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ANALYTIC OPERATOR FUNCTIONS WITH COMPACT SPECTRUM. II. SPECTRAL PAIRS AND FACTORIZATION

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Using the technique introduced in the first part of this paper, various problems concerning factorization and divisibility of analytic operator functions with compact spectrum are studied in terms of spectral pairs of operators. The basic properties of such pairs are derived. Using these properties, stability of spectral divisors is proved and necessary and sufficient conditions (in terms of moments of the inverse function) are given in order that an analytic operator function with compact spectrum admits a generalized Wiener-Hopf factorization.

INTRODUCTION

In the study of matrix and operator polynomials divisibility and factorization problems can be handled successfully by employing pairs of operators which are constructed in such a manner that they epitomize in a convenient way the spectral data of polynomials concerned. For matrix polynomials and analytic matrix functions such pairs, which we shall call spectral pairs here, may be defined in terms of eigenvectors and generalized eigenvectors (see [11,18]). To define spectral pairs for (infinite dimensional) operator polynomials the companion operator matrix can be used. For arbitrary analytic operator functions these methods are not available and one has to find other ways to define spectral pairs. Here we solve this problem by employing the notion of spectral linearization, which has been introduced in [20].

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Let $\Omega \subset \mathbb{C}$ be an open set, and let W be an analytic operator function on Ω with values in the Banach algebra $L(Y)$ of all (bounded linear) operators on the complex Banach space Y. We assume that the *spectrum* of W, i.e., the set

(0.1) $$\Sigma(W) = \{\lambda \in \Omega | W(\lambda) \text{ not invertible}\}$$

is compact in Ω. Recall (see [20]) that an operator $A : X \to X$ is a *spectral linearization* for W on Ω if its spectrum $\sigma(A)$ is a subset of Ω and the function $W(\lambda) \oplus I_X$ is equivalent on Ω to the linear function $(\lambda I_X - A) \oplus I_Y$. The latter condition means that

(0.2) $$W(\lambda) \oplus I_X = E(\lambda)[(\lambda I_X - A) \oplus I_Y]F(\lambda) \ , \quad \lambda \in \Omega \ ,$$

where $E(\lambda) : X \oplus Y \to Y \oplus X$ and $F(\lambda) : Y \oplus X \to X \oplus Y$ are invertible operators which depend analytically on $\lambda \in \Omega$. The symbols I_X and I_Y denote the identity operators on X and Y, respectively. Now define $C : X \to Y$ by setting

$$C = \frac{1}{2\pi i} \int_\Gamma \pi F(\lambda)^{-1} \tau(\lambda - A)^{-1} d\lambda .$$

Here $\pi : Y \oplus X \to Y$ is the canonical projection onto Y, the map $\tau : X \to X \oplus Y$ is the canonical embedding of X and Γ is a suitable curve in Ω around the spectrum $\Sigma(W)$. We call the pair (C,A) a *right spectral pair* for W on Ω. Left spectral pairs may be defined in an analogous way.

A more abstract, axiomatic definition of spectral pairs will be given later in this paper. For a given W right spectral pairs exist and are uniquely determined up to *similarity*, i.e., if (C_1,A_1) and (C_2,A_2) are right spectral pairs for W, then there exists an invertible operator S such that

(0.3) $$C_1 = C_2 S \quad , \quad A_1 = S^{-1} A_2 S$$

We show that with respect to divisibility the spectral pairs introduced here have the desired properties. Namely, if (C_1,A_1) and (C_2,A_2) are right spectral pairs on Ω for W_1 and W_2, respectively, then W_2 is a right divisor of W_1 on Ω, i.e., there exists an analytic operator function $Q : \Omega \to L(Y)$ such that

$$W_1(\lambda) = Q(\lambda) W_2(\lambda) \ , \ \lambda \in \Omega,$$

if and only if A_1 has an invariant subspace N such that the pair of restricted operators $(C_1|_N, A_1|_N)$ is similar to the pair

(C_2,A_2), i.e., formula (0.3) holds true with (C_1,A_1) replaced by $(C_1|_N,A_1|_N)$. In that case N is called a *supporting subspace* of W_1 with respect to the pair (C_1,A_1).

In this paper we prove that for a right spectral pair (C,A) the finite column operator

$$(0.4)\qquad \begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{m-1} \end{pmatrix} : X \to Y^m$$

is left invertible for some positive integer m, which is one of our main results. For matrix and operator polynomials this property is not difficult to prove; in fact in that case one can take m to be the degree of the polynomial. But for arbitrary analytic operator functions with compact spectrum the left invertibility of the finite column is quite unexpected and this property does not hold when the operator function has a non-compact spectrum. Here we prove the left invertibility of the finite column by using the cocycle theory from [15] (see also [4]). In fact, using the cocycle theory we first show that a spectral pair of an analytic operator function on Ω is also a spectral pair for an entire operator function, and on the basis of this extension property we prove that the finite column (0.4) is left invertible for some m > 0.

In part III of this paper we shall see that for the Hilbert space case the left invertibility of the finite column (0.4) characterizes right spectral pairs, i.e., if C and A are Hilbert space operators for which the operator (0.4) is left invertible for some m > 0, then there exists an analytic operator function which has (C,A) as its spectral pair. We do not know whether this property holds in the general Banach space setting. In the Hilbert space case it is also possible to give a more direct proof of the left invertibility of the finite column (0.4) not using the cocycle theory.

The left invertibility of the finite column (0.4) allows us to make applications to Wiener-Hopf factorization problems. We prove that an analytic operator function $W : \Omega \to L(Y)$ with continuous and invertible boundary values admits a right generalized Wiener-Hopf factorization with respect to the boundary $\partial\Omega$ of Ω if and only if there exists a sufficiently large positive integer ℓ such that all the operators

$$\frac{1}{2\pi i}\int_{\partial\Omega}\begin{bmatrix} W(\lambda)^{-1} & \lambda W(\lambda)^{-1} & \dots & \lambda^{\ell-1}W(\lambda)^{-1} \\ \lambda W(\lambda)^{-1} & \lambda^2 W(\lambda)^{-1} & \dots & \lambda^{\ell}W(\lambda)^{-1} \\ \vdots & \vdots & & \vdots \\ \lambda^{i-1}W(\lambda)^{-1} & \lambda^{i}W(\lambda)^{-1} & \dots & \lambda^{\ell+i-2}W(\lambda)^{-1} \end{bmatrix} d\lambda; \qquad i = 1, \dots, \ell-1$$

have generalized inverses (cf. [24], where this is proved for operator polynomials). We assume here that $\partial\Omega$ is a simple closed rectifiable Jordan curve. A similar theorem may be proved for left Wiener-Hopf factorization by using analogous results for left spectral pairs.

