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Exact Solutions to the Nonlinear Schrödinger Equation

Tuncay Aktosun, Theresa Busse, Francesco Demontis and Cornelis van der Mee

Dedicated to Israel Gohberg on the occasion of his eightieth birthday

Abstract. A review of a recent method is presented to construct certain exact solutions to the focusing nonlinear Schrödinger equation on the line with a cubic nonlinearity. With motivation by the inverse scattering transform and help from the state-space method, an explicit formula is obtained to express such exact solutions in a compact form in terms of a matrix triplet and by using matrix exponentials. Such solutions consist of multisolitons with any multiplicities, are analytic on the entire xt-plane, decay exponentially as $x \to \pm\infty$ at each fixed t, and can alternatively be written explicitly as algebraic combinations of exponential, trigonometric, and polynomial functions of the spatial and temporal coordinates x and t. Various equivalent forms of the matrix triplet are presented yielding the same exact solution.

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1. Introduction

Our goal in this paper is to review and further elaborate on a recent method [3, 4] to construct certain exact solutions to the focusing nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{1.1}$$

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with a cubic nonlinearity, where the subscripts denote the corresponding partial derivatives.

The NLS equation has important applications in various areas such as wave propagation in nonlinear media [15], surface waves on deep waters [14], and signal propagation in optical fibers [9–11]. It was the second nonlinear partial differential equation (PDE) whose initial value problem was discovered [15] to be solvable via the inverse scattering transform (IST) method. Recall that the IST method associates (1.1) with the Zakharov-Shabat system

$$\frac{d\varphi(\lambda,x,t)}{dx}=\begin{bmatrix}-i\lambda & u(x,t)\\ -u(x,t)^* & i\lambda\end{bmatrix}\varphi(\lambda,x,t), \tag{1.2}$$

where $u(x,t)$ appears as a potential and an asterisk is used for complex conjugation. By exploiting the one-to-one correspondence between the potential $u(x,t)$ and the corresponding scattering data for (1.2), that method amounts to determining the time evolution $u(x,0)\mapsto u(x,t)$ in (1.1) with the help of solutions to the direct and inverse scattering problems for (1.2). We note that the direct scattering problem for (1.2) consists of determining the scattering coefficients (related to the asymptotics of scattering solutions to (1.2) as $x\to\pm\infty$) when $u(x,t)$ is known for all x. On the other hand, the inverse scattering problem for (1.2) is to construct $u(x,t)$ when the scattering data is known for all λ.

Even though we are motivated by the IST method, our goal is not to solve the initial value problem for (1.1). Our aim is rather to construct certain exact solutions to (1.1) with the help of a matrix triplet and by using matrix exponentials. Such exact solutions turn out to be multisolitons with any multiplicities. Dealing with even a single soliton with multiplicities has not been an easy task in other methods; for example, the exact solution example presented in [15] for a one-soliton solution with a double pole, which is obtained by coalescing two distinct poles into one, contains a typographical error, as pointed out in [13].

In constructing our solutions we make use of the state-space method [6] from control theory. Our solutions are uniquely constructed via the explicit formula (2.6), which uses as input three (complex) constant matrices A, B, C, where A has size $p\times p$, B has size $p\times 1$, and C has size $1\times p$, with p as any positive integer. We will refer to (A,B,C) as a triplet of size p. There is no loss of generality in using a triplet yielding a minimal representation [3, 4, 6], and we will only consider such triplets. As seen from the explicit formula (2.6), our solutions are well defined as long as the matrix $F(x,t)$ defined in (2.5) is invertible. It turns out that $F(x,t)$ is invertible if and only if two conditions are met on the eigenvalues of the constant matrix A; namely, none of the eigenvalues of A are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Our solutions given by (2.6) are globally analytic on the entire xt-plane and decay exponentially as $x\to\pm\infty$ for each fixed $t\in\mathbf{R}$ as long as those two conditions on the eigenvalues of A are satisfied.

