad (N+\tfrac{1}{2})(-1+i),$

$(N+\tfrac{1}{2})(-1-i), \quad (N+\tfrac{1}{2})(1-i)$

as shown in Fig. 7-13. Prove that on C_N, $|\cot\pi z| < A$ where A is a constant.

We consider the parts of C_N which lie in the regions $y > \tfrac{1}{2}$, $-\tfrac{1}{2} \leqq y \leqq \tfrac{1}{2}$ and $y < -\tfrac{1}{2}$.

Case 1: $y > \tfrac{1}{2}$. In this case if $z = x + iy$,

$$|\cot\pi z| \;=\; \left| \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \right|$$

$$=\; \left| \frac{e^{\pi i x - \pi y} + e^{-\pi i x + \pi y}}{e^{\pi i x - \pi y} - e^{-\pi i x + \pi y}} \right|$$

$$\leqq\; \frac{|e^{\pi i x - \pi y}| + |e^{-\pi i x + \pi y}|}{|e^{-\pi i x + \pi y}| - |e^{\pi i x - \pi y}|}$$

$$=\; \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} \;=\; \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \;\leqq\; \frac{1 + e^{-\pi}}{1 - e^{-\pi}} \;=\; A_1$$

Fig. 7-13

Case 2: $y < -\tfrac{1}{2}$. Here as in Case 1,

$$|\cot\pi z| \;\leqq\; \frac{|e^{\pi i x - \pi y}| + |e^{-\pi i x + \pi y}|}{|e^{\pi i x - \pi y}| - |e^{-\pi i x + \pi y}|} \;=\; \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} \;=\; \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \;\leqq\; \frac{1 + e^{-\pi}}{1 - e^{-\pi}} \;=\; A_1$$

Case 3: $-\tfrac{1}{2} \leqq y \leqq \tfrac{1}{2}$. Consider $z = N + \tfrac{1}{2} + iy$. Then

$$|\cot\pi z| \;=\; |\cot\pi(N+\tfrac{1}{2}+iy)| \;=\; |\cot(\pi/2 + \pi i y)| \;=\; |\tanh\pi y| \;\leqq\; \tanh(\pi/2) \;=\; A_2$$

If $z = -N - \tfrac{1}{2} + iy$, we have similarly

$$|\cot\pi z| \;=\; |\cot\pi(-N-\tfrac{1}{2}+iy)| \;=\; |\tanh\pi y| \;\leqq\; \tanh(\pi/2) \;=\; A_2$$

Thus if we choose A as a number greater than the larger of A_1 and A_2, we have $|\cot\pi z| < A$ on C_N where A is independent of N. It is of interest to note that we actually have $|\cot\pi z| \leqq A_1 =$ $\coth(\pi/2)$ since $A_2 < A_1$.

25. Let $f(z)$ be such that along the path C_N of Fig. 7-13, $|f(z)| \leqq \dfrac{M}{|z|^k}$ where $k > 1$ and M are constants independent of N. Prove that

$$\sum_{-\infty}^{\infty} f(n) \;=\; -\{\text{sum of residues of } \pi\cot\pi z \, f(z) \text{ at the poles of } f(z)\}$$

Case 1: $f(z)$ has a finite number of poles.

In this case we can choose N so large that the path C_N of Fig. 7-13 encloses all poles of $f(z)$. The poles of $\cot\pi z$ are simple and occur at $z = 0, \pm 1, \pm 2, \ldots$.
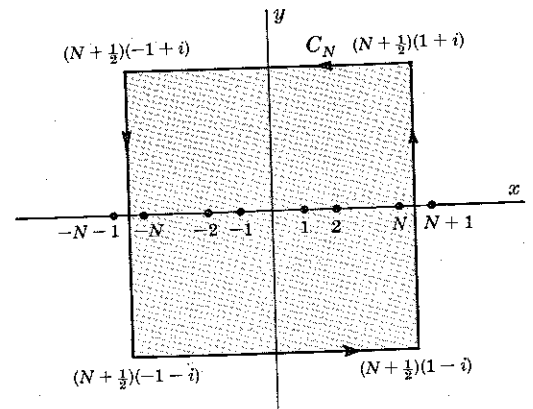
Residue of $\pi \cot \pi z \, f(z)$ at $z = n$, $n = 0, \pm1, \pm2, \ldots$, is

$$\lim_{z \to n} (z - n)\pi \cot \pi z \, f(z) \quad = \quad \lim_{z \to n} \pi \left(\frac{z - n}{\sin \pi z} \right) \cos \pi z \, f(z) \quad = \quad f(n)$$

using L'Hospital's rule. We have assumed here that $f(z)$ has no poles at $z = n$, since otherwise the given series diverges.

By the residue theorem,

$$\oint_{C_N} \pi \cot \pi z \, f(z) \, dz \quad = \quad \sum_{n = -N}^{N} f(n) \; + \; S \tag{1}$$

where S is the sum of the residues of $\pi \cot \pi z \, f(z)$ at the poles of $f(z)$. By Problem 24 and our assumption on $f(z)$, we have

$$\left| \oint_{C_N} \pi \cot \pi z \, f(z) \, dz \right| \quad \leqq \quad \frac{\pi A M}{N^k} (8N + 4)$$

since the length of path C_N is $8N + 4$. Then taking the limit as $N \to \infty$ we see that

$$\lim_{N \to \infty} \oint_{C_N} \pi \cot \pi z \, f(z) \, dz \quad = \quad 0 \tag{2}$$

Thus from (1) we have as required,

$$\sum_{-\infty}^{\infty} f(n) \quad = \quad -S \tag{3}$$

Case 2: $f(z)$ has infinitely many poles.