This paper consists of 7 sections. We start with a preliminary section in which we recall from [20] the basic properties of spectral linearizations and spectral nodes. In Section 2 we introduce spectral pairs and express divisibility in terms of restrictions of spectral pairs. Intrinsic characterizations of spectral pairs are given in the third section.

In Section 4 the notion of a supporting subspace is defined. As a first application of the theory of spectral pairs we identify the stable factorizations in terms of their corresponding supporting subspaces. A factorization

(0.5) $\quad W(\lambda) = W_2(\lambda)W_1(\lambda)\ , \quad \lambda \in \Omega\ ,$

where $W_i : \Omega \to L(Y)$, $i = 1,2$, is analytic and has compact spectrum, is called stable if after a small perturbation of W the new function still admits a factorization as in (0.5) and, moreover, the new factors are close to the original ones. We show that the factorization (0.5) is stable if and only if the corresponding supporting subspace is a stable invariant subspace (cf. [1]). As a corollary we prove the stability of spectral factorizations, i.e., of factorizations of the type (0.5) with $\Sigma(W\ )$ and $\Sigma(W_2)$ disjoint.

At the end of Section 4 we make a supplementary remark concerning the description given in [1] of stable invariant subspaces of finite dimensional operators. We characterize the spectral subspaces of an operator $A : X \to X$ acting on a finite dimensional space X as those A-invariant subspaces N of X that are stable in the following strong sense: there exist positive constants η and K such that every operator $\tilde{A} : X \to X$ with $\|A - \tilde{A}\| < \eta$ has an invariant subspace $\tilde{N}$ such that

$$\mathrm{gap}(N,\tilde{N}) \le K\,\|A - \tilde{A}\|.$$

In Section 5 we prove that a spectral pair (right or left) of an analytic operator function on some open set Ω is also a spectral pair of an entire operator function. In Section 6 for some $m > 0$ the left invertibility of the finite column (0.4) is proved. In the last section the applications to Wiener-Hopf factorization are made.

Throughout the paper the letters X and Y (with or without indices) designate complex Banach spaces. By Ω we denote an open set in the complex plane $\mathbb{C}$ (if not stated otherwise). The symbol $\partial\Delta$ denotes the boundary of a set $\Delta \subset \mathbb{C}$. The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is denoted $\mathbb{C}_\infty$. Given a compact set $K \subset \mathbb{C}$, the Banach space of all continuous Y-valued functions on K with the supremum norm is denoted $C(K,Y)$.

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1. PRELIMINARIES

In this section we recall from [20] the definition and some basic properties of spectral nodes. Let $W : \Omega \to L(Y)$ be an analytic operator function with compact spectrum $\Sigma(W)$. A quintet $\theta = (A,B,C;X,Y)$ is called a *spectral node* for W on Ω if X is a Banach space,

$$A : X \to X\ ,\ B : Y \to X\ ,\ C : X \to Y$$

are bounded linear operators and the following conditions are

(P_1) $\sigma(A) \subset \Omega$;

(P_2) $W(\lambda)^{-1} - C(\lambda I - A)^{-1}B$ has an analytic extension to Ω;

(P_3) $W(\lambda)C(\lambda I - A)^{-1}$ has an analytic extension to Ω;

(P_4) $\bigcap_{j=0}^{\infty}$ Ker $CA^j = (0)$.

A quintet $(A,B,C;X,Y)$ of spaces and operators as above is a spectral node for W on Ω if and only if (P_1), (P_2) and the following conditions (P_3'), (P_4') are satisfied:

(P_3') $(\lambda I - A)^{-1}BW(\lambda)$ has an analytic extension to Ω;

(P_4') $\overline{\text{span} \bigcup_{j=0}^{\infty} \text{Im } A^j B} = X$.

The operator A will be referred to as the *main operator* of the spectral node θ.

The following explicit construction of a spectral node ([20], Theorem 3.1) will be used in the sequel. We assume for simplicity that zero is inside Ω.

THEOREM 1.1 *Let $W : \Omega \to L(Y)$ be an analytic operator function with compact spectrum $\Sigma(W)$. Suppose that Δ is a bounded Cauchy domain containing 0 such that $\Sigma(W) \subset \Delta \subset \overline{\Delta} \subset \Omega$, and let M be the set of all continuous Y-valued functions f on the boundary $\partial\Delta$ which admit an analytic continuation to a Y-valued function in $\mathbb{C}_\infty \backslash \Sigma(W)$ vanishing at ∞, while $W(\lambda)f(\lambda)$ has an analytic continuation to Ω. The set M endowed with the supremum norm is a Banach space. Put*

$$(1.1) \qquad V : M \to M, \ (Vf)(z) = zf(z) - (2\pi i)^{-1} \int_{\partial\Delta} f(\omega)\, d\omega \ ;$$

$$(1.2) \qquad R : Y \to M, \ (Ry)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{W(w)^{-1}}{z - w}\, y \, d\omega \ ;$$

$$(1.3) \qquad Q : M \to Y, \ Qf = (2\pi i)^{-1} \int_{\partial\Delta} f(\omega)\, d\omega \ .$$

In the definition of R the contour Γ is the boundary of a bounded Cauchy domain Δ' such that $\Sigma(W) \subset \Delta' \subset \overline{\Delta'} \subset \Delta$. Then $(V,R,Q;M,Y)$ is a spectral node for W on Ω.