In our method [3, 4] we are motivated by using the IST with rational scattering data. For this purpose we exploit the state-space method [6]; namely, we use a matrix triplet (A, B, C) of an appropriate size in order to represent a rational function vanishing at infinity in the complex plane. Recall that any rational function $R(\lambda)$ in the complex plane that vanishes at infinity has a matrix realization in terms of a matrix triplet (A, B, C) as

$$R(\lambda) = -iC(\lambda I - iA)B, \tag{1.3}$$

where I denotes the identity matrix. The smallest integer p in the size of the triplet yields a minimal realization for $R(\lambda)$ in (1.3). A minimal realization is unique up to a similarity transformation. The poles of $R(\lambda)$ coincide with the eigenvalues of (iA).

The use of a matrix realization in the IST method allows us to establish the separability of the kernel of a related Marchenko integral equation [1, 2, 4, 12] by expressing that kernel in terms of a matrix exponential. We then solve that Marchenko integral equation algebraically and observe that our procedure leads to exact solutions to the NLS equation even when the input to the Marchenko equation does not necessarily come from any scattering data. We refer the reader to [3, 4] for details.

The explicit formula (2.6) provides a compact and concise way to express our exact solutions. If such solutions are desired to be expressed in terms of exponential, trigonometric (sine and cosine), and polynomial functions of x and t, this can also be done explicitly and easily by "unpacking" matrix exponentials in (2.6). If the size p in the matrices A, B, C is larger than 3, such expressions become long; however, we can still explicitly evaluate them for any matrix size p either by hand or by using a symbolic software package such as Mathematica. The power of our method is that we can produce exact solutions via (2.6) for any positive integer p. In some other available methods, exact solutions are usually tried to be produced directly in terms of elementary functions without using matrix exponentials, and hence any concrete examples that can be produced by such other methods will be relatively simple and we cannot expect those other methods to produce our exact solutions when p is large.

Our method is generalizable to obtain similar explicit formulas for exact solutions to other integrable nonlinear PDEs where the IST involves the use of a Marchenko integral equation [1, 2, 4, 12]. For example, a similar method has been used [5] for the half-line Korteweg-de Vries equation, and it can be applied to other equations such as the modified Korteweg-de Vries equation and the sine-Gordon equation. Our method is also generalizable to the matrix versions of such integrable nonlinear PDEs. For instance, a similar method has been applied in the third author's Ph.D. thesis [8] to the matrix NLS equation in the focusing case with a cubic nonlinearity.

Our method also easily handles nonsimple bound-state poles and the time evolution of the corresponding bound-state norming constants. In the literature, nonsimple bound-state poles are usually avoided due to mathematical complications. We refer the reader to [13], where nonsimple bound-state poles were investigated and complications were encountered. A systematic treatment of nonsimple bound states has recently been given in the second author's Ph.D. thesis [7].

The organization of our paper is as follows. Our main results are summarized in Section 2 and some explicit examples are provided in Section 3. For the proofs, further results, details, and a summary of other methods to solve the NLS equation exactly, we refer the reader to [3, 4].

2. Main results

In this section we summarize our method to construct certain exact solutions to the NLS equation in terms of a given triplet (A, B, C) of size p. For the details of our method we refer the reader to [3, 4]. Without any loss of generality, we assume that our starting triplet (A, B, C) corresponds to a minimal realization in (1.3). Let us use a dagger to denote the matrix adjoint (complex conjugate and matrix transpose), and let the set $\{a_j\}_{j=1}^m$ consist of the distinct eigenvalues of A, where the algebraic multiplicity of each eigenvalue may be greater than one and we use n_j to denote that multiplicity. We only impose the restrictions that no a_j is purely imaginary and that no two distinct a_j values are located symmetrically with respect to the imaginary axis on the complex plane. Let us set $\lambda_j := i a_j$ so that we can equivalently state our restrictions as that no λ_j will be real and no two distinct λ_j values will be complex conjugates of each other. Our method uses the following steps:

(i) First construct the constant $p \times p$ matrices Q and N that are the unique solutions, respectively, to the Lyapunov equations

$$Q\,A + A^\dagger Q = C^\dagger C, \tag{2.1}$$

$$A\,N + N\,A^\dagger = BB^\dagger. \tag{2.2}$$

In fact, Q and N can be written explicitly in terms of the triplet (A, B, C) as

$$Q = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda I + iA^\dagger)^{-1} C^\dagger C (\lambda I - iA)^{-1}, \tag{2.3}$$

$$N = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda I - iA)^{-1} BB^\dagger (\lambda I + iA^\dagger)^{-1}, \tag{2.4}$$

where γ is any positively oriented simple closed contour enclosing all λ_j in such a way that all λ_j^* lie outside γ. The existence and uniqueness of the solutions to (2.1) and (2.2) are assured by the fact that $\lambda_j \neq \lambda_j^*$ for all $j = 1, 2, \ldots, m$ and $\lambda_j \neq \lambda_k^*$ for $k \neq j$.

(ii) Construct the $p \times p$ matrix-valued function $F(x, t)$ as

$$F(x,t) := e^{2A^\dagger x - 4i(A^\dagger)^2 t} + Q\, e^{-2Ax - 4iA^2 t} N. \tag{2.5}$$

(iii) Construct the scalar function $u(x,t)$ via

$$u(x,t) := -2B^\dagger F(x,t)^{-1}C^\dagger. \tag{2.6}$$

Note that $u(x,t)$ is uniquely constructed from the triplet (A,B,C). As seen from (2.6), the quantity $u(x,t)$ exists at any point on the xt-plane as long as the matrix $F(x,t)$ is invertible. It turns out that $F(x,t)$ is invertible on the entire xt-plane as long as $\lambda_j \neq \lambda_j^*$ for all $j = 1,2,\ldots,m$ and $\lambda_j \neq \lambda_k^*$ for $k \neq j$.

Let us note that the matrices Q and N given in (2.3) and (2.4) are known in control theory as the observability Gramian and the controllability Gramian, respectively, and that it is well known in control theory that (2.3) and (2.4) satisfy (2.1) and (2.2), respectively. In the context of system theory, the invertibility of Q and N is described as the observability and the controllability, respectively. In our case, both Q and N are invertible due to the appropriate restrictions imposed on the triplet (A,B,C), which we will see in Theorem 1 below.

Our main results are summarized in the following theorems. For the proofs we refer the reader to [3, 4]. Although the results presented in Theorem 1 follow from the results in the subsequent theorems, we state Theorem 1 independently to clearly illustrate the validity of our exact solutions to the NLS equation.

Theorem 1. *Consider any triplet (A,B,C) of size p corresponding to a minimal representation in (1.3), and assume that none of the eigenvalues of A are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Then:*

(i) *The Lyapunov equations (2.1) and (2.2) are uniquely solvable, and their solutions are given by (2.3) and (2.4), respectively.*

(ii) *The constant matrices Q and N given in (2.3) and (2.4), respectively, are selfadjoint; i.e., $Q^\dagger = Q$ and $N^\dagger = N$. Furthermore, both Q and N are invertible.*

(iii) *The matrix $F(x,t)$ defined in (2.5) is invertible on the entire xt-plane, and the function $u(x,t)$ defined in (2.6) is a solution to the NLS equation everywhere on the xt-plane. Moreover, $u(x,t)$ is analytic on the entire xt-plane and it decays exponentially as $x \to \pm\infty$ at each fixed $t \in \mathbf{R}$.*

We will say that two triplets (A,B,C) and $(\tilde{A},\tilde{B},\tilde{C})$ are equivalent if they yield the same potential $u(x,t)$ through (2.6). The following result shows that, as far as constructing solutions via (2.6) is concerned, there is no loss of generality is choosing our starting triplet (A,B,C) of size p so that it corresponds to a minimal representation in (1.3) and that all eigenvalues a_j of the matrix A have positive real parts.