If $f(z)$ has an infinite number of poles, we can obtain the required result by an appropriate limiting procedure. See Problem 103.

26. Prove that $\displaystyle\sum_{n = -\infty}^{\infty} \frac{1}{n^2 + a^2} \; = \; \frac{\pi}{a} \coth \pi a$ **where** $a > 0$.

Let $f(z) = \dfrac{1}{z^2 + a^2}$ which has simple poles at $z = \pm ai$.

Residue of $\dfrac{\pi \cot \pi z}{z^2 + a^2}$ at $z = ai$ is

$$\lim_{z \to ai} (z - ai) \frac{\pi \cot \pi z}{(z - ai)(z + ai)} \quad = \quad \frac{\pi \cot \pi ai}{2ai} \quad = \quad -\frac{\pi}{2a} \coth \pi a$$

Similarly the residue at $z = -ai$ is $\dfrac{-\pi}{2a} \coth \pi a$, and the sum of the residues is $-\dfrac{\pi}{a} \coth \pi a$. Then by Problem 25,

$$\sum_{n = -\infty}^{\infty} \frac{1}{n^2 + a^2} \quad = \quad -(\text{sum of residues}) \quad = \quad \frac{\pi}{a} \coth \pi a$$

27. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \; = \; \frac{\pi}{2a} \coth \pi a \; - \; \frac{1}{2a^2}$ **where** $a > 0$.

The result of Problem 26 can be written in the form

$$\sum_{n = -\infty}^{-1} \frac{1}{n^2 + a^2} \; + \; \frac{1}{a^2} \; + \; \sum_{n = 1}^{\infty} \frac{1}{n^2 + a^2} \quad = \quad \frac{\pi}{a} \coth \pi a$$

or

$$2 \sum_{n = 1}^{\infty} \frac{1}{n^2 + a^2} \; + \; \frac{1}{a^2} \quad = \quad \frac{\pi}{a} \coth \pi a$$

which gives the required result.

28. Prove that $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots = \dfrac{\pi^2}{6}$.

We have $\quad F(z) = \dfrac{\pi \cot \pi z}{z^2} = \dfrac{\pi \cos \pi z}{z^2 \sin \pi z} = \dfrac{\left(1 - \dfrac{\pi^2 z^2}{2!} + \dfrac{\pi^4 z^4}{4!} - \cdots\right)}{z^3\left(1 - \dfrac{\pi^2 z^2}{3!} + \dfrac{\pi^4 z^4}{5!} - \cdots\right)}$

$$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)\left(1 + \frac{\pi^2 z^2}{3!} + \cdots\right) = \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{3} + \cdots\right)$$

so that the residue at $z = 0$ is $-\pi^2/3$.

Then as in Problems 26 and 27,

$$\oint_{C_N} \frac{\pi \cot \pi z}{z^2}\, dz = \sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}$$

$$= 2\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}$$

Taking the limit as $N \to \infty$ we have, since the left side approaches zero,

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{3} = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Another method. Take the limit as $a \to 0$ in the result of Problem 27. Then using L'Hospital's rule,

$$\lim_{a \to 0} \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{a \to 0} \frac{\pi a \coth \pi a - 1}{2a^2} = \frac{\pi^2}{6}$$

29. If $f(z)$ satisfies the same conditions given in Problem 25, prove that

$$\sum_{-\infty}^{\infty} (-1)^n f(n) = - \{\text{sum of residues of } \pi \csc \pi z\, f(z) \text{ at the poles of } f(z)\}$$

We proceed in a manner similar to that in Problem 25. The poles of $\csc \pi z$ are simple and occur at $z = 0, \pm 1, \pm 2, \ldots$.

Residue of $\pi \csc \pi z\, f(z)$ at $z = n$, $n = 0, \pm 1, \pm 2, \ldots$, is

$$\lim_{z \to n} (z - n)\pi \csc \pi z\, f(z) = \lim_{z \to n} \pi \left(\frac{z - n}{\sin \pi z}\right) f(z) = (-1)^n f(n)$$

By the residue theorem,

$$\oint_{C_N} \pi \csc \pi z\, f(z)\, dz = \sum_{n=-N}^{N} (-1)^n f(n) + S \tag{1}$$

where S is the sum of the residues of $\pi \csc \pi z\, f(z)$ at the poles of $f(z)$.

Letting $N \to \infty$, the integral on the left of (1) approaches zero (Problem 106) so that, as required, (1) becomes

$$\sum_{-\infty}^{\infty} (-1)^n f(n) = -S \tag{2}$$

30. Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n + a)^2} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$ where a is real and different from $0, \pm 1, \pm 2, \cdots$.

Let $f(z) = \dfrac{1}{(z + a)^2}$ which has a double pole at $z = -a$.

Residue of $\dfrac{\pi \csc \pi z}{(z + a)^2}$ at $z = -a$ is

$$\lim_{z \to -a} \frac{d}{dz}\left\{(z + a)^2 \cdot \frac{\pi \csc \pi z}{(z + a)^2}\right\} = -\pi^2 \csc \pi a \cot \pi a$$

Then by Problem 29,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n + a)^2} = -(\text{sum of residues}) = \pi^2 \csc \pi a \cot \pi a = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

31. Prove that if $a \neq 0, \pm 1, \pm 2, \ldots$, then

$$\frac{a^2+1}{(a^2-1)^2} - \frac{a^2+4}{(a^2-4)^2} + \frac{a^2+9}{(a^2-9)^2} - \cdots = \frac{1}{2a^2} - \frac{\pi^2 \cos \pi a}{2 \sin^2 \pi a}$$

The result of Problem 30 can be written in the form

$$\frac{1}{a^2} - \left\{ \frac{1}{(a+1)^2} + \frac{1}{(a-1)^2} \right\} + \left\{ \frac{1}{(a+2)^2} + \frac{1}{(a-2)^2} \right\} + \cdots = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

or

$$\frac{1}{a^2} - \frac{2(a^2+1)}{(a^2-1)^2} + \frac{2(a^2+4)}{(a^2-4)^2} - \frac{2(a^2+9)}{(a^2-9)^2} + \cdots = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

from which the required result follows. Note that the grouping of terms in the infinite series is permissible since the series is absolutely convergent.

