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SPECTRAL ANALYSIS OF THE TRANSPORT EQUATION
II. STABILITY AND APPLICATION TO THE MILNE PROBLEM

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0. INTRODUCTION

In this article a study is made of the integro-differential equation

$$\mu \frac{d\psi}{dx}(x,\mu) + \psi(x,\mu) =$$

$$\int_{-1}^{+1} \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{g}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)\,d\alpha\right] \psi(x,\mu')\,d\mu', \tag{0.1}$$

$$(-1 \leq \mu \leq +1,\ 0 < x < \tau \leq +\infty)$$

where τ is either finite or infinite. In astrophysics this equation describes the stationary transfer of unpolarized radiation through a homogeneous stellar or planetary atmosphere (see [5,24,13]). In neutron physics Eq.(0.1) describes the stationary transport of mutually non-interacting, undelayed neutrons with uniform speed through a homogeneous plane-parallel fuel plate of a nuclear reactor (see [6]). In both cases the function $\hat{g}$ is given and describes the scattering properties of the medium; in astrophysics $\hat{g}$ is called the phase function. (Here the albedo has been included as a factor). The problem is to determine the unknown function ψ under suitable boundary conditions. In astrophysics (resp. neutron physics) ψ represents the azimuth-averaged intensity of the radiation (resp. the angular neutron density). The variable x is a position coordinate and μ is the direction cosine of tne propagation vector. In

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astrophysics (resp. neutron physics) the parameter τ is the optical thickness (resp. the thickness of the fuel plate in units of neutron mean free path); there are no internal radiative (resp. neutron) sources.

If the parameter τ is finite, one imposes the boundary conditions

(0.2a) $\quad \psi(0,\mu) = \phi(\mu) \quad (0\le\mu\le1), \; \psi(\tau,\mu) = \phi(\mu) \quad (-1\le\mu<0),$

and calls the problem (0.1)-(0.2a) the *finite-slab problem.* If τ is infinite, one imposes the boundary conditions

(0.2b) $\quad \psi(0,\mu) = \phi_+(\mu) \quad (0\le\mu\le1), \; {}_{-1}\int^{+1}|\psi(x,\mu)|^2 d\mu = O(1) \; (x\to+\infty),$

and the problem (0.1)-(0.2b) is called the *half-space problem.* Both problems are presently investigated, both in astrophysics and neutron physics (see [24,13]; [6]). In astrophysics for infinite τ one also imposes the boundary conditions

(0.2c) $\psi(0,\mu) = 0 \; (0\le\mu\le1); \; \exists n\ge0: \; {}_{-1}\int^{+1}|\psi(x,\mu)|^2 d\mu = O(x^{2n}) \; (x\to+\infty);$

$$\lim_{x\to+\infty} {}_{-1}\int^{+1}\mu\psi(x,\mu)d\mu = -\tfrac{1}{2}F;$$

the problem (0.1)-(0.2c) is known as the *Milne problem* (see [13,5]). The functions ϕ and ϕ_+ appearing in (0.2) describe the radiative or neutron fluxes incident on the surface; F denotes the radiative flux coming from the stellar interior (see [5], Eq.(86) of Chapter I).

For physical reasons the phase function $\hat{g}$ must be nonnegative and $c = {}_{-1}\int^{+1}\hat{g}(t)dt < +\infty$. In this article we solely consider the case $0\le c\le1$, which in astrophysics always occurs and in neutron physics occurs for non-multiplying media (see [24,13]; [6]). For $c=1$ (resp. $0\le c<1$) the term "conservative" (resp. "non-conservative") case is customary.

In this paper we continue the research leading to [18] by investigating the stability of the solutions of the finite-slab and half-space problems under perturbations of the phase function $\hat{g}$. One of the perturbations of the phase function $\hat{g}$, for which the stability is established, is the truncation of its Legendre series expansion. The method we employ has been inspired by some work of Feldman [7] on a related stability problem.

Another stability result of this article is the stability of the bounded solutions of the half-space problem in the conservative case under perturbation from the non-conservative case. This result will enable us to derive analytic expressions for the solution of the Milne problem as we did in Section VI.5 of [18] for isotropic scattering. A result derived differently in a not completely rigorous way by Pahor [21] and by Busbridge and Orchard (cf. [4]) is reproduced.

Next we give a short description of the present mathematical approach to the Transport Equation (see also [17,18]). To study Eq.(0.1) in the Hilbert space $L_2[-1,+1]$ one puts $\psi(x)(\mu) = \psi(x,\mu)$ $(0<x<\tau,\ -1\le\mu\le+1)$ and one defines the operators T and B on $L_2[-1,+1]$ by

$$(0.3)\qquad \begin{aligned}(Th)(\mu) &= \mu h(\mu),\\ (Bh)(\mu) &= {}_{-1}\int^{+1}(2\pi)^{-1}{}_0\int^{2\pi}\hat{g}(\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\alpha)d\alpha h(\mu')d\mu'.\end{aligned}$$

Now Eq.(0.1) is rewritten as an operator differential equation of the form

$$(0.4)\qquad (T\psi)'(x) = -(I-B)\psi(x),\ 0<x<\tau,$$

with suitable boundary conditions. An equivalent form of the operator differential equation (0.4) with boundary conditions appears to be a vector-valued convolution equation of the form

$$(0.5)\qquad \psi(x) - {}_0\int^{\tau}H(x-y)B\psi(y)dy = \omega(x),\quad 0<x<\tau.$$

The so called propagator function $H(.)$ is given by

$$[H(t)h](\mu) = \begin{cases} +\mu^{-1}e^{-t/\mu}\ h(\mu); & 0<t<+\infty,\ 0\le\mu\le1;\\ -\mu^{-1}e^{-t/\mu}\ h(\mu); & -\infty<t<0,\ -1\le\mu<0;\\ \qquad 0 & ;\ t\mu<0.\end{cases}$$

The existence and uniqueness of the solution of finite-slab and half-space problems can be established by applying semigroup theory to Eq.(0.4) and factorization methods to Eq.(0.5) (see [18], Chapters IV and V). Problems of stability are studied by establishing the stability of the kernel $H(.)B$ of the convolution equation (0.5) in a certain operator norm.

In Section 1 we review the main elements of the theory of semi-definite admissible pairs. By means of such pairs, of which the pair (T,B) in (0.3) is just an example, the half-space, finite-slab and Milne problems can be studied in an abstract framework. In Section 2 the stability of the solution of the convolution equation (0.5) under perturbation of the operator B in the semi-definite pair (T,B) is established. By specifying the results for the case of the Transport Equation (0.1) in Section 3 we obtain stability statements in Transport Theory. In Section 4 an analytic expression for the bounded solution of the half-space problem in the conservative case is derived by means of a stability argument. In Section VI.3 in [18] such an expression has been given for the non-conservative case only. Finally in Section 5 the solution of the Milne problem is obtained.

As to notation, by $\langle .,. \rangle$ we denote the inner product of a complex Hilbert space. The kernel or null space of a linear operator T is denoted by Ker T and the image or range of T by Im T. The Banach algebra of bounded linear operators on a Banach space X is referred to as $L(X)$. The identity operator on X is denoted by I_X (or by I, if no confusion is possible). The spectrum of an operator T on X is denoted by $\sigma(T)$. For the orthogonal complement of a subset M of a Hilbert space we write $M^{\perp}$.

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1. PRELIMINARIES

Throughout this section H is a complex Hilbert space with inner product $\langle .,. \rangle$. By a *semi-definite admissible pair* on H we mean a pair (T,B) of bounded linear operators on H such that

(C.1) T is self-adjoint and has a trivial kernel;

(C.2) B is compact and A = I-B is (non-strictly) positive (i.e., $\langle Ah,h \rangle \geq 0$ for every $h \in H$);

(C.3) there exist $0<\alpha<1$ and a bounded linear operator D on H such that

$$B = |T|^{\alpha} D.$$

If A is strictly positive (i.e., $\langle Ah,h \rangle > 0$ for every $0 \neq h \in H$), the pair (T,B) is called *positive definite*. If A has a non-trivial kernel and thus A is non-strictly positive, the pair (T,B) is called *singular*.

In Chapter III of [18] a theory of semi-definite admissible pairs has been developed. Positive definite pairs have been introduced earlier in [17] under the name of self-adjoint admissible pairs. Here we review the main elements of this theory. Consider the pencil

(1.1) $L(\lambda) = A - \lambda T.$

If the pair (T,B) is singular, the point at $\lambda=0$ is an isolated point of the spectrum $\Sigma(L) = \{\lambda\in\mathbb{C}: L(\lambda) \text{ is not invertible}\}$ of $L(\lambda)$. If Γ is a positively oriented circle with centre 0 that separates the point at $\lambda=0$ from the remaining part of $\Sigma(L)$, then the operators

$$P_0 = -(2\pi i)^{-1}\int_\Gamma L(\lambda)^{-1}T d\lambda, \quad P_0^+ = -(2\pi i)^{-1}\int_\Gamma TL(\lambda)^{-1}d\lambda$$

are projections of the same finite rank. Denoting $H_0 = \operatorname{Im} P_0$, $H_0^+ = \operatorname{Im} P_0^+$, $H_1 = \operatorname{Ker} P_0$ and $H_1^+ = \operatorname{Ker} P_0^+$, one has

$$\text{(1.2)} \quad \begin{array}{l} H_0^+ = H_1^\perp,\ H_1^+ = H_0^\perp,\ T[H_0] = H_0^+,\ \overline{T[H_1]} = H_1^+; \\ \operatorname{Ker} A \subset H_0,\ A[H_1] = H_1^+. \end{array}$$

It appears that A acts as an invertible operator from H_1 onto H_1^+ and T acts as an invertible operator from the so-called *singular subspace* H_0 onto H_0^+. Thus H_0 is contained in the domain of the possibly unbounded operator $T^{-1}A$. Moreover,

(1.3) $(T^{-1}A)^2x = 0, \quad x\in H_0.$

If the pair (T,B) is positive definite, then $H_0 = H_0^+ = \{0\}$, $H_1 = H_1^+ = H$ and $P_0 = P_0^+ = 0$.

On H_1 the sesquilinear form

(1.4) $\langle x,y\rangle_A = \langle Ax,y\rangle \qquad (x,y\in H_1)$

is an inner product equivalent to the one inherited from H. Since A acts as an invertible operator from H_1 onto H_1^+ and $T[H_1] \subset H_1^+$, there exist unique bounded linear operators $S: H_1 \to H_1$ and $S^+: H_1^+ \to H_1^+$ such that

(1.5) $AS = S^+A = T.$

It appears that S is self-adjoint with respect to the inner product (1.4) on H_1. We call S the *associate operator* of the pair (T,B). One has

$$L(\lambda) = U[(I - \lambda S) \oplus (T^{-1}A - \lambda)|_{H_0}], \qquad \lambda\in\mathbb{C},$$

where $U = A(I - P_0) + TP_0$ is invertible. Hence,

$$\sigma(S) = \overline{\{\lambda^{-1}: 0\neq\lambda\in\Sigma(L)\}} \subset \mathbb{R};$$

further, $\sigma(S)\backslash\sigma(T)$ consists of isolated eigenvalues of finite multiplicity only. If the pair (T,B) is positive definite, then S and S^+ are defined on the whole of H and $S = A^{-1}T$ and $S^+ = TA^{-1}$.