Theorem 2. *Consider any triplet $(\tilde{A},\tilde{B},\tilde{C})$ of size p corresponding to a minimal representation in (1.3), and assume that none of the eigenvalues of $\tilde{A}$ are purely imaginary and that no two eigenvalues of $\tilde{A}$ are symmetrically located with respect to the imaginary axis. Then, there exists an equivalent triplet (A,B,C) of* *size p corresponding to a minimal representation in* (1.3) *in such a way that all eigenvalues of A have positive real parts.*

The next two results given in Theorems 3 and 4 show some of the advantages of using a triplet (A,B,C) where all eigenvalues of A have positive real parts. Concerning Theorem 2, we remark that the triplet (A,B,C) can be obtained from $(\tilde A,\tilde B,\tilde C)$ and vice versa with the help of Theorem 5 or Theorem 6 given below.

Theorem 3. *Consider any triplet (A,B,C) of size p corresponding to a minimal representation in* (1.3)*. Assume that all eigenvalues of A have positive real parts. Then:*

(i) *The solutions Q and N to* (2.1) *and* (2.2)*, respectively, can be expressed in terms of the triplet (A,B,C) as*

$$Q=\int_0^\infty ds\,[Ce^{-As}]^\dagger[Ce^{-As}],\qquad N=\int_0^\infty ds\,[e^{-As}B][e^{-As}B]^\dagger. \tag{2.7}$$

(ii) *Q and N are invertible, selfadjoint matrices.*

(iii) *Any square submatrix of Q containing the $(1,1)$-entry or (p,p)-entry of Q is invertible. Similarly, any square submatrix of N containing the $(1,1)$-entry or (p,p)-entry of N is invertible.*

Theorem 4. *Consider a triplet $(\tilde A,\tilde B,\tilde C)$ of size p corresponding to a minimal representation in* (1.3) *and that all eigenvalues a_j of the matrix $\tilde A$ have positive real parts and that the multiplicity of a_j is n_j for $j=1,2,\dots,m$. Then, there exists an equivalent triplet (A,B,C) of size p corresponding to a minimal representation in* (1.3) *in such a way that A is in a Jordan canonical form with each Jordan block containing a distinct eigenvalue a_j and having -1 in the superdiagonal entries, and the entries of B consist of zeros and ones. More specifically, we have*

$$A=\begin{bmatrix}A_1&0&\dots&0\\0&A_2&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&A_m\end{bmatrix},\quad B=\begin{bmatrix}B_1\\B_2\\\vdots\\B_m\end{bmatrix},\quad C=\begin{bmatrix}C_1&C_2&\dots&C_m\end{bmatrix}, \tag{2.8}$$

$$A_j:=\begin{bmatrix}a_j&-1&0&\dots&0\\0&a_j&-1&\dots&0\\0&0&a_j&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&a_j\end{bmatrix},\quad B_j:=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix},\quad C_j:=\begin{bmatrix}c_{j(n_j-1)}&\dots&c_{j1}&c_{j0}\end{bmatrix},$$

where A_j has size $n_j\times n_j$, B_j has size $n_j\times 1$, C_j has size $1\times n_j$, and the (*complex*) *constant $c_{j(n_j-1)}$ is nonzero.*

We will refer to the specific form of the triplet (A,B,C) given in (2.8) as a standard form.

The transformation between two equivalent triplets can be obtained with the help of the following two theorems. First, in Theorem 5 below we consider the transformation where all eigenvalues of A are reflected with respect to the imaginary axis. Then, in Theorem 6 we consider transformations where only some of the eigenvalues of A are reflected with respect to the imaginary axis.