32. Prove that $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \cdots = \dfrac{\pi^3}{32}$.

We have
$$F(z) = \frac{\pi \sec \pi z}{z^3} = \frac{\pi}{z^3 \cos \pi z} = \frac{\pi}{z^3(1 - \pi^2 z^2/2! + \cdots)}$$

$$= \frac{\pi}{z^3}\left(1 + \frac{\pi^2 z^2}{2} + \cdots \right) = \frac{\pi}{z^3} + \frac{\pi^3}{2z} + \cdots$$

so that the residue at $z = 0$ is $\pi^3/2$.

The residue of $F(z)$ at $z = n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ [which are the simple poles of $\sec \pi z$], is

$$\lim_{z \to n + \frac{1}{2}} \left\{ z - (n + \tfrac{1}{2}) \right\} \frac{\pi}{z^3 \cos \pi z} = \frac{\pi}{(n+\frac{1}{2})^3} \lim_{z \to n + \frac{1}{2}} \frac{z - (n+\frac{1}{2})}{\cos \pi z} = \frac{-(-1)^n}{(n+\frac{1}{2})^3}$$

If C_N is a square with vertices at $N(1+i)$, $N(1-i)$, $N(-1+i)$, $N(-1-i)$, then

$$\oint_{C_N} \frac{\pi \sec \pi z}{z^3}\, dz = -\sum_{n=-N}^{N} \frac{(-1)^n}{(n+\frac{1}{2})^3} + \frac{\pi^3}{2} = -8 \sum_{n=-N}^{N} \frac{(-1)^n}{(2n+1)^3} + \frac{\pi^3}{2}$$

and since the integral on the left approaches zero as $N \to \infty$, we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 2\left\{ \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \right\} = \frac{\pi^3}{16}$$

from which the required result follows.

MITTAG-LEFFLER'S EXPANSION THEOREM

33. Prove Mittag-Leffler's expansion theorem (see Page 175).

Let $f(z)$ have poles at $z = a_n$, $n = 1, 2, \ldots$, and suppose that $z = \zeta$ is not a pole of $f(z)$. Then the function $\dfrac{f(z)}{z - \zeta}$ has poles at $z = a_n$, $n = 1, 2, 3, \ldots$ and ζ.

Residue of $\dfrac{f(z)}{z - \zeta}$ at $z = a_n$, $n = 1, 2, 3, \ldots$, is $\lim_{z \to a_n} (z - a_n)\dfrac{f(z)}{z - \zeta} = \dfrac{b_n}{a_n - \zeta}$.

Residue of $\dfrac{f(z)}{z - \zeta}$ at $z = \zeta$ is $\lim_{z \to \zeta}(z - \zeta)\dfrac{f(z)}{z - \zeta} = f(\zeta)$.

Then by the residue theorem,

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z - \zeta}\, dz = f(\zeta) + \sum_n \frac{b_n}{a_n - \zeta} \qquad (1)$$

where the last summation is taken over all poles inside circle C_N of radius R_N (Fig. 7-14).

Suppose that $f(z)$ is analytic at $z = 0$. Then putting $\zeta = 0$ in (1), we have

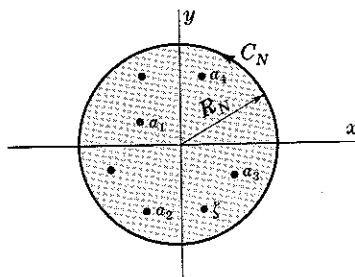$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z}\, dz = f(0) + \sum_n \frac{b_n}{a_n} \qquad (2)$$

Fig. 7-14

Subtraction of (2) from (1) yields

$$f(\zeta) - f(0) + \sum_n b_n\left(\frac{1}{a_n - \zeta} - \frac{1}{a_n}\right) = \frac{1}{2\pi i}\oint_{C_N} f(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z}\right\}dz$$

$$= \frac{\zeta}{2\pi i}\oint_{C_N} \frac{f(z)}{z(z-\zeta)}\,dz \qquad (3)$$

Now since $|z - \zeta| \geqq |z| - |\zeta| = R_N - |\zeta|$ for z on C_N, we have, if $|f(z)| \leqq M$,

$$\left|\oint_{C_N} \frac{f(z)}{z(z-\zeta)}\,dz\right| \leqq \frac{M \cdot 2\pi R_N}{R_N(R_N - |\zeta|)}$$

As $N \to \infty$ and therefore $R_N \to \infty$, it follows that the integral on the left approaches zero, i.e.,

$$\lim_{N\to\infty}\oint_{C_N} \frac{f(z)}{z(z-\zeta)}\,dz = 0$$

Hence from (3), letting $N \to \infty$, we have as required

$$f(\zeta) = f(0) + \sum_n b_n\left(\frac{1}{\zeta - a_n} + \frac{1}{a_n}\right)$$

the result on Page 175 being obtained on replacing ζ by z.

34. Prove that $\cot z = \dfrac{1}{z} + \sum_n\left(\dfrac{1}{z - n\pi} + \dfrac{1}{n\pi}\right)$ where the summation extends over $n = \pm1, \pm2, \ldots$.