Let F be the resolution of the identity of $S\in L(H_1)$ (as a self-adjoint operator with respect to the inner product (1.4)). Analogously, let F^+ be the resolution of the identity of $S^+\in L(H_1^+)$ (note that, by (1.5), S and S^+ are similar). For $x\in H$ put

$$P_p x = F((0,+\infty))(I - P_0)x, \quad P_p^+ x = F^+((0,+\infty))(I - P_0^+)x;$$

$$P_m x = F((-\infty,0))(I - P_0^+)x, \quad P_m^+ x = F^+((-\infty,0))(I - P_0^+)x.$$

Denoting $H_p = \operatorname{Im} P_p$, $H_p^+ = \operatorname{Im} P_p^+$, $H_m = \operatorname{Im} P_m$ and $H_m^+ = \operatorname{Im} P_m^+$ one has

(1.6a) $H = H_m \oplus H_p \oplus H_0, \quad H = H_m^+ \oplus H_p^+ \oplus H_0^+;$

(1.6b) $TP_p = P_p^+T, \quad TP_m = P_m^+T, \quad AP_p = P_p^+A, \quad AP_m = P_m^+A;$

(1.6c) $\overline{T[H_p]} = H_p^+, \quad \overline{T[H_m]} = H_m^+, \quad A[H_p] = H_p^+, \quad A[H_m] = H_m^+;$

(1.6d) $P_p^+ = P_p^*, \quad P_m^+ = P_m^*, \quad H_p^+ = (H_m \oplus H_0)^\perp, \quad H_m^+ = (H_p \oplus H_0)^\perp.$

Finally, the operator T is self-adjoint and $\operatorname{Ker} T = \{0\}$. So if P_+ (P_-) denotes the spectral projection of T corresponding to the positive (negative) part of its spectrum and H_+ (H_-) refers to the range of P_+ (P_-), then

$$H = H_- \oplus H_+.$$

The next decomposition theorems play an important role in establishing the existence and uniqueness of a bounded solution of the half-space problem. The second one of these theorems appeared in [18] under the extra condition of inversion symmetry and will therefore be proved here without using this condition.

THEOREM 1.1. (cf. Th. III.5.1 of [18]). *Let* (T,B) *be a positive definite admissible pair on* H. *Then*

(1.7) $H_p \oplus H_- = H_m \oplus H_+ = H.$

THEOREM 1.2. (cf. Section III.8 in [18]). *Let* (T,B) *be a semi-definite admissible pair on* H. *Then there exist subspaces* N_+ *and* N_- *of* $\operatorname{Ker} A$ *such that*

(1.8) $H_p \oplus N_+ \oplus H_- = H_m \oplus N_- \oplus H_+ = H.$

Moreover, if the finite-dimensional operator $T^{-1}A|_{H_0}$ *has Jordan blocks of order 2 only* (i.e. *if* $T^{-1}A[H_0] = \operatorname{Ker} A$), *then in* (1.8) *one may take* $N_+ = N_- = \operatorname{Ker} A$.

Proof. On H_0 one considers the indefinite inner product

(1.9) $\langle x,y\rangle_T = \langle Tx,y\rangle \qquad (x,y\in H_0)$.

According to Proposition III.5.3 of [18] the space H_0 endowed with this inner product is a Krein space (see [2] for the definition and main properties of Krein spaces) and $[H_p \oplus H_-] \cap H_0$ (resp. $[H_m \oplus H_+] \cap H_0$) is a maximal strictly negative (resp. positive) subspace of H_0. For $x = T^{-1}Ay \in T^{-1}A[H_0]$ and $z \in \operatorname{Ker} A$ one has

(1.10) $\langle Sy,z\rangle_T = \langle Ay,z\rangle = \langle y,Az\rangle = 0$,

and thus the subspace $T^{-1}A[H_0]$ of Ker A (cf. (1.3) to see this) is a neutral subspace of H_0. Let M denote a complement of $T^{-1}A[H_0]$ in Ker A. Then the indefinite inner product space M (endowed with (1.9)) has a maximal positive subspace N_+ and a maximal negative subspace N_- (c.f. [2]). Using (1.10) one sees that $T^{-1}A[H_0] \oplus N_+$ (resp. $T^{-1}A[H_0] \oplus N_-$) is a maximal positive (resp. maximal negative) subspace of H_0 that is contained in Ker A. Since $[H_p \oplus H_-] \cap H_0$ (resp. $[H_m \oplus H_+] \cap H_0$) is maximal strictly negative (resp. positive), one has

$$([H_p \oplus H_-] \cap H_0) \oplus (T^{-1}A[H_0] \oplus N_+) = ([H_m \oplus H_+] \cap H_0) \oplus (T^{-1}A[H_0] \oplus N_-) = H_0.$$

From this identity the decompositions (1.8) are clear. The second part of the theorem is straightforward. □

To study stability properties we now introduce uniform collections. A collection $\{(T,B_i)\}_{i\in I}$ of semi-definite admissible pairs on H is said to be *uniform* if in Condition (C.3) there exist one and the same $0<\alpha<1$ and a bounded set of bounded linear operators D_i on H such that

$$B_i = |T|^{\alpha} D_i, \qquad i\in I.$$

If the index set I is countable, we shall often use the term "uniform sequence".

For the non-conservative isotropic case of the Transport Equation (and later for the degenerate anisotropic case too) Hangelbroek [11] introduced the projections P_p, P_m, P_+ and P_-, and the subspaces H_p, H_m, H_+ and H_-, and proved (1.7). His work has been extended by Lekkerkerker [15] to the conservative isotropic case; in [15] a singular subspace is introduced and decompositions like (1.8) are derived.

With a semi-definite admissible pair (T,B) on H an operator differential equation with boundary conditions and a vector-valued convolution equation are connected. Their equivalence is the content of the following

THEOREM 1.3. (= Th. V.2.1 of [18]). *Let* (T,B) *be a semi-definite admissible pair on* H. *Let* $0<\tau<+\infty$, *and let* $\omega:[0,\tau] \to H$ *be a continuous vector function such that* $T\omega$ *is differentiable on* $(0,\tau)$. *Then an essentially bounded (strongly measurable) vector function* $\psi:(0,\tau) \to H$ *is a solution of the operator differential equation*

$$(1.11a) \quad (T\psi)'(t) = -(I - B)\psi(t) + (T\omega)'(t) + \omega(t) \qquad (0<t<\tau)$$

with boundary conditions

$$(1.11b) \quad \lim_{t\downarrow 0} P_+\psi(t) = P_+\omega(0), \qquad \lim_{t\uparrow\tau} P_-\psi(t) = P_-\omega(\tau),$$

if and only if ψ *is a solution of the vector-valued convolution equation*

$$(1.12) \quad \psi(t) - {}_0\int^{\tau} H(t-s)B\psi(s)ds = \omega(t) \qquad (0<t<\tau).$$

Here, in terms of the resolution of the identity E *of* T, *the propagator function* $H(.)$ *is given by*

$$(1.13) \quad H(t) = \begin{cases} +T^{-1}e^{-tT^{-1}}P_+ = +\int_0^{+\infty} \mu^{-1}e^{-t/\mu}E(d\mu), & 0<t<+\infty; \\ -T^{-1}e^{-tT^{-1}}P_- = -\int_{-\infty}^{0} \mu^{-1}e^{-t/\mu}E(d\mu), & -\infty<t<0. \end{cases}$$

For $\tau=+\infty$ an analogous theorem holds (cf. [18], Th. V.3.1). In that case one requires that $\omega:[0,+\infty) \to H$ is a bounded continuous vector function such that $T\omega$ is differentiable. Then an essentially bounded (strongly measurable) vector function $\psi:(0,+\infty) \to H$ is a solution of the operator differential equation (1.11a) on the half-line $(0,+\infty)$ with boundary condition

$$(1.14) \quad \lim_{t\downarrow 0} P_+\psi(t) = P_+\omega(0),$$

if and only if ψ is a solution of the convolution equation (1.12) (with $\tau=+\infty$). For finite τ the system of equations (1.11) is called an (abstract) *finite-slab problem*; its analogue for $\tau=+\infty$ is referred to as an (abstract) *half-space problem.* Concrete versions of these two problems are studied and applied in neutron physics and astrophysics (see [6]; [13,5,24]).

The next two results express the existence and uniqueness of (bounded) solutions of the finite-slab and half-space problems.

THEOREM 1.4. (= Th. IV.2.1 of [18]). *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* $0<\tau<+\infty$. *Then for every* $\phi\in H$ *there is a unique solution of the operator differential equation*

$$(1.15a)\quad (T\psi)'(t) = -(I - B)\psi(t) \qquad (0<t<\tau)$$

with boundary conditions

$$(1.15b)\quad \lim_{t\downarrow 0} P_+\psi(t) = P_+\phi, \qquad \lim_{t\uparrow\tau} P_-\psi(t) = P_-\phi,$$

namely

$$(1.16)\quad \psi(t) = e^{-tT^{-1}A}P_pV_\tau^{-1}\phi + e^{(\tau-t)T^{-1}A}P_mV_\tau^{-1}\phi + (I-tT^{-1}A)P_0V_\tau^{-1}\phi,$$

where the invertible operator V_τ *is given by*

$$(1.17)\quad V_\tau = P_+[P_p+e^{+\tau T^{-1}A}P_m] + P_-[P_m+e^{-\tau T^{-1}A}P_p] + P_0 - \tau P_-T^{-1}AP_0.$$

Note that Eq.(1.15a) with boundary conditions (1.15b) is equivalent to the convolution equation (1.12) with right-hand side

$$(1.18)\quad \omega(t) = e^{-tT^{-1}}P_+\phi + e^{(\tau-t)T^{-1}}P_-\phi, \qquad 0\le t\le\tau.$$

A statement of the finite-slab problem by Hangelbroek [12] stimulated the author to investigate this problem. For a case when the pair (T,B) is positive definite and parallel to the research leading to [18] Hangelbroek proved the invertibility of V_τ.

THEOREM 1.5. (= Th. IV.3.1 of [18]). *Let* (T,B) *be a positive definite admissible pair on* H. *Then for every* $\phi_+\in H_+$ *there is a unique bounded solution of the operator differential equation*

$$(1.19a)\quad (T\psi)'(t) = -(I - B)\psi(t) \qquad (0<t<+\infty)$$

with boundary condition

$$(1.19b)\quad \lim_{t\downarrow 0} P_+\psi(t) = \phi_+,$$

namely

$$(1.20)\quad \psi(t) = e^{-tT^{-1}A}P\phi_+ \qquad (0<t<+\infty).$$

Here P *denotes the projection of* H *onto* H_p *along* H_-.