The space M of Theorem 1.1 can be described as follows (see [5]; also [20], Lemma 3.3). Consider the bounded linear operator $T : C(\partial\Delta,Y) \to C(\partial\Delta,Y)$ given by the formula

$$(Tf)(z) = zf(z) + \frac{1}{2\pi i}\int_{\partial\Delta} [W(\lambda) - I]f(\lambda)d\lambda \quad .$$

Then $\sigma(T) = \Sigma(W) \cup \partial\Delta$, and

$$M = \operatorname{Im}\left[\frac{1}{2\pi i}\int_{\Gamma} (\lambda I - T)^{-1}d\lambda\right] ,$$

where the contour Γ is defined as in Theorem 1.1 . Moreover, $V = T|_M$.

For a given analytic operator function $W : \Omega \to L(Y)$ with compact spectrum spectral nodes exist (as shown in Theorem 1.1) and are uniquely determined up to similarity (see [20], Theorem 1.2). Here similarity means the following: Two spectral nodes $\theta_i = (A_i,B_i,C_i;X_i,Y)$, $i = 1,2$, are *similar* if there exists an invertible operator $S : X_1 \to X_2$ such that $A_1 = S^{-1}A_2S$, $B_1 = S^{-1}B_2$ and $C_1 = C_2S$. In particular the main operator of a spectral node for a given W is defined uniquely up to similarity; moreover, the spectrum of the main operator coincides with $\Sigma(W)$. The connection between linearization (cf. [5,22,3]) and spectral nodes is explained and used in [20] to solve problems concerning equivalence and similarity of analytic operator functions.

The notion of a spectral node is a natural generalization of the notions of standard triples and Γ-spectral triples for operator polynomials which have been introduced and studied in [11,12, 13,16,24]. On the other hand, spectral nodes are related to realizations for analytic operator functions (cf. [2], Section 2.3; also [9], Section III.1).

2. DIVISIBILITY AND SPECTRAL PAIRS

In this section we express divisibility in terms of restrictions of spectral pairs. Particular attention is given to spectral divisors.

Let $W : \Omega \to L(Y)$ be an analytic operator function with compact spectrum $\Sigma(W)$. A pair (C,A) (resp. (A,B)) of operators $C : X \to Y$ and $A : X \to X$ (resp. $A : X \to X$ and $B : Y \to X$) is called a *right* (resp. *left*) *spectral pair* for W on Ω if there exists an operator $B : Y \to X$ (resp. $C : X \to Y$) such that $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω. The formal definition of a right spectral pair which is given here coincides with the one employed in Introduction, which is evident from the connections between spectral nodes and spectral linearizations (cf. [20], Theorem 5.1).

Two right (resp. left) spectral pairs (C_1,A_1) and (C_2,A_2) (resp. (A_1,B_1) and (A_2,B_2)) are called *similar* if $C_1S = C_2$, $A_1 = SA_2S^{-1}$ (resp. $SB_1 = B_2$, $A_1 = S^{-1}A_2S$) for some invertible operator S. From what we know about spectral nodes it is clear that a right (resp. left) spectral pair for W on Ω exists and is unique up to similarity.

To describe divisibility in terms of spectral pairs we need a notion of restriction. For $i = 1,2$ let (C_i,A_i) be a pair of operators $A_i : X_i \to X_i$ and $C_i : X_i \to Y$. The pair (C_2,A_2) is called a *right restriction* of (C_1,A_1) if there exists a left invertible operator $S : X_2 \to X_1$ such that $C_1S = C_2$, $A_1S = SA_2$. Analogously, let (A_i,B_i) be a pair of operators $A_i : X_i \to X_i$ and $B_i : Y \to X_i$ ($i = 1,2$). We call (A_2,B_2) a *left restriction* of (A_1,B_1) if there exists a right invertible operator $S : X_1 \to X_2$ such that $SB_1 = B_2$, $SA_1 = A_2S$. The notion of restriction for pairs of operators acting between finite dimensional spaces has been introduced in [7] and further studied in [6].

The next theorems give a full description of the connection between divisibility and spectral pairs. We adopt the following definition of divisibility: an analytic function $W_1 : \Omega \to L(Y)$ with compact spectrum is called a *right* (*left*) divisor of an analytic function $W : \Omega \to L(Y)$ on Ω if there exists an analytic function $Q : \Omega \to L(Y)$ such that $W(\lambda) = Q(\lambda)W_1(\lambda)$ $\left(\text{resp. } W(\lambda) = W_1(\lambda)Q(\lambda)\right)$ for all $\lambda \in \Omega$. In that case Q necessarily has a compact spectrum.

THEOREM 2.1. *For $i = 1,2$, let $W_i : \Omega \to L(Y)$ be an analytic operator function with compact spectrum and let (C_i,A_i) be a right spectral pair for W_i on Ω. Then the pair (C_2,A_2) is a right restriction of the pair (C_1,A_1) if and only if the operator function W_2 is a right divisor of the operator function W_1 (on Ω).*

THEOREM 2.2 *For $i = 1,2$, let $W_i : \Omega \to L(Y)$ be an analytic operator function with compact spectum, and let (A_i,B_i) be a left spectral pair for W_i on Ω. Then the pair (A_2,B_2) is a left restriction of the pair (A_2,B_2) if and only if the operator function W_2 is a left divisor of the operator function W_1 (on Ω).*

In the context of polynomials results analogous to Theorems 2.1 and 2.2 have been obtained in [16,18,24], and in the context of analytic matrix functions results of this type were obtained in [19,23]. Note, however, that spectral pairs for polynomials are defined not in terms of spectral nodes but intrinsically in terms of maximality. In the next section we provide an intrinsic characterization of spectral pairs for analytic operator functions.

We shall prove Theorem 2.1 only (Theorem 2.2 can be proved by an analogous argument).

Proof of Theorem 2.1. Let $B_i : Y \to X_i$ be the (unique) operator such that $\theta_i = (A_i, B_i, C_i; X_i, Y)$ is a spectral node for W_i on Ω, $i = 1,2$. Suppose (C_2, A_2) is a right restriction of (C_1, A_1). Then there exists a left invertible operator $S : X_2 \to X_1$ such that

(2.1) $\quad C_1 S = C_2, \quad A_1 S = S A_2$.