Consider the function $f(z) = \cot z - \dfrac{1}{z} = \dfrac{z\cos z - \sin z}{z\sin z}$. Then $f(z)$ has simple poles at $z = n\pi$, $n = \pm1, \pm2, \pm3, \ldots$, and the residue at these poles is

$$\lim_{z\to n\pi}(z - n\pi)\left(\frac{z\cos z - \sin z}{z\sin z}\right) = \lim_{z\to n\pi}\left(\frac{z - n\pi}{\sin z}\right)\lim_{z\to n\pi}\left(\frac{z\cos z - \sin z}{z}\right) = 1$$

At $z = 0$, $f(z)$ has a removable singularity since

$$\lim_{z\to0}\left(\cot z - \frac{1}{z}\right) = \lim_{z\to0}\left(\frac{z\cos z - \sin z}{z\sin z}\right) = 0$$

by L'Hospital's rule. Hence we can define $f(0) = 0$.

By Problem 110 it follows that $f(z)$ is bounded on circles C_N having center at the origin and radius $R_N = (N + \frac{1}{2})\pi$. Hence by Problem 33,

$$\cot z - \frac{1}{z} = \sum_n\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)$$

from which the required result follows.

35. Prove that $\cot z = \dfrac{1}{z} + 2z\left\{\dfrac{1}{z^2 - \pi^2} + \dfrac{1}{z^2 - 4\pi^2} + \cdots\right\}$.

We can write the result of Problem 34 in the form

$$\cot z = \frac{1}{z} + \lim_{N\to\infty}\left\{\sum_{n=-N}^{-1}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{N}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)\right\}$$

$$= \frac{1}{z} + \lim_{N\to\infty}\left\{\left(\frac{1}{z + \pi} + \frac{1}{z - \pi}\right) + \left(\frac{1}{z + 2\pi} + \frac{1}{z - 2\pi}\right) + \cdots + \left(\frac{1}{z + N\pi} + \frac{1}{z - N\pi}\right)\right\}$$

$$= \frac{1}{z} + \lim_{N\to\infty}\left\{\frac{2z}{z^2 - \pi^2} + \frac{2z}{z^2 - 4\pi^2} + \cdots + \frac{2z}{z^2 - N^2\pi^2}\right\}$$

$$= \frac{1}{z} + 2z\left\{\frac{1}{z^2 - \pi^2} + \frac{1}{z^2 - 4\pi^2} + \cdots\right\}$$

MISCELLANEOUS PROBLEMS

36. Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty}\dfrac{e^{zt}}{\sqrt{z+1}}\,dz$ where a and t are any positive constants.

The integrand has a branch point at $z=-1$. We shall take as branch line that part of the real axis to the left of $z=-1$. Since we cannot cross this branch line, let us consider

$$\oint_C \frac{e^{zt}}{\sqrt{z+1}}\,dz$$

where C is the contour $ABDEFGHJKA$ shown in Fig. 7-15. In this figure EF and HJ actually lie on the real axis but have been shown separated for visual purposes. Also, FGH is a circle of radius ϵ while BDE and JKA represent arcs of a circle of radius R.

Since $e^{zt}/\sqrt{z+1}$ is analytic inside and on C, we have by Cauchy's theorem

$$\oint_C \frac{e^{zt}}{\sqrt{z+1}}\,dz \;=\; 0 \qquad\qquad (1)$$

Fig. 7-15

Omitting the integrand, this can be written

$$\int_{AB} + \int_{BDE} + \int_{EF} + \int_{FGH} + \int_{HJ} + \int_{JKA} \;=\; 0 \qquad\qquad (2)$$

Now on BDE and JKA, $z = Re^{i\theta}$ where θ goes from θ_0 to π and π to $2\pi - \theta_0$ respectively.

On EF, $z+1 = ue^{\pi i}$, $\sqrt{z+1} = \sqrt{u}\,e^{\pi i/2} = i\sqrt{u}$; whereas on HJ, $z+1 = ue^{-\pi i}$, $\sqrt{z+1} = \sqrt{u}\,e^{-\pi i/2} = -i\sqrt{u}$. In both cases $z = -u - 1$, $dz = -du$, where u varies from $R-1$ to ϵ along EF and ϵ to $R-1$ along HJ.

On FGH, $z+1 = \epsilon e^{i\phi}$ where ϕ goes from $-\pi$ to π.

Thus (2) can be written

$$\int_{a-iR}^{a+iR}\frac{e^{zt}}{\sqrt{z+1}}\,dz \;+\; \int_{\theta_0}^{\pi}\frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}\,iRe^{i\theta}\,d\theta \;+\; \int_{R-1}^{\epsilon}\frac{e^{-(u+1)t}\,(-du)}{i\sqrt{u}}$$

$$+\; \int_{\pi}^{-\pi}\frac{e^{(\epsilon e^{i\phi}-1)t}}{\sqrt{\epsilon e^{i\phi}+1}}\,i\epsilon e^{i\phi}\,d\phi \;+\; \int_{\epsilon}^{R-1}\frac{e^{-(u+1)t}\,(-du)}{-i\sqrt{u}}$$

$$+\; \int_{\pi}^{2\pi-\theta_0}\frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}\,iRe^{i\theta}\,d\theta \;=\; 0 \qquad\qquad (3)$$

Let us now take the limit as $R \to \infty$ and $\epsilon \to 0$. We can show (see Problem 111) that the second, fourth and sixth integrals approach zero. Hence we have

$$\int_{a-i\infty}^{a+i\infty}\frac{e^{zt}}{\sqrt{z+1}}\,dz \;=\; \lim_{\substack{\epsilon \to 0 \\ R \to \infty}} 2i\int_{\epsilon}^{R-1}\frac{e^{-(u+1)t}}{\sqrt{u}}\,du \;=\; 2i\int_{0}^{\infty}\frac{e^{-(u+1)t}}{\sqrt{u}}\,du$$

or letting $u = v^2$,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{zt}}{\sqrt{z+1}}\,dz \;=\; \frac{1}{\pi}\int_{0}^{\infty}\frac{e^{-(u+1)t}}{\sqrt{u}}\,du \;=\; \frac{2e^{-t}}{\pi}\int_{0}^{\infty}e^{-v^2t}\,dv \;=\; \frac{e^{-t}}{\sqrt{\pi t}}$$