The existence of the projection P is clear from Theorem 1.1. If the pair (T,B) is only assumed to be semi-definite, then still the half-space problem (1.19) has at least one bounded solution for every $\phi\in H$. Uniqueness holds if and only if the finite-dimensional operator $T^{-1}A|_{H_0}$ has Jordan blocks of order 2 only. Assuming inversion symmetry this has been shown in [18] (see Th. IV.3.4 there). With the help of Theorem 1.2 this additional hypothesis can be dropped.

Finally, to deal with the Milne problem we need the following

THEOREM 1.6. (= Th. IV.3.5 of [18]). *Let* (T,B) *be a semi-definite admissible pair on* H. *Then under the boundary conditions*

$$\lim_{t\downarrow 0} P_+\psi(t) = 0; \qquad \exists n\geq 0: \ \|\psi(t)\| = O(t^n) \quad (t\to+\infty)$$

the complete solution of the operator differential equation (1.19a) *is given by*

$$(1.21) \qquad \psi(t) = e^{-tT^{-1}A}x_p + (I - tT^{-1}A)x_0 \qquad (0<t<+\infty),$$

where $x_0\in[H_p \oplus H_-] \cap H_0$ *and* $x_0+x_p\in H_-$. *The vector* x_0 *is uniquely determined by* x_p, *and conversely.*

A concrete version of this (abstract) Milne problem appears in astrophysics (see [5,13,3]).

In the next section we shall need certain Banach spaces of strongly measurable functions on $(0,\tau)$. Given $0<\tau\leq+\infty$, $1\leq p\leq+\infty$ and a Banach space H, by $L_p((0,\tau);H)$ we denote the Banach space of all strongly measurable functions $\psi:(0,\tau)\to H$ that are bounded with respect to the norm

$$\|\psi\| = \begin{cases} \left[\int_0^\tau \|\psi(t)\|dt\right]^{1/p}, & 1\leq p<+\infty; \\ \operatorname*{ess\,sup}_{0<t<\tau} \|\psi(t)\|, & p=+\infty. \end{cases}$$

By strong measurability we mean measurability with respect to Lebesgue measure as treated in Section VI.31 of [26]. Finally, by $BC((0,\tau);H)$ we mean the closed subspace of $L_\infty((0,\tau);H)$ consisting of all bounded continuous functions $\psi:(0,\tau)\to H$.

2. STABILITY PROPERTIES: SEMI-DEFINITE ADMISSIBLE PAIRS

In this section a detailed study is made of the stability of the solution of the convolution equation

$$(2.1) \qquad \psi(t) - \int_0^\tau H(t-s)B\psi(s)ds = \omega(t) \qquad (0<t<\tau),$$

where $H(t)$ is the propagator function of the semi-definite admissible pair (T,B) on the Hilbert space H. Using the equivalence of Eq.(2.1) to a finite-slab problem (for finite τ) or to a half-space problem (for $\tau=+\infty$) (see Theorem 1.3 and the paragraph following its statement), one gets the stability of the solution of these two problems.

THEOREM 2.1. *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* $\{(T,B_N)\}_{N=0}^{+\infty}$ *be a uniform sequence of semi-definite admissible pairs on* H *such that* $\lim_{N\to+\infty}\|B-B_N\| = 0$. *Fix* $0<\tau<+\infty$ *and* $1\le p\le+\infty$. *Then for every right-hand side* $\omega\in L_p((0,\tau);H)$ *the unique solution* ψ *of* Eq.(2.1) *in the space* $L_p((0,\tau);H)$ *is the limit in the norm of* $L_p((0,\tau);H)$ *of the unique solution* $\psi_N\in L_p((0,\tau);H)$ *of the convolution equation*

$$\psi_N(t) - \int_0^\tau H(t-s)B_N\psi_N(s)ds = \omega(t) \qquad (0<t<+\infty) \tag{2.2}$$

as $N\to+\infty$. *The convergence is uniform in* ω *on bounded subsets of* $L_p((0,\tau);H)$.

Proof. For $N = 0,1,2,\ldots$ consider the operators K and K_N given by

$$(K\phi)(t) = \int_0^\tau H(t-s)B\phi(s)ds,\ (K_N\phi)(t) = \int_0^\tau H(t-s)\, B_N\phi(s)ds. \tag{2.3}$$

Then the operators K and K_N are bounded on $L_p((0,\tau);H)$ and

$$\|K-K_N\| \le \int_{-\tau}^{+\tau}\|H(t)(B-B_N)\|\,dt \le \|D^\alpha-D_N^\alpha\|\int_{-\tau}^{+\tau}\|\,|T|^\alpha H(t)\|dt. \tag{2.4}$$

Here $0<\alpha<1$ is some constant with the property that for certain operators D^α and D_N^α in $L(H)$ the identities $B = |T|^\alpha D^\alpha$ and $B_N = |T|^\alpha D_N^\alpha$ hold true $(N = 0,1,2,\ldots)$. Such α exists because of the uniformity of the sequence $\{(T,B_N)\}_{N=0}^{+\infty}$, and α can be chosen in such a way that the sequence $\{\|D^\alpha-D_N^\alpha\|\}_{N=0}^{+\infty}$ is bounded.

Note that for $0<\beta\le\alpha$ one has $B = |T|^\beta D^\beta$ and $B_N = |T|^\beta D_N^\beta$, where $D^\beta = |T|^{\alpha-\beta}D^\alpha$ and $D_N^\beta = |T|^{\alpha-\beta}D_N^\alpha$ $(N = 0,1,2,\ldots)$.

Let E denote the resolution of the identity of the self-adjoint operator T, and let $0<\beta<\alpha$ and $x\in H$. A straightforward application of Hölders inequality yields

$$\begin{aligned}\|\,|T|^{\alpha-\beta}x\|^2 &= \int_{-\infty}^{+\infty}|t|^{2(\alpha-\beta)}\|E(dt)x\|^2 \le \\ &\le \left[\int_{-\infty}^{+\infty}|t|^{2\alpha}\ \|E(dt)x\|^2\right]^{1-\beta/\alpha}\left[\int_{-\infty}^{+\infty}\|E(dt)x\|^2\right]^{\beta/\alpha} = \\ &= \|\,|T|^\alpha x\|^{2(1-\beta/\alpha)}\|x\|^{2\beta/\alpha}.\end{aligned}$$

With the help of the estimate (2.4) (with α replaced by β) one gets

(2.5) $\|K-K_N\| \le \|B-B_N\|^{1-\beta/\alpha}\|D^\alpha-D_N^\alpha\|^{\beta/\alpha} \int_{-\tau}^{+\tau} \| |T|^\beta H(t)\| dt.$

Since $\{\|D^\alpha-D_N^\alpha\|\}_{N=0}^{+\infty}$ is a bounded sequence, one obtains

$$\lim_{N\to+\infty} \|K-K_N\|_{L_p((0,\tau);H)} = 0.$$

By Theorem V.4.1 (valid for positive definite pairs only) and the remark at the end of Section V.4 of [18] the operator I-K is invertible on $L_p((0,\tau);H)$. Thus

$$\|\psi-\psi_N\| = \|[I-K)^{-1} - (I-K_N)^{-1}]\omega\| \to 0 \qquad (N\to+\infty),$$

uniformly in ω on bounded subsets of $L_p((o,\tau);H)$.□

THEOREM 2.2. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $\{(T,B_N)\}_{N=0}^{+\infty}$ *be a uniform sequence of positive definite admissible pairs on* H *such that* $\lim_{N\to+\infty}\|B-B_N\| = 0$. *Fix* $1\le p\le+\infty$. *Then for every right-hand side* $\omega \in L_p((0,+\infty);H)$ *the unique solution* ψ *of* Eq.(2.1) *(with* $\tau=+\infty$*) in the space* $L_p((0,+\infty);H)$ *is the limit in the norm of* $L_p((0,+\infty);H)$ *of the unique solution* $\psi_N \in L_p((0,+\infty);H)$ *of the convolution equation*

$$\psi_N(t) - \int_0^{+\infty} H(t-s)B_N\psi_N(s)ds = \omega(t) \qquad (0<t<+\infty)$$

as $N\to+\infty$. *The convergence is uniform in* ω *on bounded subsets of* $L_p((0,+\infty);H)$.

The proof of this theorem is the same as the proof of Theorem 2.1 and is therefore omitted. The restriction to positive definite pairs is needed for the application of Theorem V.5.1 of [18], according to which the operator I-K (defined analogously to (2.3)) is invertible on $L_p((0,+\infty);H)$ for $1\le p\le+\infty$.

COROLLARY 2.3. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $\{(T,B_N)\}_{N=0}^{+\infty}$ *be a uniform sequence of positive definite pairs on* H *such that* $\lim_{N\to+\infty}\|B-B_N\| = 0$. *Then the following identities hold true:*

(2.6a) $\lim_{N\to+\infty} \mathrm{gap}(H_p,H_p^N) = 0, \qquad \lim_{N\to+\infty} \mathrm{gap}(H_m,H_m^N) = 0;$

(2.6b) $\lim_{N\to+\infty} \|P-P^N\| = 0, \qquad \lim_{N\to+\infty} \|Q-Q^N\| = 0;$

(2.6c) $\lim_{N\to+\infty} \|V-V^N\| = 0, \qquad \lim_{N\to+\infty} \|V_\tau-V_\tau^N\| = 0.$

In these expressions the superschript ^{N} *refers to subspaces and* *operators connected to the pair* (T,B_N) $(N = 0,1,2,\ldots)$.

Proof. Apply Theorem 2.2 for $p=+\infty$ and $\omega(t) = e^{-tT^{-1}}\phi_+$ $(0<t<+\infty)$, where $\phi_+ \in H_+$. In the norm of $L_\infty((0,+\infty);H)$ one has $\|\omega\| = \|\phi_+\|$. According to the equivalence of Eq.(2.1) to an operator differential equation (see the paragraph following the statement of Th. 1.3; see also Th. V.3.1 of [18]) and Theorem 1.5 one has

$$\psi(t) = e^{-tT^{-1}A}P\phi_+, \qquad \psi_N(t) = e^{-tT^{-1}A_N}P^N\phi_+ \qquad (0<t<+\infty),$$

where $A_N = I-B_N$. Therefore, for $0<t<+\infty$ one has

$$(2.7) \qquad \left\|\left[e^{-tT^{-1}A}P - e^{-tT^{-1}A_N}P^N\right]\phi_+\right\| \le \|\psi-\psi_N\|_{L_\infty((0,+\infty);H)} \to 0,$$

uniformly in ϕ_+ on bounded subsets of H_+. Inserting $t=0$ one gets the first one of the identities (2.6b). The second one follows by considering the pair $(-T,B)$ and the uniform sequence $\{(-T,B_N)\}_{N=0}^{+\infty}$ rather than (T,B) and the uniform sequence $\{(T,B_N)\}_{N=0}^{+\infty}$.