Theorem 5. *Assume that the triplet* (A, B, C) *of size* p *corresponds to a minimal realization in* (1.3) *and that all eigenvalues of* A *have positive real parts. Consider the transformation*

$$(A, B, C, Q, N, F) \mapsto (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N}, \tilde{F}), \tag{2.9}$$

where (Q, N) *corresponds to the unique solution to the Lyapunov system in* (2.1) *and* (2.2)*, the quantity* F *is as in* (2.5)*,*

$$\tilde{A} = -A^\dagger, \quad \tilde{B} = -N^{-1}B, \quad \tilde{C} = -CQ^{-1}, \quad \tilde{Q} = -Q^{-1}, \quad \tilde{N} = -N^{-1},$$

and $\tilde{F}$ *and* $\tilde{u}$ *are as in* (2.5) *and* (2.6)*, respectively, but by using* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{Q}, \tilde{N})$ *instead of* (A, B, C, Q, N) *on the right-hand sides. We then have the following:*

(i) *The matrices* $\tilde{Q}$ *and* $\tilde{N}$ *are selfadjoint and invertible. They satisfy the respective Lyapunov equations*

$$\begin{cases} \tilde{Q}\tilde{A} + \tilde{A}^\dagger \tilde{Q} = \tilde{C}^\dagger \tilde{C}, \\ \tilde{A}\tilde{N} + \tilde{N}\tilde{A}^\dagger = \tilde{B}\tilde{B}^\dagger. \end{cases} \tag{2.10}$$

(ii) *The quantity* F *is transformed as* $\tilde{F} = Q^{-1}FN^{-1}$*. The matrix* $\tilde{F}$ *is invertible at every point on the* xt*-plane.*

To consider the case where only some of eigenvalues of A are reflected with respect to the imaginary axis, let us again start with a triplet (A, B, C) of size p and corresponding to a minimal realization in (1.3), where the eigenvalues of A all have positive real parts. Without loss of any generality, let us assume that we partition the matrices A, B, C as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{2.11}$$

so that the $q \times q$ block diagonal matrix A_1 contains the eigenvalues that will remain unchanged and A_2 contains the eigenvalues that will be reflected with respect to the imaginary axis on the complex plane, the submatrices B_1 and C_1 have sizes $q \times 1$ and $1 \times q$, respectively, and hence A_2, B_2, C_2 have sizes $(p-q) \times (p-q)$, $(p-q) \times 1$, $1 \times (p-q)$, respectively, for some integer q not exceeding p. Let us clarify our notational choice in (2.11) and emphasize that the partitioning in (2.11) is not the same partitioning used in (2.8). Using the partitioning in (2.11), let us write the corresponding respective solutions to (2.1) and (2.2) as

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{2.12}$$

where Q_1 and N_1 have sizes $q\times q$, Q_4 and N_4 have sizes $(p-q)\times(p-q)$, etc. Note that because of the selfadjointness of Q and N stated in Theorem 1, we have

$$Q_1^\dagger = Q_1, \quad Q_2^\dagger = Q_3, \quad Q_4^\dagger = Q_4, \quad N_1^\dagger = N_1, \quad N_2^\dagger = N_3, \quad N_4^\dagger = N_4.$$

Furthermore, from Theorem 3 it follows that Q_1, Q_4, N_1, and N_4 are all invertible.

Theorem 6. *Assume that the triplet* (A,B,C) *partitioned as in* (2.11) *corresponds to a minimal realization in* (1.3) *and that all eigenvalues of* A *have positive real parts. Consider the transformation* (2.9) *with* $(\tilde A,\tilde B,\tilde C)$ *having similar block representations as in* (2.11), (Q,N) *as in* (2.12) *corresponding to the unique solution to the Lyapunov system in* (2.1) *and* (2.2),

$$\tilde A_1 = A_1, \quad \tilde A_2 = -A_2^\dagger, \quad \tilde B_1 = B_1 - N_2N_4^{-1}B_2, \quad \tilde B_2 = -N_4^{-1}B_2,$$