37. Prove that $\displaystyle\int_{0}^{\infty}\frac{(\ln u)^2}{u^2+1}\,du = \frac{\pi^3}{8}$.

Let C be the closed curve of Fig. 7-16 below where Γ_1 and Γ_2 are semicircles of radii ϵ and R respectively and center at the origin. Consider

$$\oint_C \frac{(\ln z)^2}{z^2+1}\,dz$$
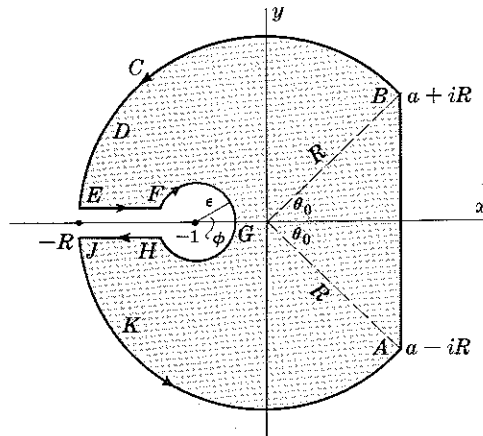
Since the integrand has a simple pole $z = i$ inside C and since the residue at this pole is

$$\lim_{z \to i} (z-i) \frac{(\ln z)^2}{(z-i)(z+i)} = \frac{(\ln i)^2}{2i}$$

$$= \frac{(\pi i/2)^2}{2i}$$
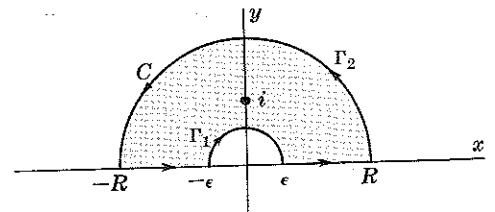
$$= \frac{-\pi^2}{8i}$$

Fig. 7-16

we have by the residue theorem

$$\oint_C \frac{(\ln z)^2}{z^2+1}\, dz = 2\pi i \left(\frac{-\pi^2}{8i} \right) = \frac{-\pi^3}{4} \tag{1}$$

Now

$$\oint_C \frac{(\ln z)^2}{z^2+1}\, dz = \int_{-R}^{-\epsilon} \frac{(\ln z)^2}{z^2+1}\, dz + \int_{\Gamma_1} \frac{(\ln z)^2}{z^2+1}\, dz + \int_{\epsilon}^{R} \frac{(\ln z)^2}{z^2+1}\, dz + \int_{\Gamma_2} \frac{(\ln z)^2}{z^2+1}\, dz \tag{2}$$

Let $z = -u$ in the first integral on the right so that $\ln z = \ln(-u) = \ln u + \ln(-1) = \ln u + \pi i$ and $dz = -du$. Also let $z = u$ (so that $dz = du$ and $\ln z = \ln u$) in the third integral on the right. Then using (1), we have

$$\int_{\epsilon}^{R} \frac{(\ln u + \pi i)^2}{u^2+1}\, du + \int_{\Gamma_1} \frac{(\ln z)^2}{z^2+1}\, dz + \int_{\epsilon}^{R} \frac{(\ln u)^2}{u^2+1}\, du + \int_{\Gamma_2} \frac{(\ln z)^2}{z^2+1}\, dz = \frac{-\pi^3}{4}$$

Now let $\epsilon \to 0$ and $R \to \infty$. Since the integrals around Γ_1 and Γ_2 approach zero, we have

$$\int_0^{\infty} \frac{(\ln u + \pi i)^2}{u^2+1}\, du + \int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\, du = \frac{-\pi^3}{4}$$

or

$$2 \int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\, du + 2\pi i \int_0^{\infty} \frac{\ln u}{u^2+1}\, du - \pi^2 \int_0^{\infty} \frac{du}{u^2+1} = \frac{-\pi^3}{4}$$

Using the fact that $\displaystyle\int_0^{\infty} \frac{du}{u^2+1} = \tan^{-1} u \Big|_0^{\infty} = \frac{\pi}{2}$,

$$2 \int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\, du + 2\pi i \int_0^{\infty} \frac{\ln u}{u^2+1}\, du = \frac{\pi^3}{4}$$

Equating real and imaginary parts, we find

$$\int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\, du = \frac{\pi^3}{8}, \qquad \int_0^{\infty} \frac{\ln u}{u^2+1}\, du = 0$$

the second integral being a by-product of the evaluation.

38. Prove that

$$\frac{\coth \pi}{1^3} + \frac{\coth 2\pi}{2^3} + \frac{\coth 3\pi}{3^3} + \cdots = \frac{7\pi^3}{180}$$

Consider

$$\oint_{C_N} \frac{\pi \cot \pi z \coth \pi z}{z^3}\, dz$$

taken around the square C_N shown in Fig. 7-17. The poles of the integrand are located at: $z = 0$ (pole of order 5); $z = \pm 1, \pm 2, \ldots$ (simple poles); $z = \pm i, \pm 2i, \ldots$ (simple poles).

By Problem 5 (replacing z by πz) we see that:

Residue at $z = 0$ is $\dfrac{-7\pi^3}{45}$.