Since $\|P-P^N\| \to 0$ as $N\to+\infty$, $H_p = \operatorname{Im} P$ and $H_p^N = \operatorname{Im} P^N$, one has $\operatorname{gap}(H_p,H_p^N) \to 0$ as $N\to+\infty$ (cf. [14,9], where the gap between subspaces was introduced). The second one of the inequalities (2.6a) is proved likewise. From (2.6a) one derives, with the aid of [14,9],

$$(2.8) \qquad \lim_{N\to+\infty}\|P_p-P_p^N\| = 0, \qquad \lim_{N\to+\infty}\|P_m-P_m^N\| = 0.$$

From (2.8) and (2.7) (with ϕ_+ replaced by $P_+P_p\phi$) one obtains

$$(2.9a) \qquad \lim_{N\to+\infty}\left\|\left[e^{-tT^{-1}A}P_p - e^{-tT^{-1}A_N}P_p^N\right]\phi\right\| = 0, \qquad 0\le t<+\infty,$$

uniformly in ϕ on bounded subsets of H. This is clear from the identities $P_p\phi = P(P_+P_p)\phi$ and $P_p^N\phi = P^N(P_+P_p^N)\phi$, $\phi\in H$. Considering the pair $(-T,B)$ and the uniform sequence $\{(-T,B_N)\}_{N=0}^{+\infty}$ one gets

$$(2.9b) \qquad \lim_{N\to+\infty}\left\|\left[e^{+tT^{-1}A}P_m - e^{+tT^{-1}A_N}P_m^N\right]\phi\right\| = 0, \qquad 0\le t<+\infty,$$

uniformly in ϕ on bounded subsets of H. Since

$$V_\tau = P_+\left[P_p + e^{+\tau T^{-1}A}P_m\right] + P_-\left[P_m + e^{-\tau T^{-1}A}P_p\right]$$

and V_τ^N admits a similar representation, the second part of (2.6c) is clear. □

Next a stability theorem is derived for solutions of Eq.(2.1) on the half-line in case (T,B) is only assumed to be a semi-definite pair. This theorem will play an essential role in the solution of the half-space problem in the conservative case (cf. Section 4).

THEOREM 2.4. *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* $A = I-B$, $\dim \operatorname{Ker} A = 1$ *and* $\dim H_0 = 2$, *where* H_0 *denotes the singular subspace of* (T,B). *Suppose that* $\{(T,B_N)\}_{N=0}^{+\infty}$ *is a uniform sequence of positive definite admissible pairs on* H *such that* $\lim_{N\to+\infty} \|B-B_N\| = 0$. *Then for* N *large enough the operator* $T^{-1}(I-B_N)$ *has exactly two eigenvalues* λ_N^- *and* λ_N^+ *tending to zero, where* $\lambda_N^- < 0 < \lambda_N^+$. *Further,*

$$(2.10)\quad \lim_{N\to+\infty} \operatorname{gap}(H_p^N\ ,\ H_p \oplus \operatorname{Ker} A) = \lim_{N\to+\infty} \operatorname{gap}(H_m^N\ ,\ H_m \oplus \operatorname{Ker} A) = 0.$$

Here $H_p^N(H_m^N)$ *denotes the spectral subspace of* $T^{-1}(I-B_N)$ *corresponding to the positive (negative) part of its spectrum.*

Proof. According to Theorem 1.2 one has

$$H_p \oplus \operatorname{Ker} A \oplus H_- = H_m \oplus \operatorname{Ker} A \oplus H_+ = H,$$

indeed. Consider the operator polynomials

$$L(\lambda) = A - \lambda T, \quad L_N(\lambda) = (I - B_N) - \lambda T.$$

Then $\lim_{N\to+\infty} \|L(\lambda) - L_N(\lambda)\| = 0$, uniformly in $\lambda\in\mathbb{C}$, and $\lambda=0$ is an isolated point of the spectrum of $L(\lambda)$. Let Γ be a positively oriented circle with centre 0 such that the point at $\lambda=0$ is separated by Γ from the remaining part of the spectrum of $L(\lambda)$. By a result of Gohberg and Sigal (cf. [10]) there exists $M\geq 0$ such that for $N\geq M$ the sum of the algebraic multiplicities of the eigenvalues of the pencil $L_N(\lambda)$ within Γ equals the algebraic multiplicity of the eigenvalue of $L(\lambda)$ at $\lambda=0$ (which coincides with $\dim H_0 = 2$). More precisely, one chooses M in such a way that for $N\geq M$

$$(2.11)\quad \max_{\lambda\in\Gamma} \|L(\lambda)^{-1}[L(\lambda) - L_N(\lambda)]\| < 1.$$

For $N\geq M$ put

$$R = -(2\pi i)^{-1}\int_\Gamma L(\lambda)^{-1}d\lambda, \quad R_N = -(2\pi i)^{-1}\int_\Gamma L_N(\lambda)^{-1}d\lambda;$$

then R and R_N are self-adjoint and $\lim_{N\to+\infty}\|R-R_N\| = 0$. It is straightforward to prove that for $N\geq M$ the operators TR, TR_N, RT and R_NT are projections. By definition, $H_0 = \operatorname{Im} RT$, and thus $\dim RT = 2$. But then (2.11) yields that for $N\geq M$ all these four projections have rank two. Putting $H^{0,N} := \operatorname{Im} R_NT$ and $H^{1,N} := \operatorname{Ker} R_NT$ one gets

$$\operatorname{Im} TR_N = (H^{1,N})^\perp, \quad \operatorname{Ker} TR_N = (H^{0,N})^\perp; \quad N\geq M.$$

Put

$$W^N = P_0(R_NT) + (I - P_0)(I - R_NT), \quad N\geq M,$$

where $P_0 = RT$. Then W^N is invertible, maps $H^{0,N}$ onto H_0 and $H^{1,N}$ onto H_1 and satisfies $\lim_{N\to+\infty} \|W^N-I\| = 0$. For $N\geq M$ put

(2.12a) $S^{0,N} = W^N T^{-1}(I-B_N)\Big|_{H^{0,N}}(W^N)^{-1}, \quad S^0 = T^{-1}A|_{H_0},$

defined as operators on H_0. Then in the norm of H_0 one has

(2.12b) $\lim_{N\to+\infty} \|S^{0,N}-S^0\| = 0;$

further, $\sigma(S_0) = \{0\}$ and S_0 has one Jordan block at $\lambda=0$ of order 2. However, $A_N^{-1}T$ is similar to a self-adjoint operator. (Note that A_N is strictly positive and $A_N^{-1}T = A_N^{-\frac{1}{2}}(A_N^{-\frac{1}{2}}TA_N^{-\frac{1}{2}})A_N^{+\frac{1}{2}}$). So $S^{0,N}$ can be diagonalized for N large enough and $S^{0,N}$ has two different eigenvalues, to be denoted by λ_N^+ and λ_N^- (with $\lambda_N^+ > \lambda_N^-$). Thus $S^{0,N}$ has exactly two non-trivial (one-dimensional) invariant subspaces, namely Ker $(S^{0,N} - \lambda_N^-)$ and Ker $(S^{0,N} - \lambda_N^+)$. As an application of Theorem 8.2 of [1] one finds

(2.13) $\lim_{N\to+\infty} \text{gap}(\text{Ker } (S^{0,N} - \lambda_N^{\pm}) , \text{Ker } A) = 0.$

Let us prove that $\lambda_N^- < 0 < \lambda_N^+$ for N large enough. Since $\dim H_0 = 2$ and $\dim \text{Ker } A = 1$, there exist vectors $0\neq p_0$ and p_1 in H_0 such that $T^{-1}Ap_1 = p_0$, $T^{-1}Ap_0 = 0$. Then $\langle Tp_0,p_0\rangle = \langle Ap_1,p_0\rangle = \langle p_1,Ap_0\rangle = 0$, $\alpha := \langle Tp_0,p_1\rangle = \langle Ap_1,p_1\rangle > 0$ (because $A \geq 0$ and $p_1 \notin \text{Ker } A$) and $\gamma := \langle Tp_1,p_1\rangle \in \mathbb{R}$. A straightforward calculation yields that $\langle T(p_1+\xi p_0),(p_1+\xi p_0)\rangle$ is strictly positive (resp. strictly negative) for $\xi>-\gamma/2\alpha$ (resp. $\xi<-\gamma/2\alpha$). Therefore, the two-dimensional vector space H_0 endowed with the inner product (1.9) is a Krein space with a maximal positive and a maximal negative subspace of dimension one (see [2] for the terminology). For N large enough the operator $S^{0,N}$ in (2.12a) is strictly positive in the inner product

(2.14) $\{x,y\}_N = \langle T(W^N)^{-1}x,(W^N)^{-1}y\rangle$

on H_0, while $\|S^{0,N} - S^0\| \to 0$ and $\|W^N - I\| \to 0$ as $N \to +\infty$. Therefore, for N large enough the space H_0 with inner product (2.14) is a Krein space with a maximal positive and a maximal negative subspace of dimension one. Using Theorem VII.1.2 of [2] one easily sees that $\lambda_N^- < 0 < \lambda_N^+$ for N large enough.

Next, put

$$\tilde{B}_N = B_N(I - R_N T) + (I + T)R_N T; \ \tilde{B} = B(I - P_0) + (I + T)P_0.$$

As in the proof of Theorem III.6.3 of [18] one shows that $(T,\tilde{B}_N)$ and $(T,\tilde{B})$ are positive definite admissible pairs on H and that the spectral subspaces of their associate operators corresponding to the positive part of their spectrum coincide with $H_p^N \cap H^{1,N}$ and H_p, respectively. Further, $\{(T,\tilde{B}_N)\}_{N=0}^{+\infty}$ is a uniform sequence and $\lim_{N\to+\infty} \|\tilde{B}-\tilde{B}_N\| = 0$. From (2.6a) it is clear that

$$\lim_{N\to+\infty} \operatorname{gap}(H_p^N \cap H^{1,N}, H_p) = 0.$$

With the help of (2.13) (for $\lambda_N^+>0$) one obtains the first identity in (2.10). The second part of (2.10) is proved likewise. □

In [7] Feldman investigated the stability of discrete eigenvalues and eigenfunctions of the operator polynomial $L(\lambda) = A-\lambda T$. The similarity of his approach to the present one consists of showing the stability of the operator K in (2.3) (for $\tau=+\infty$) in the norm of $L_p((0,+\infty);H)$ under perturbations of B in the operator norm.