$$\tilde C_1 = C_1 - C_2Q_4^{-1}Q_3, \quad \tilde C_2 = -C_2Q_4^{-1},$$

and $\tilde Q$ *and* $\tilde N$ *partitioned in a similar way as in* (2.12) *and given as*

$$\tilde Q_1 = Q_1 - Q_2Q_4^{-1}Q_3, \quad \tilde Q_2 = -Q_2Q_4^{-1}, \quad \tilde Q_3 = -Q_4^{-1}Q_3, \quad \tilde Q_4 = -Q_4^{-1},$$

$$\tilde N_1 = N_1 - N_2N_4^{-1}N_3, \quad \tilde N_2 = -N_2N_4^{-1}, \quad \tilde N_3 = -N_4^{-1}N_3, \quad \tilde N_4 = -N_4^{-1},$$

and $\tilde F$ *and* $\tilde u$ *as in* (2.5) *and* (2.6), *respectively, but by using* $(\tilde A,\tilde B,\tilde C,\tilde Q,\tilde N)$ *instead of* (A,B,C,Q,N) *on the right-hand sides. We then have the following:*

(i) *The matrices* $\tilde Q$ *and* $\tilde N$ *are selfadjoint and invertible. They satisfy the respective Lyapunov equations given in* (2.10).

(ii) *The quantity* F *is transformed according to*

$$\tilde F = \begin{bmatrix} I & -Q_2Q_4^{-1} \\ 0 & -Q_4^{-1}\end{bmatrix} F \begin{bmatrix} I & 0 \\ -N_4^{-1}N_3 & -N_4^{-1}\end{bmatrix},$$

and the matrix $\tilde F$ *is invertible at every point on the* xt*-plane.*

(iii) *The triplets* (A,B,C) *and* $(\tilde A,\tilde B,\tilde C)$ *are equivalent; i.e.,* $\tilde u(x,t) = u(x,t)$.

3. Examples

In this section we illustrate our method of constructing exact solutions to the NLS equation with some concrete examples.

Example 1. The well-known "n-soliton" solution to the NLS equation is obtained by choosing the triplet (A,B,C) as

$$A = \operatorname{diag}\{a_1,a_2,\dots,a_n\}, \quad B^\dagger = \begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}, \quad C = \begin{bmatrix}c_1 & c_2 & \dots & c_n\end{bmatrix},$$

where a_j are distinct (complex) nonzero constants with positive real parts, B contains n entries, and the quantities c_j are complex constants. Note that diag is used to denote the diagonal matrix. In this case, using (2.5) and (2.7) we evaluate the (j,k)-entries of the $n\times n$ matrix-valued functions Q, N, and $F(x,t)$ as

$$N_{jk}=\frac{1}{a_j+a_k^*},\quad Q_{jk}=\frac{c_j^*c_k}{a_j^*+a_k},\quad F_{jk}=\delta_{jk}e^{2a_j^*x-4i(a_j^*)^2t}+\sum_{s=1}^{n}\frac{c_j^*c_s\,e^{-2a_sx-4ia_s^2t}}{(a_j^*+a_s)(a_s+a_k^*)},$$

where δ_{jk} denotes the Kronecker delta. Having obtained Q, N, and $F(x,t)$, we construct the solution $u(x,t)$ to the NLS equation via (2.6) or equivalently as the ratio of two determinants as

$$u(x,t)=\frac{2}{\det F(x,t)}\begin{vmatrix}0 & B^\dagger\\ C^\dagger & F(x,t)\end{vmatrix}. \tag{3.1}$$