Residue at $z = n$ $(n = \pm 1, \pm 2, \ldots)$ is
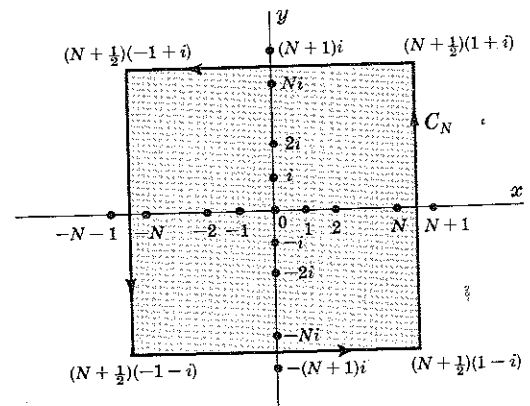

Fig. 7-17

$$\lim_{z \to n} \left\{ \frac{(z-n)}{\sin \pi z} \cdot \frac{\pi \cos \pi z \coth \pi z}{z^3} \right\} = \frac{\coth n\pi}{n^3}$$

Residue at $z = ni$ $(n = \pm 1, \pm 2, \ldots)$ is

$$\lim_{z \to ni} \left\{ \frac{(z-ni)}{\sinh \pi z} \cdot \frac{\pi \cot \pi z \cosh \pi z}{z^3} \right\} = \frac{\coth n\pi}{n^3}$$

Hence by the residue theorem,

$$\oint_{C_N} \frac{\pi \cot \pi z \coth \pi z}{z^3} \, dz = \frac{-7\pi^3}{45} + 4 \sum_{n=1}^{N} \frac{\coth n\pi}{n^3}$$

Taking the limit as $N \to \infty$, we find as in Problem 25 that the integral on the left approaches zero and the required result follows.

Supplementary Problems

RESIDUES AND THE RESIDUE THEOREM

39. For each of the following functions determine the poles and the residues at the poles:

(a) $\dfrac{2z+1}{z^2 - z - 2}$,   (b) $\left(\dfrac{z+1}{z-1}\right)^2$,   (c) $\dfrac{\sin z}{z^2}$,   (d) sech z,   (e) cot z.

Ans. (a) $z = -1, 2$; $1/3, 5/3$

   (b) $z = 1$; 4

   (c) $z = 0$; 1

   (d) $z = \frac{1}{2}(2k+1)\pi i$; $(-1)^{k+1} i$ where $k = 0, \pm 1, \pm 2, \ldots$

   (e) $z = k\pi i$; 1 where $k = 0, \pm 1, \pm 2, \ldots$

40. Prove that $\oint_C \dfrac{\cosh z}{z^3} \, dz = \pi i$ if C is the square with vertices at $\pm 2 \pm 2i$.

41. Show that the residue of $(\csc z \operatorname{csch} z)/z^3$ at $z = 0$ is $-1/60$.

42. Evaluate $\oint_C \dfrac{e^z \, dz}{\cosh z}$ around the circle C defined by $|z| = 5$.   Ans. $8\pi i$

43. Find the zeros and poles of $f(z) = \dfrac{z^2 + 4}{z^3 + 2z^2 + 2z}$ and determine the residues at the poles.

   Ans. Zeros: $z = \pm 2i$ Res; at $z = 0$ is 2   Res; at $z = -1 + i$ is $-\frac{1}{2}(1 - 3i)$ Res; at $z = -1 - i$ is $-\frac{1}{2}(1 + 3i)$

44. Evaluate $\oint_C e^{-1/z} \sin(1/z) \, dz$ where C is the circle $|z| = 1$.   Ans. $2\pi i$

45. Let C be a square bounded by $x = \pm 2$, $y = \pm 2$. Evaluate $\oint_C \dfrac{\sinh 3z}{(z - \pi/4)^3} \, dz$.   Ans. $-9\pi\sqrt{2}/2$

46. Evaluate $\oint_C \dfrac{2z^2 + 5}{(z+2)^3 (z^2+4) z^2} \, dz$ where C is (a) $|z - 2i| = 6$, (b) the square with vertices at $1 + i, 2 + i, 2 + 2i, 1 + 2i$.

47. Evaluate $\oint_C \dfrac{2 + 3 \sin \pi z}{z(z-1)^2} \, dz$ where C is a square having vertices at $3 + 3i, 3 - 3i, -3 + 3i, -3 - 3i$.
   Ans. $-6\pi i$

48. Evaluate $\dfrac{1}{2\pi i} \oint_C \dfrac{e^{zt}}{z(z^2+1)} \, dz$, $t > 0$ around the square with vertices at $1 + i, -1 + i, -1 - i, 1 - i$.
   Ans. $1 - \cos t$

DEFINITE INTEGRALS

49. Prove that $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

50. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$.   *Ans.* $5\pi/288$

51. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin 3\theta}{5-3\cos\theta}\, d\theta$.   *Ans.* 0

52. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta$.   **53.** Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta = \frac{3\pi}{8}$.

54. Prove that if $m > 0$, $\displaystyle\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(1+m)}{4}$.

55. (*a*) Find the residue of $\dfrac{e^{iz}}{(z^2+1)^5}$ at $z = i$.   (*b*) Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+1)^5}\, dx$.

56. If $a^2 > b^2 + c^2$, prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}$.

57. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{(5-3\cos\theta)^4}\, d\theta = \frac{135\pi}{16{,}384}$.

58. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4+x^2+1}$.   *Ans.* $\pi\sqrt{3}/6$

59. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2}$.   *Ans.* $\pi/2$

60. Prove that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$.

61. Discuss the validity of the following solution to Problem 19. Let $u = (1+i)x/\sqrt{2}$ in the result $\displaystyle\int_0^\infty e^{-u^2}\, du = \frac{1}{2}\sqrt{\pi}$ to obtain $\displaystyle\int_0^\infty e^{-ix^2}\, dx = \frac{1}{2}(1-i)\sqrt{\pi/2}$ from which $\displaystyle\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$ on equating real and imaginary parts.