3. STABILITY PROPERTIES: SPECIFICATION FOR THE TRANSPORT EQUATION

In this section the stability theorems 2.1 and 2.2 are applied to the linear Transport Equation

$$\mu \frac{d\psi}{dx}(x,\mu) + \psi(x,\mu) = \tag{3.1}$$
$$\int_{-1}^{+1} \left[\frac{1}{2\pi} \int_0^{2\pi} \hat{g}(\mu\mu'+\sqrt{1-\mu^2}\ \sqrt{1-\mu'^2}\ \cos\alpha)d\alpha\right] \psi(x,\mu')d\mu'$$
$$+ f(x,\mu); \quad 0<x<\tau,\ -1\le\mu\le+1.$$

On this integro-differential equation one imposes the boundary conditions (0.2a) (for finite τ) or (0.2b) (for infinite τ). It is assumed that the phase function $\hat{g}$ is real-valued, belongs to $L_r[-1,+1]$ for some r>1 and satisfies $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1}\hat{g}(t)P_n(t) \le 1$ $(n = 0,1,2,\ldots)$. Here $P_n(\mu) = (2^n\cdot n!)^{-1}(\frac{d}{d\mu})^n(\mu^2-1)^n$ is the usual Legendre polynomial of degree n. As to the inhomogeneous term $f(x,\mu)$, which in astrophysics (resp. neutron physics) describes internal radiative (resp. neutron) sources, one supposes that f acts as a bounded continuous vector function from $(0,\tau)$ into $L_p[-1,+1]$ for some $p>2$.

If one defines the operators T and B on $L_2[-1,+1]$ by

$$(3.2)\qquad \begin{aligned}(Th)(\mu) &= \mu h(\mu),\\ (Bh)(\mu) &= \int_{-1}^{+1}\left[(2\pi)^{-1}\int_0^{2\pi}\hat{g}(\mu\mu'+\sqrt{1-\mu^2}\ \sqrt{1-\mu'^2}\ \cos\alpha)d\alpha\right]h(\mu')d\mu',\end{aligned}$$

then (T,B) is a semi-definite admissible pair on $L_2[-1,+1]$ (cf. [18], Theorem VI.1.1). In fact, B is an integral operator on $L_2[-1,+1]$ with the property that

$$BP_n = a_n P_n,\quad a_n = (n+\tfrac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}(t)P_n(t)dt;\quad n = 0,1,2,\ldots$$

(see [25], Appendix XII.8). So the pair (T,B) is positive definite if and only if all coefficients a_n are strictly less than +1. Putting

$$\psi(x)(\mu) = \psi(x,\mu),\quad f(x)(\mu) = f(x,\mu)\quad (0<x<\tau,\ -1\le\mu\le+1),$$

Eq.(3.1) is easily rewritten as an operator differential equation on $L_2[-1,+1]$ of the form

$$(3.3)\qquad (T\psi)'(x) = -(I - B)\psi(x) + f(x)\qquad (0<x<\tau).$$

As a consequence of the condition on f, for finite τ (resp. for $\tau=+\infty$) one can find a unique continuous function $\omega:[0,\tau]\to L_2[-1,+1]$ (resp. a unique bounded continuous function $\omega:[0,+\infty)\to L_2[-1,+1]$) such that $T\omega$ is differentiable on $(0,\tau)$ and satisfies the system of equations

$$(3.4a)\qquad (T\omega)'(t) + \omega(t) = f(t)\qquad (0<t<\tau);$$

$$(3.4b)\qquad \begin{cases}P_+\omega(0) = P_+\phi,\ P_-\omega(\tau) = P_-\phi; & 0<\tau<+\infty;\\ P_+\omega(0) = \phi_+; & \tau=+\infty.\end{cases}$$

This unique vector function ω is given by

$$(3.5)\qquad \omega(t) = \begin{cases}e^{-tT^{-1}}P_+\phi + e^{(\tau-t)T^{-1}}P_-\phi + \int_0^\tau H(t-s)f(s)ds, & 0<t<\tau;\\ e^{-tT^{-1}}\phi_+ + \int_0^{+\infty}H(t-s)f(s)ds, & \tau=+\infty.\end{cases}$$

For the proof we refer to Sections V.2 and V.3 and to Theorem VI.1.2 of [18]. (The uniqueness of ω has not been shown there, but this is easily derived from Theorems 1.4 and 1.5 applied to the pair $(T,0)$ on $L_2[-1,+1]$). In (3.5) the vectors ϕ and ϕ_+ are arbitrary vectors of $L_2[-1,+1]$ and $L_2[0,1]$, respectively.

The expression (3.5) makes sense (i.e., contains the absolutely convergent Bochner integral ${}_0\int^\tau H(t-s)f(s)ds$), if f is only assumed to be an element of $L_\infty((0,\tau);L_p[-1,+1])$ for some $p>2$. Further, with the aid of the parts of [18] mentioned in the preceding paragraph one easily shows that the operator mapping (ϕ,f) into ω acts as bounded linear operator from $L_2[-1,+1] \oplus L_\infty((0,\tau);L_p[-1,+1])$ into $L_\infty((0,\tau);L_2[-1,+1])$ ($p>2$ fixed, $0<\tau<+\infty$; an analogous statement holds true for $\tau=+\infty$).

The next two theorems specify Theorems 2.1 and 2.2 for the pair (T,B) in (3.2) and amount to the stability of the finite-slab and half-space problems.

THEOREM 3.1. *For* $r>1$ *let* $(g_N)_{N=0}^{+\infty}$ *be a sequence of real-valued functions in* $L_r[-1,+1]$ *whose expansion coefficients* $a_n^N = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}_N(t)P_n(t)dt$ *do not exceed* $+1$ ($N,n = 0,1,2,\ldots$). *Let* $\hat{g}$ *be a function in* $L_r[-1,+1]$ *such that* $\|\hat{g} - \hat{g}_N\|_r \to 0$ as $N\to+\infty$. *Fix* $0<\tau<+\infty$ *and* $\phi \in L_2[-1,+1]$, *and let* $f:(0,\tau) \to L_p[-1,+1]$ *be a bounded continuous vector function for some* $p>2$. *Then for* $N\to+\infty$ *the unique solution* $\psi^N:(0,\tau) \to L_2[-1,+1]$ *of the integro-differential equation*

$$\mu\frac{d\psi^N}{dx}(x,\mu) + \psi^N(x,\mu) = \int_{-1}^{+1}\left[\frac{1}{2\pi}\int_0^{2\pi}\hat{g}_N(\mu\mu'+\sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)d\alpha\right]\psi(x,\mu')d\mu' + f(x,\mu); \quad 0<x<\tau,\ -1\le\mu\le+1,$$

with boundary conditions

$$\psi^N(0,\mu) = \phi(\mu) \quad (0\le\mu\le+1), \qquad \psi^N(\tau,\mu) = \phi(\mu) \quad (-1\le\mu<0),$$

converges to the unique solution $\psi:(0,\tau) \to L_2[-1,+1]$ *of* Eq.(3.1) *with boundary conditions* (0.2a). *In fact,*

$$\lim_{N\to+\infty}\ \sup_{0<x<\tau}\int_{-1}^{+1}|\psi(x,\mu) - \psi^N(x,\mu)|^2 d\mu = 0$$

and the convergence is uniform in ϕ *on bounded subsets of* $L_2[-1,+1]$ *and in* f *on bounded subsets of* $BC((0,\tau);L_p[-1,+1])$.

Proof. Introduce the pair (T,B) by (3.2) and the pair (T,B_N) by an analogous set of formulas in terms of $\hat{g}_N$. Then for $N = 0,1,2,\ldots$ the pair (T,B_N) is a semi-definite admissible pair on $L_2[-1,+1]$. It is well-known that there is a constant c (not depending on N) such that for N = 0,1,2,...

(3.6) $\quad \|B\| \le c\|\hat{g}\|_r,\ \|B-B_N\| \le c\|\hat{g}-\hat{g}_N\|_r,\ \|B_N\| \le c\|\hat{g}_N\|_r.$

Using the proof of Theorem VI.1.1 in [18] one sees that for a fixed $0<\alpha<(2r)^{-1}(r-1)$ there exist bounded operators D and D^N such that $B = |T|^{\alpha}D$, $B^N = |T|^{\alpha}D^N$ and $\|D^N\| \le d\|\hat{g}_N\|_r$ (N = 0,1,2,...). Here d is a constant not depending on N. Hence, $\{(T,B_N)\}_{N=0}^{+\infty}$ is a uniform sequence of semi-definite admissible pairs on $L_2[-1,+1]$ such that $\|B-B_N\| \to 0$ as $N\to+\infty$. Thus the stability theorem 2.1 may be applied.

With the help of Theorem 1.3 it is clear that the present theorem merely is an application of Theorem 2.1 with ω taken as in (3.5). The statement on uniform convergence is clear from the corresponding statement in Theorem 2.1, because the operator mapping (ϕ,f) into the right-hand side of (3.5) is a bounded linear operator from $L_2[-1,+1] \oplus L_\infty((0,\tau);L_p[-1,+1])$ into $L_\infty((0,\tau);L_2[-1,+1])$. □

THEOREM 3.2. *For* $r>1$ *let* $(\hat{g}_N)_{N=0}^{+\infty}$ *be a sequence of real-valued functions in* $L_r[-1,+1]$ *whose expansion coefficients* $a_n^N = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}_N(t)P_n(t)dt$ *are strictly less than* +1 (N,n = 0,1,2,...). *Let* $\hat{g}$ *be a function in* $L_r[-1,+1]$ *such that* $\|\hat{g}-\hat{g}_N\|_r \to 0$ *as* $N\to+\infty$. *Fix* $\phi_+ \in L_2[0,1]$, *and let* $f:(0,+\infty) \to L_p[-1,+1]$ *be a bounded continuous vector function for some* $p>2$. *Then for* $N\to+\infty$ *the unique solution* $\psi^N:(0,+\infty) \to L_2[-1,+1]$ *of the integro-differential equation*

$$\mu \frac{d\psi^N}{dx}(x,\mu) + \psi^N(x,\mu) = \int_{-1}^{+1}\left[\frac{1}{2\pi}\int_0^{2\pi}\hat{g}_N(\mu\mu'+\sqrt{1-\mu^2}\ \sqrt{1-\mu'^2}\ \cos\alpha)d\alpha\right]\psi(x,\mu')d\mu' + f(x,\mu);\quad 0<x<+\infty,\ -1\le\mu\le+1,$$

with boundary conditions

$$\psi^N(0,\mu) = \phi_+(\mu)\quad (0\le\mu\le+1),\quad \int_{-1}^{+1}|\psi(x,\mu)|^2d\mu = O(1)\ (x\to+\infty),$$

converges to the unique solution $\psi:(0,+\infty) \to L_2[-1,+1]$ *of* Eq.(3.1) *with boundary conditions* (0.2b). *In fact,*

$$\lim_{N\to+\infty}\ \sup_{0<x<+\infty} \int_{-1}^{+1} |\psi(x,\mu) - \psi^N(x,\mu)|^2 d\mu = 0$$

and the convergence is uniform in ϕ *on bounded subsets of* $L_2[-1,+1]$ *and in* f *on bounded subsets of* $BC((0,+\infty);L_p[-1,+1])$.

This theorem follows from Theorem 2.2 in the same way as Theorem 3.1 follows from Theorem 2.1. Therefore, the proof is omitted.