So for $\lambda \in \Omega\backslash\{\Sigma(W_1) \cup \Sigma(W_2)\}$ we have:

$$W_1(\lambda)W_2(\lambda)^{-1} =$$
$$= W_1(\lambda)[W_2(\lambda)^{-1} - C_2(\lambda - A_2)^{-1}B_2] + W_1(\lambda)C_1(\lambda - A_1)^{-1}SB_2 .$$

By Property (P_2) for θ_2 and Property (P_3) for θ_1, it is clear that $W_1 W_2^{-1}$ has an analytic continuation to Ω. So W_2 is a right divisor of W_1 on Ω.

Conversely, let W_2 be a right divisor of W_1 on Ω, and let $H = W_1 W_2^{-1}$, which is an analytic operator funtion on Ω. Since $\Sigma(W_1)$ and $\Sigma(W_2)$ are compact subsets of Ω, clearly $\Sigma(H) = \{\lambda \in \Omega \mid H(\lambda)$ is not invertible$\}$ is compact too. Let $\theta_0 = (A_0, B_0, C_0; X_0, Y)$ be a spectral node for H on Ω. Using Theorem 2.2 in [20] we construct a spectral node $\theta = (A,B,C;X,Y)$ for $W_1 = HW_2$ on Ω from the spectral nodes θ_0 and θ_2 . We have:

$$X = X_2 \oplus X_0, \quad A = \begin{pmatrix} A_2 & B_2 C_0 \\ 0 & A_0 \end{pmatrix}, \quad C = \begin{pmatrix} C_2 & Q \end{pmatrix} ,$$

where

$$Q = (2\pi i)^{-1} \int_{\partial\Delta} \{W_2(\lambda)^{-1} - C_2(\lambda - A_2)^{-1}B_2\} C_0 (\lambda - A_0)^{-1}\, d\lambda ,$$

and Δ is a bounded Cauchy domain such that $(\Sigma(W_1) \cup \Sigma(W_2)) \subset \Delta \subset \bar{\Delta} \subset \Omega$. Using the uniqueness of spectral nodes (Theorem 1.2 in [20]), we conclude that the spectral nodes θ and θ_1 for W_1 on Ω are similar.

So there exists an invertible operator $\tilde{S} : X_2 \oplus X_0 \to X_1$ such that $\tilde{S}A = A_1\tilde{S}$ and $C_1\tilde{S} = C$. Define $S : X_2 \to X_1$ by $Sx_2 = \tilde{S}(x_2,0)$. Then S is left invertible and satisfies the identities (2.1). Hence (C_2,A_2) is a left restriction of (C_1,A_1). □

COROLLARY 2.3. *Let $W_1,W_2 : \Omega \to L(Y)$ be analytic operator functions with compact spectra. Then W_1 and W_2 have equal right (resp. left) spectral pairs if and only if there exists an invertible operator $E(\lambda) \in L(Y)$ depending analytically on $\lambda \in \Omega$ such that $W_1(\lambda) = E(\lambda)W_2(\lambda)$ $\big($resp. $W_1(\lambda) = W_2(\lambda)E(\lambda)\big)$ for all $\lambda \in \Omega$.*

Proof. If (C,A) is a right spectral pair for both W_1 and W_2 on Ω , then, in view of Theorem 2.1, the functions W_1 and W_2 are right divisors of each other on Ω. Hence, the operator functions E and F, which are defined on $\Omega\backslash\big(\Sigma(W_1) \cup \Sigma(W_2)\big)$ by $E = W_1W_2^{-1}$ and $F = W_2W_1^{-1}$, have an analytic continuation to Ω. But then these continuations take invertible values on all of Ω. □

Let $W(\lambda) = W_1(\lambda)W_2(\lambda)$, $\lambda \in \Omega$, where $W_1,W_2 : \Omega \to L(Y)$ are analytic operator functions with compact spectra. The function W_2 (resp. W_1) is called a *right* (resp. *left*) *spectral divisor* of W if $\Sigma(W_1) \cap \Sigma(W_2) = \emptyset$. Note that in this case $\Sigma(W)$ is the union of the disjoint compact sets $\Sigma(W_1)$ and $\Sigma(W_2)$, and hence $\Sigma(W_2)$ is a compact and relatively open subset of $\Sigma(W)$. Conversely, if σ is a compact and relatively open subset of $\Sigma(W)$, then there exists a right spectral divisor W_2 of W on Ω such that $\Sigma(W_2) = \sigma$. If Ω is simply connected or the group of invertible elements of $L(Y)$ is connected, this fact follows from the theory developed in [15]; for an arbitrary open set we prove this statement at the end of Section 5.

To describe spectral divisors, we use the notion of a spectral subspace. A subspace $M \subset X$ is called a *spectral subspace* for an operator $A : X \to X$, if M is the image of a Riesz projection for A, i.e.,

$$M = \operatorname{Im}\left(\frac{1}{2\pi i}\int_\Gamma (\lambda-A)^{-1}\, d\lambda\right) .$$

for some simple rectifiable contour Γ such that $\Gamma \cap \sigma(A) = \emptyset$ (in this case we say that the spectral subspace M corresponds to the part of $\sigma(A)$ which is inside Γ).

THEOREM 2.4. *Let* $W, W_2 : \Omega \to L(Y)$ *be analytic operator functions with compact spectra, and let* (C,A) *and* (C_2, A_2) *be right spectral pairs for* W *and* W_2 *on* Ω*, respectively. Denote by* X *(resp.* X_2*) the space on which* A *(resp.* A_2*) acts. Then* W_2 *is a right spectral divisor of* W *on* Ω *if and only if there exists a left invertible operator* $S : X_2 \to X$ *such that*

$$CS = C_2, \quad AS = SA_2, \tag{2.2}$$

and Im S *is a spectral subspace of the operator* A.