For example, when $n=1$, from (3.1) we obtain the single soliton solution

$$u(x,t)=\frac{-8c_1^*(\mathrm{Re}[a_1])^2\,e^{-2a_1^*x+4i(a_1^*)^2t}}{4(\mathrm{Re}[a_1])^2+|c_1|^2\,e^{-4x(\mathrm{Re}[a_1])+8t(\mathrm{Im}[a_1^2])}},$$

where Re and Im denote the real and imaginary parts, respectively. From (1.1) we see that if $u(x,t)$ is a solution to (1.1), so is $e^{i\theta}u(x,t)$ for any real constant θ. Hence, the constant phase factor $e^{i\theta}$ can always be omitted from the solution to (1.1) without any loss of generality. As a result, we can write the single soliton solution also in the form

$$u(x,t)=2\,\mathrm{Re}[a_1]\;e^{i\beta(x,t)}\mathrm{sech}\left(2\,\mathrm{Re}[a_1](x-4t\,\mathrm{Im}[a_1])-\log\left(\frac{|c_1|}{2\mathrm{Re}[a_1]}\right)\right),$$

where it is seen that $u(x,t)$ has amplitude $2\,\mathrm{Re}[a_1]$ and moves with velocity $4\,\mathrm{Im}[a_1]$ and we have

$$\beta(x,t):=2x\mathrm{Im}[a_1]+4t\,\mathrm{Re}[a_1^2].$$

Example 2. For the triplet (A,B,C) given by

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},\quad B=\begin{bmatrix}1\\ 1\end{bmatrix},\quad C=\begin{bmatrix}1 & -1\end{bmatrix}, \tag{3.2}$$

we evaluate Q and N explicitly by solving (2.1) and (2.2), respectively, as

$$N=\begin{bmatrix}1/4 & 1\\ 1 & -1/2\end{bmatrix},\quad Q=\begin{bmatrix}1/4 & -1\\ -1 & -1/2\end{bmatrix},$$

and obtain $F(x,t)$ by using (2.5) as

$$F(x,t)=\begin{bmatrix}e^{4x-16it}-e^{2x-4it}+\frac{1}{16}e^{-4x-16it} & \frac{1}{4}e^{-4x-16it}+\frac{1}{2}e^{2x-4it}\\ -\frac{1}{4}e^{-4x-16it}-\frac{1}{2}e^{2x-4it} & e^{-2x-4it}-e^{-4x-16it}+\frac{1}{4}e^{2x-4it}\end{bmatrix}.$$

Finally, using (2.6), we obtain the corresponding solution to the NLS equation as

$$u(x,t)=\frac{8e^{4it}(9e^{-4x}+16e^{4x})-32e^{16it}(4e^{-2x}+9e^{2x})}{-128\cos(12t)+4e^{-6x}+16e^{6x}+81e^{-2x}+64e^{2x}}. \tag{3.3}$$

It can independently be verified that $u(x,t)$ given in (3.3) satisfies the NLS equation on the entire xt-plane.

With the help of the results stated in Section 2, we can determine triplets $(\tilde A,\tilde B,\tilde C)$ that are equivalent to the triplet in (3.2).

The following triplets all yield the same $u(x,t)$ given in (3.3):

(i) $\tilde A=\begin{bmatrix}2&0\\0&1\end{bmatrix},\ \tilde B=\begin{bmatrix}9/\alpha_1\\-4/\alpha_2\end{bmatrix},\ \tilde C=\begin{bmatrix}\alpha_1&\alpha_2\end{bmatrix},$

where α_1 and α_2 are arbitrary (complex) nonzero parameters. Note that both eigenvalues of $\tilde A$ are positive, whereas only one of the eigenvalues of A in (3.2) is positive.

(ii) $\tilde A=\begin{bmatrix}-2&0\\0&1\end{bmatrix},\ \tilde B=\begin{bmatrix}16/(9\alpha_3)\\-4/(9\alpha_4)\end{bmatrix},\ \tilde C=\begin{bmatrix}\alpha_3&\alpha_4\end{bmatrix},$

where α_3 and α_4 are arbitrary (complex) nonzero parameters. Note that the eigenvalues of $\tilde A$ in this triplet are negatives of the eigenvalues of A given in (3.2).