The situation that $\|\hat{g}-\hat{g}_N\|_r \to 0$ as $N\to+\infty$, occurs in several instances. Feldman studied a general nonnegative phase function $\hat{g}$ in $L_r[-1,+1]$ such that ${}_{-1}\int^{+1}\hat{g}(t)dt \le 1$ (see [7]), and approximated it by the nonnegative degenerate phase functions

$$\hat{g}_N(\mu) = \sum_{n=0}^{N}(n+\tfrac{1}{2})(1-\tfrac{n}{N+1})(1-\tfrac{n}{N+2})a_nP_n(\mu) \qquad (N = 0,1,2,\ldots)$$

in the L_r-norm. The second instance is the case when $\hat{g}\in L_2[-1,+1]$. In this case the condition that $\|\hat{g}-\hat{g}_N\|_r \to 0$ as $N\to+\infty$ for some $r>1$ is satisfied, if

$$\lim_{N\to+\infty}\sum_{n=0}^{+\infty}|a_n-a_n^N|^2 = 0.$$

To see this, note that $\|\hat{g}-\hat{g}_N\|_2^2 = \sum_{n=0}^{N}\|a_n-a_n^N\|^2 = \|B-B_N\|_2^2$, where $\|B-B_N\|_2$ denotes the Hilbert-Schmidt norm of $B-B_N$. The third example is provided by the cut-off of the Legendre series expansion of the phase function, an approximation often implicitly used by physicists (see [13], Section 6.4.2). For this cut-off the approximants $\hat{g}_N$ of $\hat{g}$ are given by

$$\hat{g}_N(\mu) = \sum_{n=0}^{N}a_n(n+\tfrac{1}{2})P_n(\mu); \qquad N = 0,1,2,\ldots,\ -1\le\mu\le+1.$$

By a well-known result of Pollard [22] one has $\|\hat{g}-\hat{g}_N\|_r \to 0$ as $N\to+\infty$, provided $r > \frac{4}{3}$. For $1\le r\le\frac{4}{3}$ this need not be true (cf. [20]).

To generalize the approximation by truncating the Legendre series expansion to phase functions in $L_r[-1,+1]$ $(1<r\le\frac{4}{3})$ we introduce the orthogonal projections P^N of $L_2[-1,+1]$ onto the (N+1)-dimensional subspace spanned by the Legendre polynomials $P_0,\ldots P_N$. Defining B by (3.2) and B_N analogously in terms of $\hat{g}_N$, one has $B_N = BP_N$. However, for some $0<\alpha<(2r)^{-1}(r-1)$ one has $B = |T|^\alpha D$, where D is a bounded operator ([18], Theorem VI.1.1). So $B_N = |T|^\alpha DP^N$ $(N = 0,1,2,\ldots)$. Now it is clear that $\{(T,B_N)\}_{N=0}^{+\infty}$ is a uniform sequence of semi-definite admissible pairs on

$L_2[-1,+1]$ such that $\lim_{N\to+\infty} \|B-B_N\| = 0$. The conclusions of Theorem 3.1 (for the case $a_n \le +1$) and Theorem 3.2 (for the case $a_n < +1$) appear to be straightforward applications of Theorems 2.1 and 2.2, respectively. Hence, the approximation by cutting off the Legendre series expansion of the phase function has been mathematically justified.

We conclude this section by pointing out two other applications of the stability theorems for semi-definite admissible pairs. One of these applications is the stability of the solution of the symmetric multigroup Transport Equation in a non-multiplying medium. This equation may be described by a semi-definite admissible pair on $L_2((-1,+1);\mathbb{C}^N)$, where N is the number of groups considered (see [18], Section VI.7). For the physical aspects of the multigroup Transport Equation we refer to [6].

In astrophysics Eq.(3.1) is obtained from the more general integro-differential equation

$$(\cos\theta)\frac{d\psi}{dx}(x,\omega) + \psi(x,\omega) = (2\pi)^{-1}\int_{\Omega}\hat{g}(\omega\cdot\omega')\psi(x,\omega')d\omega' + f(x,\omega); \quad 0<x<\tau,\ \omega\in\Omega, \tag{3.7}$$

by averaging the solution ψ over azimuth (see [5,24,13]). In the above equation Ω denotes the unit sphere in $\mathbb{R}^3$ and ω is a point of Ω with spherical coordinates θ and ϕ. It is easy to see that Theorems 3.1 and 3.2 also hold true for Eq.(3.7) under analogous hypotheses on the phase function $\hat{g}$.

4. THE CONSERVATIVE CASE

In this section we compute the unique solution $\psi:(0,+\infty)\to L_2[-1,+1]$ of the integro-differential equation

$$\mu\frac{d\psi}{dx}(x,\mu) + \psi(x,\mu) = \int_{-1}^{+1}\left[\frac{1}{2\pi}\int_0^{2\pi}\hat{g}(\mu\mu'+\sqrt{1-\mu^2}\ \sqrt{1-\mu'^2}\ \cos\alpha)d\alpha\right]\psi(x,\mu')d\mu' \quad (0<x<+\infty;\ -1\le\mu\le+1) \tag{4.1a}$$

under the boundary conditions

$$\psi(0,\mu)=\phi_+(\mu)\quad(0\le\mu\le1),\qquad \int_{-1}^{+1}|\psi(x,\mu)|^2d\mu = O(1)\ \ (x\to+\infty). \tag{4.1b}$$

The phase function $\hat{g}$ is nonnegative, degenerate and conservative, i.e.,

(4.2) $\hat{g}(\mu) = \sum_{n=0}^{N} a_n(n+\frac{1}{2})P_n(\mu); \quad a_0 = 1, \ -1 \le a_n < +1 \ (n = 1,2,\ldots,N).$

Note that the situation that for some $1 \le M \le N$ the coefficients a_n satisfy $a_n = +1$ $(n = 0,1,\ldots,M)$ and $-1 \le a_n < +1$ $(n = M+1,\ldots,N)$, is excluded.

First we introduce some functions and terminology. No matter possible restrictions on the expansion coefficients of a phase function $\hat{g}$ one associates with $\hat{g}$ its *Kuščer polynomials* $(H_n)_{n=0}^{+\infty}$, which are given by the recurrence relation

(4.3a) $(2n+1)(1-a_n)\mu H_n(\mu) = (n+1)H_{n+1}(\mu) + nH_{n-1}(\mu);$

(4.3b) $H_{-1}(\mu) \equiv 0, \quad H_0(\mu) = 1, \quad H_1(\mu) = (1-a_0)\mu.$

In case $a_n \neq 1$ $(n = 0,1,2,\ldots)$, in the non-conservative case, for instance, the Kuščer polynomial has degree n $(n = 0,1,2,\ldots)$. In case $\hat{g}$ satisfies (4.2), one has

(4.4) $H_0(\mu) = 1, \quad H_1(\mu) \equiv 0, \quad H_2(\mu) \equiv -\frac{1}{2}, \quad \deg H_n = n-2$ $(n = 2,3,4,\ldots),$

where $\deg H_n$ denotes the degree of H_n. Moreover, comparing (4.3a) with the recurrence relation of the Legendre polynomials one sees that $H_n(0) = P_n(0)$ $(n = 0,1,2,\ldots)$.

By $\Psi(\nu,\mu)$ one denotes the *characteristic binomial*, which is given by

(4.5a) $\Psi(\nu,\mu) = \sum_{n=0}^{N} a_n(n+\frac{1}{2})H_n(\nu)P_n(\mu);$

by $\Psi(\mu)$ we mean the *characteristic function*, i.e.,

(4.5b) $\Psi(\mu) = \Psi(\mu,\mu) = \sum_{n=0}^{N} a_n(n+\frac{1}{2})H_n(\mu)P_n(\mu).$

In terms of $\Psi(\mu)$ (or $\Psi(\nu,\mu)$) one defines the *dispersion function* Λ by

(4.5c) $\Lambda(\lambda) = 1 + \lambda \int_{-1}^{+1} (\mu-\lambda)^{-1}\Psi(\mu)d\mu, \quad \lambda \in \mathbb{C}_\infty \setminus [-1,+1].$

All these functions have appeared in literature (see [13]). The term "dispersion function" and its notation by Λ are customary in neutron physics; in astrophysics Λ is usually denoted by T and the equation $T(z) = 0$ is called the *characteristic equation* and its roots *characteristic roots*. If (4.2) holds, Λ has a double zero at infinity and the other zeros of Λ outside $[-1,+1]$ are real-valued and simple. This follows by writing $\Lambda(\lambda)$ as the determinant

$$\Lambda(\lambda) = \det[I - \lambda(\lambda-T)^{-1}B],$$

where T and B are given by (3.2) (cf. [12], for instance).

In the non-conservative case (i.e., if $a_n<+1$ for $n=0,1,\ldots,N$ and $a_n=0$ for $n\geq N+1$) the dispersion function Λ is uniformly Hölder continuous on the extended imaginary line and there exists a unique, so-called *H-function* H that is continuous and non-zero on the closed right half-plane and analytic on the open right half-plane, satisfies $H(0) = 1$ and the factorization formula

$$\Lambda(\lambda) = H(\lambda)^{-1}H(-\lambda)^{-1}, \quad \operatorname{Re}\lambda = 0 \tag{4.6}$$

(cf. [18], Th.2.1 and Eq.(2.10)). Factorization results like (4.6) have appeared in literature several times; for the references we refer to Section VI.2 of [18].

PROPOSITION 4.1. *Let the phase function* $\hat{g}$ *satisfy* (4.2), *and for* $0<c<1$ *let* H_c *denote the H-function associated with the phase function* $c\hat{g}$. *Then there exists a unique function* H, *defined, continuous and non-zero on the closed right half-plane with the exception of the point at infinity, and analytic on the open right half-plane, such that the factorization formula* (4.6) *holds. The function* H *satisfies*

$$\lim_{c\uparrow 1} H_c(\lambda) = H(\lambda), \tag{4.7}$$

uniformly in λ *on bounded subsets of the closed right half-plane, and is a solution of the H-equation*

$$\frac{1}{H(\mu)} = 1 - \mu\int_0^1 \frac{\Psi(\nu)H(\nu)}{\nu+\mu}d\nu; \quad \operatorname{Re}\mu > 0. \tag{4.8}$$

Proof. Let Γ be the positively oriented circle with centre 0 and radius $R>1$. Then there exists $0<c_0<1$ such that for $c_0\leq c<1$ the dispersion function Λ_c associated with the phase function $c\hat{g}$ has precisely two zeros outside Γ, namely $\nu_c>R$ and $-\nu_c<-R$. This is clear, because $\lim_{c\uparrow 1}\Lambda_c(\lambda) = \Lambda(\lambda)$, uniformly in λ on compact subsets of $\mathbb{C}_\infty\setminus[-1,+1]$, and Λ has a double zero at infinity.