THEOREM 2.5. *Let* $W, W_2 : \Omega \to L(Y)$ *be analytic operator functions with compact spectra, and let* (A,B) *and* (A_2, B_2) *be left spectral pairs for* W *and* W_2 *on* Ω*, respectively. Denote by* X *(resp.* X_2*) the space on which* A *(resp.* A_2*) acts. Then* W_2 *is a left spectral divisor of* W *on* Ω *if and only if there exists a right invertible operator* $S : X \to X_2$ *such that* $SB = B_2$*,* $SA = A_2 S$ *and* Ker S *is a spectral subspace of the operator* A.

We prove Theorem 2.4 only (the proof of Theorem 2.5 is analogous).

<u>Proof of Theorem 2.4</u>. Let W_2 be a right spectral divisor of W on Ω, and let $W_1 = WW_2^{-1}$ be the quotient. Let $\theta_1 = (A_1, B_1, C_1; X_1, Y)$ be a spectral node for W_1 on Ω. As in the proof of Theorem 2.1, we construct two spectral nodes for W on Ω: one of the form $(A,B,C;X,Y)$ and the other one of the form $(\tilde{A}, \tilde{B}, \tilde{C}; X_1 \oplus X_2, Y)$, where

$$\tilde{A} = \begin{pmatrix} A_2 & * \\ 0 & A_1 \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} C_2 & * \end{pmatrix}. \tag{2.3}$$

Then these two spectral nodes are similar (Theorem 1.2 in [20]); so

$$C\tilde{S} = \tilde{C}, \qquad A\tilde{S} = \tilde{S}\tilde{A} \tag{2.4}$$

for some invertible operator $\tilde{S} : X_2 \oplus X_1 \to X$. Define $S : X_2 \to X$ by $Sx_2 = \tilde{S}(x_2, 0)$. Then S is left invertible and satisfies Eqs (2.2). From (2.3) it is clear that $X_2 \oplus (0)$ is the spectral subspace of $\tilde{A}$ corresponding to $\sigma(A_2) = \Sigma(W_2)$ (because $\sigma(A_1) = \Sigma(W_1)$). Hence, Im S is the spectral subspace of A corresponding to the same set $\sigma(A_2) = \Sigma(W_2)$.

The converse statement is proved by reversing this argument. □

3. CHARACTERIZATIONS OF SPECTRAL PAIRS

Spectral pairs for analytic operator functions with compact spectrum were defined above via spectral nodes. However, in the context of polynomials spectral pairs can be defined independently of the notion of spectral nodes (see [16,24]). In this section we give such intrinsic characterizations of spectral pairs of analytic operator functions.

THEOREM 3.1. *A pair (C,A) of operators $C : X \to Y$ and $A : X \to X$ is a right spectral pair on Ω for an analytic function $W : \Omega \to L(Y)$ with compact spectrum if and only if the following four conditions are fulfilled:*

(Q_1) $\sigma(A) \subset \Omega$;

(Q_2) $W(\lambda)C(\lambda-A)^{-1}$ *has an analytic continuation to* Ω;

(Q_3) *for some (and hence every) bounded Cauchy domain Δ such that $\left(\Sigma(W) \cup \sigma(A)\right) \subset \Delta \subset \overline{\Delta} \subset \Omega$, the operator $S : X \to C(\partial\Delta, Y)$, defined by*

$$(Sx)(z) = C(z-A)^{-1}x \qquad (z \in \partial\Delta),$$

is left invertible;

(Q_4) *every other pair of operators satisfying (Q_1), (Q_2) and (Q_3) is a restriction of (C,A).*

More explicitly, condition (Q_4) means the following. Let (C_0,A_0) be a pair of operators $C_0 : X_0 \to Y$; $A_0 : X_0 \to X_0$ with the following properties: (i) $\sigma(A_0) \subset \Omega$; (ii) the function $W(\lambda)C_0(\lambda-A_0)^{-1}$ has an analytic continuation to Ω; (iii) the operator $S_0 : X_0 \to C(\partial\Delta_0, Y)$ defined by

$$(S_0x)(z) = C_0(z-A_0)^{-1}x \qquad (z \in \partial\Delta_0) \tag{3.1}$$

is left invertible, where Δ_0 is a bounded Cauchy domain such that $\left(\Sigma(W) \cup \sigma(A_0)\right) \subset \Delta_0 \subset \overline{\Delta_0} \subset \Omega$. Then there exists a left invertible operator $T : X_0 \to X$ such that $CT = C_0$ and $AT = TA_0$.

THEOREM 3.2. *A pair (A,B) of operators $A : X \to X$ and $B : Y \to X$ is a left spectral pair on Ω for some analytic function $W : \Omega \to L(Y)$ with compact spectrum if and only if the following four conditions are fulfilled:*

(Q_1) $\sigma(A) \subset \Omega$;

(Q_2') $\quad$ *$(\lambda-A)^{-1}BW(\lambda)$ has an analytic continuation to Ω;*

(Q_3') $\quad$ *for some (and hence every) bounded Cauchy domain Δ such that $\left(\Sigma(W) \cup \sigma(A)\right) \subset \Delta \subset \overline{\Delta} \subset \Omega$, the operator $S : C(\partial\Delta, Y) \to X$, defined by*

$$Sf = (2\pi i)^{-1} \int_{\partial\Delta} (\lambda-A)^{-1}Bf(\lambda)\, d\lambda ,$$

is right invertible;

(Q_4') $\quad$ *every other pair of operators satisfying (Q_1), (Q_2') and (Q_3') is a left restriction of (A,B).*

Proof of Theorem 3.1. Let (C,A) be a right spectral pair for W on Ω. Choose $B : Y \to X$ such that $\theta_0 = (A,B,C;X,Y)$ is a spectral node for W on Ω. Then conditions (Q_1) and (Q_2) follow from (P_1) and (P_2) of the definition of a spectral node. To derive (Q_3), consider the spectral node $\theta_0 = (V,R,Q;M,Y)$ described in Theorem 1.1 (assuming that zero is inside Δ). Let $J : X \to M$ be the operator that establishes the similarity between θ and θ_0. Then (by [20], Theorem 1.2) we have $Jx = Sx$ for each $x \in X$. So Ker $S = \{0\}$ and Im $S = M$. Since M is a complemented subspace of $C(\partial\Delta, Y)$ (cf. [20], Lemma 3.3), the left invertibility of S is evident. This proves (Q_3).