62. Show that $\displaystyle\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\, dx = \frac{-\pi}{2\sqrt{3}}\, e^{-\pi/\sqrt{3}}$.

SUMMATION OF SERIES

63. Prove that $\displaystyle\sum_{n=1}^\infty \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}$.

64. Prove that (*a*) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, (*b*) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$.

65. Prove that $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}\, n\sin n\theta}{n^2+a^2} = \frac{\pi}{2}\frac{\sinh a\theta}{\sinh a\pi}$, $-\pi < \theta < \pi$.

66. Prove that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$.

67. Prove that $\displaystyle\sum_{n=-\infty}^\infty \frac{1}{n^4+4a^4} = \frac{\pi}{4a^3}\left\{\frac{\sinh 2\pi a + \sin 2\pi a}{\cosh 2\pi a - \cos 2\pi a}\right\}$.

68. Prove that $\displaystyle\sum_{n=-\infty}^\infty \sum_{m=-\infty}^\infty \frac{1}{(m^2+a^2)(n^2+b^2)} = \frac{\pi^2}{ab}\coth\pi a \coth\pi b$.

MITTAG-LEFFLER'S EXPANSION THEOREM

69. Prove that $\csc z = \dfrac{1}{z} - 2z\left(\dfrac{1}{z^2-\pi^2} - \dfrac{1}{z^2-4\pi^2} + \dfrac{1}{z^2-9\pi^2} - \cdots\right).$

70. Prove that $\operatorname{sech} z = \pi\left(\dfrac{1}{(\pi/2)^2+z^2} - \dfrac{3}{(3\pi/2)^2+z^2} + \dfrac{5}{(5\pi/2)^2+z^2} - \cdots\right).$

71. (a) Prove that $\tan z = 2z\left(\dfrac{1}{(\pi/2)^2-z^2} + \dfrac{1}{(3\pi/2)^2-z^2} + \dfrac{1}{(5\pi/2)^2-z^2} + \cdots\right).$

(b) Use the result in (a) to show that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \cdots = \dfrac{\pi^2}{8}.$

72. Prove the expansions (a) 2, (b) 4, (c) 5, (d) 7, (e) 8 on Page 175.

73. Prove that $\displaystyle\sum_{k=1}^{\infty} \dfrac{1}{z^2+4k^2\pi^2} = \dfrac{1}{2z}\left\{\dfrac{1}{2} - \dfrac{1}{z} + \dfrac{1}{e^z-1}\right\}.$

74. Prove that $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \cdots = \dfrac{\pi^4}{96}.$

MISCELLANEOUS PROBLEMS

75. Prove that Cauchy's theorem and integral formulas can be obtained as special cases of the residue theorem.

76. Prove that the sum of the residues of the function $\dfrac{2z^5-4z^2+5}{3z^6-8z+10}$ at all the poles is 2/3.

77. If n is a positive integer, prove that $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)d\theta = \dfrac{2\pi}{n!}.$

78. Evaluate $\displaystyle\oint_C z^3 e^{1/z}\,dz$ around the circle C with equation $|z-1| = 4$.   *Ans.* 1/24

79. Prove that under suitably stated conditions on the function:

(a) $\displaystyle\int_0^{2\pi} f(e^{i\theta})\,d\theta = 2\pi f(0)$,   (b) $\displaystyle\int_0^{2\pi} f(e^{i\theta})\cos\theta\,d\theta = -\pi f'(0)$.

80. Show that   (a) $\displaystyle\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\,d\theta = 2\pi$

(b) $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\cos\theta\,d\theta = \pi.$

81. Prove that $\displaystyle\int_0^{\infty} \dfrac{\sin ax}{e^{2\pi x}-1}\,dx = \dfrac{1}{4}\coth\dfrac{a}{2} - \dfrac{1}{2a}.$

[*Hint.* Integrate $e^{aiz}/(e^{2\pi z}-1)$ around a rectangle with vertices at $0,\ R,\ R+i,\ i$ and let $R\to\infty$.]

82. Prove that $\displaystyle\int_0^{\infty} \dfrac{\sin ax}{e^x+1}\,dx = \dfrac{1}{2a} - \dfrac{\pi}{2\sinh\pi a}.$

83. If a, p and t are positive constants, prove that $\displaystyle\int_{a-i\infty}^{a+i\infty} \dfrac{e^{zt}}{z^2+p^2}\,dz = \dfrac{\sin pt}{p}.$

84. Prove that $\displaystyle\int_0^{\infty} \dfrac{\ln x}{x^2+a^2}\,dx = \dfrac{\pi\ln 2}{2a}.$

85. If $-\pi < a < \pi$, prove that $\displaystyle\int_{-\infty}^{\infty} e^{i\lambda x}\dfrac{\sinh ax}{\sinh\pi x}\,dx = \dfrac{\sin a}{\cos a + \cosh\lambda}.$

86. Prove that $\displaystyle\int_0^\infty \frac{dx}{(4x^2 + \pi^2)\cosh x} = \frac{\ln 2}{2\pi}$.

87. Prove that   (a) $\displaystyle\int_0^\infty \frac{\ln x}{x^4 + 1}\, dx = \frac{-\pi^2\sqrt{2}}{16}$,   (b) $\displaystyle\int_0^\infty \frac{(\ln x)^2}{x^4 + 1}\, dx = \frac{3\pi^3\sqrt{2}}{64}$.