From Theorem 2.1 of [18] it follows that for $0<c<1$ and $0<\alpha<\frac{1}{2}$ the dispersion function Λ_c is Hölder continuous of exponent α on the extended imaginary line; the same is true for Λ itself. Further, Λ_c and Λ are strictly positive for imaginary λ. Hence, for

$c_0<c<1$ and $0<\alpha<\frac{1}{2}$ the functions $\nu_c{}^2(\nu_c{}^2-\lambda^2)^{-1}(1-\lambda^2)\Lambda_c(\lambda)$ and $(1-\lambda^2)\Lambda(\lambda)$ are Hölder continuous of exponent α and strictly positive on the extended imaginairy line, while

$$\lim_{c\uparrow 1}\ \sup_{\mathrm{Re}\lambda=0}\left|\frac{\nu_c{}^2(1-\lambda^2)}{\nu_c{}^2-\lambda^2}\Lambda_c(\lambda)-(1-\lambda^2)\Lambda(\lambda)\right|=0.$$

By Theorem III.4.1 of [23] the function $(1-\lambda^2)\Lambda(\lambda)$ has a canonical factorization with respect to the imaginary line, namely

$$(1-\lambda^2)\Lambda(\lambda)=\hat{H}(\lambda)^{-1}\hat{H}(-\lambda)^{-1},\quad \mathrm{Re}\ \lambda=0;$$

here $\hat{H}$ is continuous and non-zero on the closed right half-plane and analytic on the open right half-plane. Put $H(\lambda)=(1+\lambda)\hat{H}(\lambda)$, $\mathrm{Re}\ \lambda\geq 0$. Then H has all the properties described in the statement of the present proposition and is unique in this respect. Using the stability of a canonical factorization (cf. [8]; Section I.5) one sees that

$$\lim_{c\uparrow 1}\left|\frac{\nu_c(1+\lambda)}{\nu_c+\lambda}H_c(\lambda)^{-1}-(1+\lambda)H(\lambda)^{-1}\right|=0,$$

uniformly in λ on the closed right half-plane. Since $\lim_{c\uparrow 1}\nu_c=+\infty$, formula (4.7) is clear. The H-equation (4.8) is clear from (4.7) ant its analogue for the non-conservative case (cf. [18], Section VI.2). □

The H-equation (4.8) is frequently employed (see [5,13], for instance); its rigorous derivation is due to Busbridge [3], also for the conservative case. Here (4.8) is mentioned to link up the function H to the H-function studied by Chandrasekhar [5].

Next we compute the bounded solutions of the half-space problem for phase functions of the form (4.2). For such phase functions the pair (T,B) (defined by (3.2)) is semi-definite and singular. It has the property that $\mathrm{Ker}\ A=\mathrm{span}\ \{P_0\}$; its singular subspace H_0 is two-dimensional and is given by $H_0=\mathrm{span}\ \{P_0,P_1\}$ (cf. [16]). According to Theorem 1.2 one has

$$H_p\oplus\mathrm{span}\ \{P_0\}\oplus H_-=H_m\oplus\mathrm{span}\ \{P_0\}\oplus H_+=L_2[-1,+1].$$

For the conservative isotropic case these decompositions have been found by Lekkerkerker [15].

THEOREM 4.2. *Let the phase function* $\hat{g}$ *satisfy* (4.2). *Then the projection* P of $L_2[-1,+1]$ *onto* $H_p\oplus\mathrm{span}\ \{P_0\}$ *along* H_- *is given by*

$$(4.9)\qquad (Ph)(\mu)=\begin{cases}\sum_{n=0}^{N} a_n(n+\tfrac{1}{2})(-1)^n q_n(-\mu)H(-\mu)\int_0^1 \nu(\nu-\mu)^{-1}q_n(\nu)H(\nu)h(\nu)d\nu; & (-1\le\mu<0)\\ h(\mu), & 0\le\mu\le+1.\end{cases}$$

For $N = 0,1$ *one has* $q_0(\mu) = 1$ *and* $q_1(\mu) \equiv 0$. *For* $N \ge 2$ *the function* q_1 *vanishes and* $q_0,q_2,q_3,\ldots q_N$ *are certain polynomials of degree* $\le N-1$, *which are the unique polynomial solutions of the equations*

$$(4.10a)\quad q_n(\lambda) = H_n(\lambda) + \lambda\int_0^1 \frac{\Psi(\lambda,\mu)q_n(\mu)-\Psi(\mu,\mu)q_n(\lambda)}{\mu-\lambda} H(\mu)d\mu,\qquad \lambda \notin [0,1];$$

$$(4.10b)\quad \int_0^1 q_n(\mu)H(\mu)d\mu = 2\delta_{n0}\qquad (n = 0,2,3,\ldots,N).$$

Proof. For $0<c<1$ to the phase function $c\hat{g}$ there corresponds the positive definite admissible pair (T,cB), with T and B defined by (3.2). Obviously, $\{(T,cB)\}_{0<c<1}$ is a uniform collection of positive definite admissible pairs on $L_2[-1,+1]$ and $\lim_{c\uparrow 1}\|B-cB\| = 0$; further, one has $\dim \operatorname{Ker} A = 1$ and $\dim H_0 = 2$. So the conditions of Theorem 2.4 are fulfilled.

Let us apply Theorem 2.4. If H_p^c (resp. H_m^c) is the spectral subspace of $(I-cB)^{-1}T$ corresponding to the positive (resp. negative) part of its spectrum, one has

$$(4.11)\quad \lim_{c\uparrow 1} \operatorname{gap}(H_p^c,H_p \oplus \operatorname{span}\{P_0\}) = \lim_{c\uparrow 1} \operatorname{gap}(H_m^c,H_m \oplus \operatorname{span}\{P_0\}) = 0.$$

If P^c (Q^c) denotes the projection of $L_2[-1,+1]$ onto H_p^c (H_m^c) along H_- (H_+) (which exists; see Th.1.1), then (4.11) implies that

$$(4.12)\quad \lim_{c\uparrow 1}\|P^c - P\| = \lim_{c\uparrow 1}\|Q^c - Q\| = 0$$

(the implication (4.11) $\Rightarrow$ (4.12) follows from results on the gap between subspaces of a Banach space; see [14,9]).

If H_c denotes the H-fuction corresponding to the phase function $c\hat{g}$, then (4.7) holds uniformly in λ on $[0,1]$. A formula for P^c is provided by Theorem VI.3.1 of [18] (applied for $c\hat{g}$). In this formula (i.e., formula (VI 3.2) of [18]) there appear the H-function H_c and certain polynomials $q_0^c,\ldots,q_N^c$ of degree $\le N$ that satisfy the equations

$$(4.13)\quad q_n^c(\lambda) = H_n^c(\lambda) + \lambda \int_0^1 \frac{\Psi^c(\lambda,\mu)q_n^c(\mu)-\Psi^c(\mu,\mu)q_n^c(\lambda)}{\mu-\lambda} H_c(\mu)d\mu; \quad \lambda \notin [0,1].$$

Here H_n^c is the Kuščer polynomial of degree n and $\Psi^c(\nu,\mu)$ the characteristic binomial, both of them corresponding to $c\hat{g}$. By Proposition VI.3.3 of [18] one has

$$q_n^c(\mu) = H_c(\mu)^{-1}[(P^c)^*P_n](\mu); \quad 0\le\mu\le+1,\ n = 0,1,\ldots,N.$$

Using (4.7) and (4.12) one sees that there exist polynomials $q_0,\ldots,q_N$ of degree $\le$ N such that

$$(4.14)\quad \lim_{c\uparrow 1} \max_{0\le\mu\le 1} |q_n(\mu)-q_n^c(\mu)| = 0, \quad q_n(\mu) = H(\mu)^{-1}[P^*P_n](\mu).$$

Using (4.13) formula (4.10a) is clear. From Eq.(VI 3.2) of [18] and (4.12) one obtains (4.9).

In general, Ker $P^* = (\text{Im } P)^\perp = H_p^\perp \cap \text{Im } A = A[H_m] \oplus \text{span}\{P_1\}$ (see (1.2)). So, by (4.14), $q_1(\mu) \equiv 0$. Further, in the usual inner product of $L_2[-1,+1]$ one has

$$\int_0^1 q_n(\mu)H(\mu)d\mu = \langle P^*P_n,P_0\rangle = \langle P_n,PP_0\rangle = \langle P_n,P_0\rangle = 2\delta_{n0},$$

and thus (4.10b) is clear. It remains to prove that for N = 0,1 one has $q_0(\mu) = 1$, and that for N $\ge$ 2 the functions $q_0,q_2,\ldots,q_N$ are the unique polynomial solutions of the system of equations (4.10).

For N = 0,1 one has $\Psi(\mu) = \frac{1}{2}a_0 + \frac{3}{2}a_1(1-a_0)\mu^2 = \frac{1}{2}$ (see (4.3b) & (4.5b)). Using (4.8) for $\mu\to+\infty$, one gets $\int_0^1 H(\mu)d\mu = 2\int_0^1 \Psi(\mu)H(\mu)d\mu = 2$. Now q_0 satisfies (4.10a), and thus $q_0(0) = H_0(0) = 1$. Since q_0 is a polynomial of degree at most 1, it has the form $q_0(\mu) = A\mu + 1$. By (4.10b) one has $A\int_0^1 \mu H(\mu)d\mu = \int_0^1 q_0(\mu)H(\mu)d\mu - \int_0^1 H(\mu)d\mu = 2 - 2 = 0$. Since the integrand of $\int_0^1 \mu H(\mu)d\mu$ is strictly positive, one has A = 0. Hence, $q_0(\mu) = 1$.

Let us consider the case N $\ge$ 2, and let $\mathcal{P}_n$ denote the (n+1)-dimensional space of polynomials of degree $\le$ n. On the polynomials one defines the linear map V by

$$(Vp)(\lambda) = p(\lambda) - \lambda \int_0^1 \frac{\Psi(\lambda,\mu)p(\mu)-\Psi(\mu,\mu)p(\lambda)}{\mu-\lambda} H(\mu)d\mu; \quad \lambda \notin [0,1]$$

Note that $Vq_n = H_n$ (n$\ge$0). Rewriting $(Vp)(\lambda)$ in the form

$$(Vp)(\lambda) = p(\lambda) - \lambda \int_0^1 \frac{\Psi(\lambda,\mu)-\Psi(\mu,\mu)}{\mu-\lambda} p(\mu)H(\mu)d\mu -$$
$$- \lambda \int_0^1 \frac{p(\lambda)-p(\mu)}{\lambda-\mu} \Psi(\mu)H(\mu)d\mu \quad (\lambda \notin [0,1]),$$

observing that $1 - {}_0\int^1 \Psi(\mu)H(\mu)d\mu = 0$ (employ (4.8) for $\mu\to+\infty$), and noticing that for $n\geq 2$ the Kuščer polynomial H_n has degree $n-2$, it is clear that

(4.15) $\quad V[P_k] \subset P_{\max(N-2,k-1)}$; $\quad k = 0,1,2,\ldots$.

Let us first prove the linear independence of $q_0,q_2,\ldots,q_n$ ($n\geq 2$). Take constants $\xi_0,\xi_2,\ldots,\xi_n$ such that $\xi_0 q_0+\xi_2 q_2+\ldots\xi_n q_n = 0$. Applying V and (4.4) one gets $\xi_0+2\xi_2 = \xi_3 = \xi_4 = \ldots = \xi_n = 0$. But (4.10b) yields $\xi_0 = \frac{1}{2}\,{}_0\int^1 \{\xi_0 q_0(\mu)+\xi_2 q_2(\mu)+\ldots+\xi_n q_n(\mu)\}H(\mu)d\mu = 0$. Hence, $\xi_0 = \xi_2 = \ldots = \xi_n = 0$, and the linear independence of $q_0,q_2,\ldots,q_n$ has been established.