Next we derive (Q_4). Let (C_0, A_0) be a pair of operators $C_0 : X_0 \to Y$ and $A_0 : X_0 \to X_0$ with the properties (i), (ii), and (iii). It is sufficient to show that (C_0, A_0) is a right restriction of (Q,V), where Q and V are defined by (1.3) and (1.1), respectively (replacing there Δ by Δ_0). Let S be the operator defined by (3.1). According to properties (i) and (ii) we have Im $S_0 \subset M$ (= domain of definition of V). Further one easily verifies that $QS_0 = C_0$ and $VS_0 = S_0A$. Since S_0 is left invertible (property (iii)), we may conclude that (C_0, A_0) is a right restriction of (Q,V), and (Q_4) is proved.

Now conversely, let (C,A) be a pair of operators $C : X \to Y$ and $A : X \to X$ which satisfies the conditions (Q_1), (Q_2), (Q_3) and (Q_4). From the first part of the proof we know that the operator $S : X \to M$, defined by $(Sx)(z) = C(z-A)^{-1}x$ $(z \in \partial\Delta)$, is left invertible and $QS = C$ and $VS = SA$. Because of property (Q_4) for the pair (C,A), the pair (Q,V) is a right restriction of the pair (C,A).

Hence there exists a left invertible $T : M \to X$ such that $Q = ST$ and $TV = AT$. But then

$$QV^n(ST) = CA^nT = QV^n , \quad n = 0,1,2, \dots .$$

From property (P_4) of the spectral node $\theta_0 = (V,Q,R;M,Y)$ we conclude that $ST = I$, and thus T is invertible. Now define $B : Y \to X$ by $B = TR$. Then the node $(A,B,C;X,Y)$ is similar to θ_0, and hence (C,A) is a right spectral pair. □

Proof of Theorem 3.2. Let (A,B) be a left spectral pair for W on Ω. Let $C : X \to Y$ be such that $\theta = (A,B,C;X,Y)$ is a spectral node for W on Ω. Then conditions (Q_1) and (Q_2') follow from the properties (P_1) and (P_3') for θ. To derive (Q_3') consider the operator $T : C(\partial\Delta,Y) \to X$, defined by

$$(3.2) \qquad Tf = \frac{1}{2\pi i}\int_{\partial\Delta} (\lambda-A)^{-1}BW(\lambda)f(\lambda)d\lambda .$$

Since $W(\lambda)$ is invertible for all $\lambda \in \partial\Delta$, it suffices to show that T is right invertible. Let $\theta_0 = (V,R,Q;M,Y)$ be the spectral node for W on Ω constructed in Theorem 1.1 (assuming 0 to be inside Δ), and let $J : X \to M$ be the operator that establishes the similarity between θ and θ_0. Then (by [20], Theorem 1.2)

$$(Jx)(z) = C(z-A)^{-1}x, \quad z \in \partial\Delta ,$$

for each $x \in X$. But then we can apply Corollary 1.3 in [20] to show that $TJx = x$ $(x \in X)$, which implies that T is right invertible.

Next, we deduce (Q_4'). Let (A_0,B_0) be a pair of operators $A_0 : X_0 \to X_0$ and $B_0 : Y \to X_0$ with the following properties: (i) $\sigma(A_0) \subset \Omega$; (ii') $(\lambda-A_0)^{-1}B_0W(\lambda)$ has an analytic continuation to Ω; (iii') for some bounded Cauchy domain Δ_0 such that $(\Sigma(W) \cup \sigma(A_0)) \subset \Delta_0 \subset \overline{\Delta_0} \subset \Omega$, the operator $S_0 : C(\partial\Delta_0,Y) \to X$, defined by

$$S_0f = (2\pi i)^{-1}\int_{\partial\Delta_0} (\lambda-A_0)^{-1}B_0f(\lambda)d\lambda ,$$

is right invertible. We have to show that (A_0,B_0) is a left restriction of (A,B). Define $T_0 : C(\partial\Delta_0,Y) \to X_0$ by

$$T_0f = \frac{1}{2\pi i}\int_{\partial\Delta_0} (\lambda-A_0)^{-1}B_0W(\lambda)f(\lambda)d\lambda .$$

Property (iii') and the invertibility of the operator $W(\lambda)$ for each $\lambda \in \partial\Delta_0$ imply that T_0 is right invertible. Consider the operator $G : X \to X_0$ defined by $G = T_0 J$. Clearly, G is bounded and one easily checks that $GA = A_0 G$ and $GB = B_0$. It remains to prove that G is right invertible. To this end we compute GT, where T is defined by (3.2) replacing Δ by Δ_0. We have

$$GTf = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Delta_0} (\lambda-A_0)^{-1}B_0W(\lambda)C(\lambda-A)^{-1}\cdot \cdot\left(\int_{\partial\Delta_0} (z-A)^{-1}BW(z)f(z)dz\right) d\lambda \ .$$

In the first integral we replace Δ_0 by a somewhat smaller Cauchy domain Δ_0' such that $\{\sigma(A) \cup \sigma(A_0)\} \subset \Delta_0' \subset \overline{\Delta_0'} \subset \Delta_0$. Further, using the resolvent identity, we can write GTf as $\alpha - \beta$, where

$$\alpha = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Delta_0'} \left(\int_{\partial\Delta_0} (z-\lambda)^{-1}(\lambda-A_0)^{-1}B_0W(\lambda)C(\lambda-A)^{-1}\cdot \cdot BW(z)f(z)dz\right) d\lambda \ ,$$

$$\beta = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\Delta_0'} \left(\int_{\partial\Delta_0} (z-\lambda)^{-1}(\lambda-A_0)^{-1}B_0W(\lambda)C(z-A)^{-1}\cdot \cdot BW(z)f(z)dz\right) d\lambda \ .$$

In the expression for α we replace $C(\lambda-A)^{-1}B$ by $W(\lambda)^{-1} - H(\lambda)$, where $H(\lambda)$ depends analytically on λ in Ω. Next we interchange the order of integration and use property (ii'). In this way one finds that $\alpha = T_0 f$. Similarly, using Fubini's theorem and property (ii'), one sees that $\beta = 0$. Thus $GT = T_0$. Since T_0 is right invertible, the same is true for G and property (Q_4') is verified.