(iii) $\tilde A=\begin{bmatrix}2&0\\0&-1\end{bmatrix},\ \tilde B=\begin{bmatrix}1/\alpha_5\\-1/\alpha_6\end{bmatrix},\ \tilde C=\begin{bmatrix}\alpha_5&\alpha_6\end{bmatrix},$

where α_5 and α_6 are arbitrary (complex) nonzero parameters. Note that $\tilde A$ here agrees with A in (3.2).

(iv) $\tilde A=\begin{bmatrix}-2&0\\0&-1\end{bmatrix},\ \tilde B=\begin{bmatrix}16/\alpha_7\\-9/\alpha_8\end{bmatrix},\ \tilde C=\begin{bmatrix}\alpha_7&\alpha_8\end{bmatrix},$

where α_7 and α_8 are arbitrary (complex) nonzero parameters. Note that both eigenvalues of $\tilde A$ are negative.

(v) Equivalent to (3.2) we also have the triplet $(\tilde A,\tilde B,\tilde C)$ given by

$$\tilde A=\begin{bmatrix}\alpha_9&\alpha_{10}\\ \dfrac{(1-\alpha_9)(\alpha_9-2)}{\alpha_{10}}&3-\alpha_9\end{bmatrix},$$

$$\tilde B=\frac{\begin{bmatrix}5\alpha_{10}^2\alpha_{11}+\alpha_{10}\alpha_{12}-5\alpha_9\alpha_{10}\alpha_{12}\\ 14\alpha_{10}\alpha_{11}-5\alpha_9\alpha_{10}\alpha_{11}+10\alpha_{12}-15\alpha_9\alpha_{12}+5\alpha_9^2\alpha_{12}\end{bmatrix}}{\alpha_{10}^2\alpha_{11}^2+3\alpha_{10}\alpha_{11}\alpha_{12}-2\alpha_9\alpha_{10}\alpha_{11}\alpha_{12}+2\alpha_{12}^2-3\alpha_9\alpha_{12}^2+\alpha_9^2\alpha_{12}^2},$$

$$\tilde C=\begin{bmatrix}\alpha_{11}&\alpha_{12}\end{bmatrix},$$

where $\alpha_9,\ \dots,\ \alpha_{12}$ are arbitrary parameters with the restriction that $\alpha_{10}\alpha_{11}\alpha_{12}\neq 0$, which guarantees that the denominator of $\tilde B$ is nonzero; when $\alpha_{10}=0$ we must have $\alpha_{11}\alpha_{12}\neq 0$ and choose α_9 as 2 or 1. In fact, the minimality of the triplet $(\tilde{A}, \tilde{B}, \tilde{C})$ guarantees that $\tilde{B}$ is well defined. For example, the triplet is not minimal if $\alpha_{11}\alpha_{12} = 0$. We note that the eigenvalues of $\tilde{A}$ are 2 and 1 and that $\tilde{A}$ here is similar to the matrix $\tilde{A}$ in the equivalent triplet given in (i).

Other triplets equivalent to (3.2) can be found as in (v) above, by exploiting the similarity for the matrix $\tilde{A}$ given in (ii), (iii), and (iv), respectively, and by using (1.3) to determine the corresponding $\tilde{B}$ and $\tilde{C}$ in the triplet.

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Tuncay Aktosun
Department of Mathematics
University of Texas at Arlington
Arlington, TX 76019, USA
e-mail: aktosun@uta.edu

Theresa Busse
Department of Mathematics and Computer Science
Northeast Lakeview College
Universal City, TX 78148, USA
e-mail: tbusse@alamo.edu

Francesco Demontis and Cornelis van der Mee
Dipartimento di Matematica e Informatica
Università di Cagliari
Viale Merello 92
I-09123 Cagliari, Italy
e-mail: fdemontis@unica.it
cornelis@krein.unica.it

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