For $n\geq N$ consider V on P_n. Since $V[P_n] \subset V_{n-1}$ (cf. (4.15)), one has $\operatorname{Ker} V\cap P_n \neq \{0\}$. However, $\{q_0,q_2,\ldots,q_n\}$ spans an n-dimensional subspace of P_n. So $\operatorname{Ker} V\cap P_n$ is a one-dimensional subspace of V_n. Note that $q_0+2q_2 \neq 0$ (otherwise q_0 and q_2 are linear dependent) and that $V(q_0+2q_2) = H_0+2H_2 = 0$. Thus

(4.16) $\quad \operatorname{Ker} V = \{\xi(q_0+2q_2) : \xi\in\mathbb{C}\} \neq \{0\}$.

Hence, the functions $q_0,q_2,\ldots q_N$ are the unique polynomial solutions of the system of equations (4.10). Finally, by (4.16) one has $V[P_n] = P_{n-1}$ for $n\geq N$. Thus $q_0,q_2,\ldots,q_N$ are polynomials of degree $\leq N-1$. □

For a phase function satisfying (4.2) the Transport Equation (4.1a) with boundary conditions (4.1b) has a unique (bounded) solution ψ. This follows from Theorem III 3.3 of [18] and the decomposition

$$H_p \oplus \operatorname{Ker} A \oplus H_- = L_2[-1,+1].$$

In terms of the projection P of $L_2[-1,+1]$ onto $H_p \oplus \operatorname{Ker} A$ along H_- (described by (4.9)) one has $\psi(0,\mu) = (P\phi)(\mu)$ $(0\leq\mu\leq 1)$.

The projection P is connected to the scattering function $S(\nu,\mu)$ of Chandrasekhar [5] and the brightness coefficient $\rho(\nu,\mu)$ of Sobolev [24]. The connection is given by

(4.17) $\quad (Ph)(-\mu) = \frac{1}{2} \int_0^1 \mu^{-1} S(\mu,\nu)h(\nu)d\nu = 2 \int_0^1 \nu\rho(\nu,\mu)h(\nu)d\nu;\ 0\leq\mu\leq+1.$

For the non-conservative case an expression for $S(\nu,\mu)$ has been provided by Busbridge [3]. This expression has been improved and generalized to the conservative case by Pahor [21] (see also [4]). Up to the translation (4.17) this expression coincides with (4.9).

5. THE MILNE PROBLEM

In this section we derive the solution of the integro-differential equation (4.1a) with boundary conditions

$$(5.1a)\quad \psi(0,\mu) = 0 \quad (-1\le\mu\le 0); \qquad \int_{-1}^{+1} |\psi(x,\mu)|^2 d\mu = O(1) \quad (x\to+\infty);$$

$$(5.1b)\quad \lim_{x\to+\infty} \int_{-1}^{+1} \mu\psi(x,\mu)d\mu = -\tfrac{1}{2}F,$$

for phase functions $\hat{g}$ of the form (4.2). Eq.(4.1a) with boundary conditions (5.1) is usually called the *Milne problem*.

According to Theorem 1.6 the solutions of the Milne problem (with the constant F in (5.1b) unspecified) all have the form

$$(5.2)\quad \psi(x) = e^{-xT^{-1}A}x_p + (I-xT^{-1}A)x_0, \quad 0\le x<+\infty,$$

where $x_p \in H_p$, $x_0 \in [H_p \oplus H_-] \cap H_0$ and $x_p+x_0 \in H_-$. Here $[H_p \oplus H_-] \cap H_0$ is a one-dimensional subspace of H_0 (note that $\dim H_0 = 2$, and use Theorem III.7.1 of [18]). If P denotes the projection of $L_2[-1,+1]$ onto $H_p \oplus \ker A$ along H_- (which is described in detail in Section 4), then $x_p = -Px_0$ and $\psi(0) = x_p+x_0 = (I-P)x_0$. But for $e(\mu) \equiv 1$ one has

$$\begin{aligned} -\tfrac{1}{2}F &= \lim_{x\to+\infty} \int_{-1}^{+1} \mu\psi(x,\mu)d\mu = \lim_{x\to+\infty} \langle T\psi(x),e\rangle = \\ &= \lim_{x\to+\infty} \{\langle Te^{-xT^{-1}A}x_p,e\rangle + \langle T(I-xT^{-1}A)x_0,e\rangle\} = \\ &= \langle Tx_0,e\rangle - \lim_{x\to+\infty} x\langle Ax_0,e\rangle = \langle Tx_0,e\rangle, \end{aligned}$$

because $e \in \operatorname{Ker} A$; here we have used (1.6c) - (1.6d). Writing $x_0 = \xi P_0+\eta P_1$ (i.e., $x_0(\mu) = \xi-\eta\mu$) for certain constants $\xi,\eta \in \mathbb{C}$, one gets $-\tfrac{1}{2}F = \xi\langle TP_0,e\rangle + \eta\langle TP_1,e\rangle = \tfrac{2}{3}\eta$, and thus $\eta = -\tfrac{3}{4}F$. So $\psi(0) = (I-P)x_0 = (I-P)(\xi P_0-\tfrac{3}{4}FP_1) = -\tfrac{3}{4}F(I-P)P_1$. Hence,

$$(5.3)\quad \psi(0,\mu) = -\tfrac{3}{4}F[(I-P)P_1](\mu), \quad -1\le\mu\le+1.$$

One way to find an expression for $\psi(0,\mu)$ is to apply Theorem 4.2. For N = 0,1 one finds

$$\psi(0,\mu) = -\tfrac{3}{4}F\{\mu-\tfrac{1}{2}H(-\mu)\int_0^1 \nu^2(\nu-\mu)^{-1}H(\nu)d\nu\} =$$

$$= -\tfrac{3}{4}F\{\mu-\tfrac{1}{2}H(-\mu)\int_0^1 \nu H(\nu)d\nu-\tfrac{1}{2}\mu H(-\mu)\int_0^1 \nu(\nu-\mu)^{-1}H(\nu)d\nu\}.$$

Substituting (4.8) for $\Psi(\nu) \equiv \frac{1}{2}$ one gets

$$\psi(0,\mu) = \tfrac{3}{8}H(-\mu)\int_0^1 \nu H(\nu)d\nu, \quad -1\le\mu<0.$$

However, the H-function H is the same as the H-function in the conservative isotropic case (note that $\Psi(\nu) \equiv \frac{1}{2}$ does not depend on a_1). So $\int_0^1 \nu H(\nu)d\nu = \frac{2}{3}\sqrt{3}$ (cf. [3], Eq.(12.15); see also [18], the paragraph preceding Th.VI.5.2). Hence, for N = 0,1 one has

(5.4) $\psi(0,\mu) = \frac{1}{4}\sqrt{3}\ H(-\mu), \quad -1\le\mu\le0.$

Another way to find $\psi(0,\mu)$ rests upon the intertwining property

(5.5) $T(I-P) = Q^*T,$

where Q is the projection of $L_2[-1,+1]$ onto $H_m \oplus \text{Ker } A$ along H_+. For the non-conservative case an analogous property is due to Hangelbroek. To prove (5.5), take $h \in L_2[-1,+1]$. Then $h = h_p+h_0+h_-$ for unique vectors $h \in H_p$, $h_0 \in \text{Ker } A = \text{span }\{P_0\}$ and $h_- \in H_-$. So $T(I-P)h = Th_-$. On the other hand, $\text{Im } Q^* = (\text{Ker } Q)^\perp = H_-$ and $\text{Ker } Q^* = (\text{Im } Q)^\perp = H_m^\perp \cap \text{Im } A = [H_p^+ \oplus H_0^+] \cap \text{Im } A = H_p^+ \oplus A[H_0] = H_p^+ \oplus T[\text{Ker } A]$; here we employed (1.6d) and $A[H_0] = T[\text{Ker } A] = \text{span }\{P_1\}$. So $Q^*Th = Q^*(Th_p+Th_0) + Q^*(Th_-) = Q^*(Th_-) = Th_-$. Thus the intertwining property (5.5) is clear.

THEOREM 5.1. *Let the phase function* $\hat{g}$ *satisfy* (4.2) *with* $N\ge2$. *Then the integro-differential equation* (4.1a) *with boundary conditions* (5.1) *has a unique solution* ψ *and*

(5.6) $\psi(0,-\mu) = \frac{1}{4}F\ \mu^{-1}[q_0(\mu) + 2q_2(\mu)]H(\mu) \quad (0\le\mu\le1),$

where $\tilde{q} = q_0+2q_2$ *is the unique polynomial solution of the system of equations*

(5.7a) $\tilde{q}(\lambda) = \lambda\int_0^1 \frac{\Psi(\lambda,\mu)\tilde{q}(\mu)-\Psi(\mu,\mu)\tilde{q}(\lambda)}{\mu-\lambda} H(\mu)d\mu, \quad \lambda \notin [0,1];$

(5.7b) $\int_0^1 \tilde{q}(\mu)H(\mu)d\mu = 2.$

Proof. For n = 0,1,2,... one defines a function r_n by

$$[(I-Q)P_n](\mu) = r_n(\mu)H(\mu) \quad (0\le\mu\le+1).$$

Since $(2n+1)TP_n = (n+1)P_{n+1}+nP_{n-1}$ (which is the recurrence relation for the Legendre polynomials), one easily sees that

$$\mu r_n(\mu)H(\mu) = \left\{\frac{n+1}{2n+1}\, q_{n+1}(\mu) + \frac{n}{2n+1}\, q_{n-1}(\mu)\right\} H(\mu), \qquad 0\le\mu\le 1$$

(see (4.14)), and thus r_n is a polynomial of degree $\le \max(N,n)$. So

$$[(I-P)P_1](-\mu) = -[(I-Q)P_1](\mu) = -r_1(\mu)H(\mu), \qquad 0\le\mu\le +1.$$

So for $0\le\mu\le 1$ one gets

$$\psi(0,-\mu) = \tfrac{3}{4}Fr_1(\mu)H(\mu) = \frac{1}{4}F\mu^{-1}\{q_0(\mu)+2q_2(\mu)\}H(\mu),$$

and (5.6) is clear. From the proof of Theorem 4.2 it is clear that $\tilde{q} = q_0+2q_2$ is the unique solution of the system of equations (5.7). □

The first results on the solution of the isotropic Milne problem are due to Milne [19], who reduced it to a convolution equation; formula (5.4) has been found by Chandrasekhar [5]. The expression (5.6) for the anisotropic case is due to Pahor [21]. (Note the difference in Condition (5.1b)). Another derivation of (5.6) has been given by Busbridge and Orchard [4]. Here we present a mathematical justification of these results.

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