Conversely, let (A,B) be a pair of operators $A : X \to X$ and $B : Y \to X$ that satisfy conditions (Q_1), (Q_2'), (Q_3') and (Q_4'). Again consider the spectral node $\theta_0 = (V,R,Q;M,Y)$ for W on Ω, which we have constructed in Theorem 1.1. Since (V,R) is a left spectral pair, we know from the first part of the proof that the operator $H : M \to X$ defined by

$$H = \frac{1}{2\pi i} \int_{\partial\Delta} (\lambda-A)^{-1}BW(\lambda)Q(\lambda-V)^{-1}d\lambda$$

is right invertible, $AH = HV$ and $HR = B$. It suffices to show that H has a left inverse too. Because of property (Q_4') for the pair (A,B) the pair (V,R) is a right restriction of the pair (A,B).

Hence, there exists a right invertible $F : X \to M$ such that $FA = VF$ and $FB = R$. But then

$$FH = \frac{1}{2\pi i}\int_{\partial\Delta}(\lambda-V)^{-1}RW(\lambda)Q(\lambda-V)^{-1}d\lambda \ .$$

Applying Corollary 1.3 in [20], one sees that $FH = I$, and the proof is complete. □

Let $\theta_0 = (V,R,Q;M,Y)$ be the spectral node for W on Ω constructed in Theorem 1.1. From the proof of Theorem 3.2 it is not difficult to see that the operator $\Pi : C(\partial\Delta,Y) \to C(\partial\Delta,Y)$ defined by

$$\Pi f = \frac{1}{2\pi i}\int_{\partial\Delta}(\lambda-V)^{-1}RW(\lambda)f(\lambda)d\lambda$$

is a projection operator whose image is equal to M. It can be proved that Ker Π is the closure in $C(\partial\Delta,Y)$ of the linear subspace N of all $f \in C(\partial\Delta,Y)$ such that

$$f(\lambda) = g(\lambda) + W^{-1}(\lambda)h(\lambda) \ , \quad \lambda \in \partial\Delta \ ,$$

where g has an analytic continuation to Δ and h has an analytic continuation to $\mathbb{C}_\infty \setminus \overline{\Delta}$ with $h(\infty) = 0$.

4. STABILITY OF ANALYTIC RIGHT DIVISORS

In this section, as a first application of the theory of spectral pairs, we describe the stable analytic right divisors in terms of certain stable invariant subspaces. As a corollary the stability of spectral divisors is obtained. Let $W : \Omega \to L(Y)$ be an analytic operator function with compact spectrum, and let (C,A) be a right spectral pair for W on Ω. The space on which A acts is denoted by X. A closed complemented subspace N of X is called a *supporting subspace* of the pair (C,A) if N is invariant under A and the pair of restricted operators $(C|_N, A|_N)$ is a right spectral pair on Ω for some analytic operator function $W_1 : \Omega \to L(Y)$ with compact spectrum. Since N is complemented, the pair $(C|_N, A|_N)$ is a right restriction of the pair (C,A) and hence W_1 is a right divisor of W on Ω (cf. Theorem 2.1). The next proposition concerns the converse statement.

PROPOSITION 4.1. *Let (C,A) be a right spectral pair for W on Ω, and let $W_1 : \Omega \to L(Y)$ be a right divisor of W on Ω. Then there* *exists a unique supporting subspace* N *of the pair* (C,A) *such that the pair* $(C|_N, A|_N)$ *is a right spectral pair for* W_1 *on* Ω. *This subspace* N *is given by*

$$(4.1) \qquad N = \mathrm{Im}\left(\frac{1}{2\pi i}\int_{\partial\Delta}(\lambda-A)^{-1}BW(\lambda)C_1(\lambda-A_1)^{-1}d\lambda\right).$$

Here (C_1, A_1) *is a right spectral pair for* W_1 *on* Ω, *the map* $B : Y \to X$ *is an operator such that* $(A,B,C;X,Y)$ *is a spectral node for* W *on* Ω, *and* Δ *is a bounded Cauchy domain such that* $\{\sigma(A)\cup\sigma(A_1)\} \subset \Delta \subset \overline{\Delta} \subset \Omega$.

<u>Proof</u>. Let X_1 be the space on which A_1 acts. Since W_1 is a right divisor of W, the pair (C_1, A_1) is a right restriction of the pair (C,A), and so there exists a left invertible map $S : X_1 \to X$ such that

$$(4.2) \qquad CS = C_1, \quad AS = SA_1.$$

Obviously, Im S is a supporting subspace of W with respect to (C,A), and $(C|_{\mathrm{Im}\,S}, A|_{\mathrm{Im}\,S})$ is a right spectral pair for W_1 on Ω.

Next, let N be a supporting subspace such that $(C|_N, A|_N)$ is a right spectral pair for W_1 on Ω. Then there exists an invertible operator $T : X_1 \to N$ such that

$$(4.3) \qquad CTx = C_1x, \quad ATx = TA_1x \quad (x \in X_1).$$

From (4.2) and (4.3) it is clear that $CA^nSx = C_1A_1{}^nx = CA^nTx$ for each $x \in X$ and $n = 0,1,2,\dots$. So, by property (P_4) of a spectral node, we have $Tx = Sx$ for $x \in X_1$. Hence $N = \mathrm{Im}\, S$, and the uniqueness of the supporting subspace is proved.

Finally, note that $C_1(\lambda-A_1)^{-1} = C(\lambda-A)^{-1}S$, because of (4.2). But then we can apply [20], Corollary 1.3, to show that

$$S = \frac{1}{2\pi i}\int_{\partial\Delta}(\lambda-A)^{-1}BW(\lambda)C_1(\lambda-A_1)^{-1}d\lambda.$$

It follows that $N = \mathrm{Im}\, S$ is given by formula (4.1). □

Note that the previous proposition allows us to speak about <u>the</u> supporting subspace of (C,A) associated with a given right divisor.