 [*Hint.* Consider $\displaystyle\oint_C \frac{(\ln z)^2}{z^4 + 1}\, dz$ around a semicircle properly indented at $z = 0$.]

88. Evaluate $\displaystyle\int_0^\infty \frac{\ln x}{(x^2 + 1)^2}\, dx$.   *Ans.* $\frac{1}{4}\pi \ln 2$

89. Prove that if $|a| < 1$ and $b > 0$, $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh x}\cos bx\, dx = \frac{\pi}{2}\left(\frac{\sin a\pi}{\cos a\pi + \cosh b\pi}\right)$.

90. Prove that if $-1 < p < 1$, $\displaystyle\int_0^\infty \frac{\cos px}{\cosh x}\, dx = \frac{\pi}{2\cosh(p\pi/2)}$.

91. Prove that $\displaystyle\int_0^\infty \frac{\ln(1 + x)}{1 + x^2}\, dx = \frac{\pi \ln 2}{2}$.

92. If $\alpha > 0$ and $-\pi/2 < \beta < \pi/2$, prove that

 (a) $\displaystyle\int_0^\infty e^{-\alpha x^2 \cos\beta} \cos(\alpha x^2 \sin\beta)\, dx = \frac{1}{2}\sqrt{\pi/\alpha}\,\cos(\beta/2)$.

 (b) $\displaystyle\int_0^\infty e^{-\alpha x^2 \cos\beta} \sin(\alpha x^2 \sin\beta)\, dx = \frac{1}{2}\sqrt{\pi/\alpha}\,\sin(\beta/2)$.

93. Prove that $\displaystyle\csc^2 z = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n\pi)^2}$.

94. If α and p are real and such that $0 < |p| < 1$ and $0 < |\alpha| < \pi$, prove that
$$\int_0^\infty \frac{x^{-p}\, dx}{x^2 + 2x\cos\alpha + 1} = \left(\frac{\pi}{\sin p\pi}\right)\left(\frac{\sin p\alpha}{\sin\alpha}\right)$$

95. Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x^2 - x^3}} = \frac{2\pi}{\sqrt{3}}$. [*Hint.* Consider the contour of Fig. 7-18.]

96. Prove the residue theorem for multiply-connected regions.

97. Find sufficient conditions under which the residue theorem (Problem 2) is valid if C encloses infinitely many isolated singularities.

98. Let C be a circle with equation $|z| = 4$. Determine the value of the integral
$$\oint_C z^2 \csc\frac{1}{z}\, dz$$
if it exists.

Fig. 7-18

99. Give an analytical proof that $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$. [*Hint.* Consider the derivative of $(\sin\theta)/\theta$, showing that it is a decreasing function.]

100. Prove that $\displaystyle\int_0^\infty \frac{x}{\sinh \pi x}\, dx = \frac{1}{4}$.

101. Verify that the integral around Γ in equation (2) of Problem 22 goes to zero as $R \to \infty$.

102. (a) If r is real, prove that $\displaystyle\int_0^\pi \ln(1 - 2r\cos\theta + r^2)\, d\theta = \begin{cases} 0 & \text{if } |r| \leq 1 \\ \pi \ln r^2 & \text{if } |r| \geq 1 \end{cases}$.

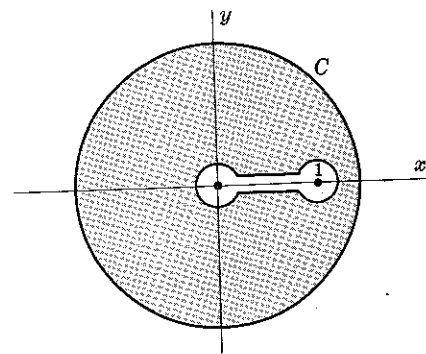 (b) Use the result in (a) to evaluate $\displaystyle\int_0^{\pi/2} \ln\sin\theta\, d\theta$ (see Problem 23).

103. Complete the proof of Case 2 in Problem 25.

104. If $0 < p < 1$, prove that $\displaystyle\int_0^\infty \frac{x^{-p}}{x-1}\,dx = \pi \cot p\pi$ in the Cauchy principal value sense.

105. Show that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^4 + n^2 + 1} = \frac{\pi\sqrt{3}}{3}\tanh\left(\frac{\pi\sqrt{3}}{2}\right)$

106. Verify that as $N \to \infty$ the integral on the left of (1) in Problem 29 goes to zero.

107. Prove that $\displaystyle\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536}$.

108. Prove the results given on Page 175 for (a) $\displaystyle\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right)$ and (b) $\displaystyle\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right)$.

109. If $-\pi \leqq \theta \leqq \pi$, prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\theta}{n^3} = \frac{\theta(\pi - \theta)(\pi + \theta)}{12}$.

110. Prove that the function $\cot z - 1/z$ of Problem 34 is bounded on the circles C_N.

111. Show that the second, fourth and sixth integrals in equation (3) of Problem 36 approach zero as $\epsilon \to 0$ and $R \to \infty$.

112. Prove that $\displaystyle\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh(3\pi/2)} + \frac{1}{5\cosh(5\pi/2)} - \cdots = \frac{\pi}{8}$.

113. Prove that $\displaystyle\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z}}\,dz = \frac{1}{\sqrt{\pi t}}$ where a and t are any positive constants.

114. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56{,}700}$.

115. Prove that $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)\cosh \pi x} = \frac{4-\pi}{2}$.

116. Prove that $\displaystyle\frac{1}{1^3 \sinh \pi} - \frac{1}{2^3 \sinh 2\pi} + \frac{1}{3^3 \sinh 3\pi} - \cdots = \frac{\pi^3}{360}$.

117. Prove that if a and t are any positive constants,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{zt}\cot^{-1} z\,dz = \frac{\sin t}{t}$$