It will be convenient to state the main results of this section for functions with continuous boundary values. So in the remaining part of this section Ω stands for a bounded Cauchy domain.

By $F(\Omega)$ we denote the class of functions $\overline{\Omega} \to L(Y)$ which are analytic in Ω, continuous on $\overline{\Omega}$ and have invertible values on $\partial\overline{\Omega}$. On $F(\Omega)$ there is a natural distance, namely

$$|||W_1 - W_2||| \quad := \max_{\lambda \in \partial\Omega} ||W_1(\lambda) - W_2(\lambda)|| \quad .$$

Let $W \in F(\Omega)$, and let $W_1 \in F(\Omega)$ be a right divisor of W on $\overline{\Omega}$. The right divisor W_1 is called *stable* if for each $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: every operator function $\widetilde{W} \in F(\Omega)$ with $|||\widetilde{W} - W_1||| < \delta$ has a right divisor $\widetilde{W}_1 \in F(\Omega)$ such that $|||\widetilde{W}_1 - W_1||| < \varepsilon$. Since the map $(W,W_1) \mapsto W(.)W_1(.)^{-1}$ is continuous with respect to the distance on $F(\Omega)$, it is clear that a right divisor W_1 of W is stable if and only if the factorization $W = W_1W_2$ is stable in the following sense: given $\varepsilon > 0$ there exists $\eta > 0$ such that every operator function $\widetilde{W} \in F(\Omega)$ with $|||\widetilde{W} - W||| < \eta$ admits a factorization $\widetilde{W} = \widetilde{W}_2\widetilde{W}_1$ with factors $\widetilde{W}_1$, $\widetilde{W}_2$ in $F(\Omega)$ and

$$|||\widetilde{W}_2 - W_2||| < \varepsilon \ , \qquad |||\widetilde{W}_1 - W_1||| < \varepsilon \ .$$

For monic matrix polynomials (see [1], cf. [14]) and for rational matrix functions (see [2]) stable factorizations have been described in terms of so-called stable invariant subspaces. The main result of this section shows that a similar description also holds true for analytic operator functions with compact spectrum.

Let A be a bounded linear operator on a Banach space X. Recall (see [1]) that a closed A-invariant subspace N of X is called *stable* if given $\varepsilon > 0$ there exists $\delta > 0$ such that every operator $\widetilde{A} : X \to X$ with $||\widetilde{A} - A|| < \delta$ has a closed invariant subspace $\widetilde{N}$ with $\mathrm{gap}(\widetilde{N},N) < \varepsilon$. Here $\mathrm{gap}(\widetilde{N},N)$ denotes the gap or maximal opening between $\widetilde{N}$ and N (see [17,21]).

For the finite dimensional case ($n = \dim X < \infty$) the description of all stable A-invariant subspaces is given in [2], Section 8.1, and reads as follows. Let $\lambda_1, \ldots, \lambda_r$ be the different eigenvalues of A and let $X_1, \ldots, X_r$ be the corresponding spectral subspaces, i.e.,

$$X_i = \{x \in X | (A-\lambda_i I)^n x = 0\} \ , \qquad i = 1, \ldots, r \ .$$

An A-invariant subspace N is stable if and only if

$N = N_1 \oplus \dots \oplus N_r$, where N_j is an arbitrary A-invariant subspace of X_j whenever dim Ker $(A-\lambda_j I) = 1$, while otherwise $N_j = (0)$ or $N_j = X_j$.

THEOREM 4.2. *Let (C,A) be a right spectral pair for $W \in F(\Omega)$, and let $W_1 \in F(\Omega)$ be a right divisor of W on $\overline{\Omega}$. Then W is a stable right divisor of W if and only if the supporting subspace N of (C,A) associated with W_1 is a stable A-invariant subspace.*

The proof of Theorem 4.2 is based on the description of divisibility in terms of spectral pairs (Section 2), as well as on some stability properties of analytic operator functions and their spectral nodes, which are of independent interest. Namely, small changes in a spectral node (resp. in an analytic function) imply small changes in the corresponding function (resp. in the corresponding spectral node). We make this statement precise in Theorems 4.3 and 4.5 below.

THEOREM 4.3. *Let $(A,B,C;X,Y)$ be a spectral node for $W \in F(\Omega)$. Then there exist positive constants η and K with the following property: for every triple of operators $\tilde{A} : X \to X$, $\tilde{B} : Y \to X$, $\tilde{C} : X \to Y$ such that*

$$||A - \tilde{A}|| + ||B - \tilde{B}|| + ||C - \tilde{C}|| < \eta ,$$

the quintet $(\tilde{A},\tilde{B},\tilde{C};X,Y)$ is a spectral node for some $\tilde{W} \in F(\Omega)$ and

$$|||W - \tilde{W}||| \le K(||A - \tilde{A}|| + ||B - \tilde{B}|| + ||C - \tilde{C}||) . \tag{4.4}$$

Proof. By Theorem 4.1 in [20] (which holds also for analytic operator functions with continuous boundary values) there exists a continuous function $H : \overline{\Omega} \to L(Y)$, which is analytic in Ω, such that the operator

$$E(\lambda) = \begin{pmatrix} A-\lambda & B \\ C & H(\lambda) \end{pmatrix} : \overline{\Omega} \to L(X \oplus Y)$$

is invertible for all $\lambda \in \overline{\Omega}$, and

$$E^{-1}(\lambda) = \begin{pmatrix} * & * \\ * & W(\lambda) \end{pmatrix}, \quad \lambda \in \overline{\Omega}.$$

Clearly, the operator

$$\tilde{E}(\lambda) = \begin{pmatrix} \tilde{A}-\lambda & \tilde{B} \\ \tilde{C} & H(\lambda) \end{pmatrix}$$

is also invertible for all $\lambda \in \Omega$ provided η is small enough and

$$\sup_{\lambda \in \Omega} \|\tilde{E}^{-1}(\lambda) - E^{-1}(\lambda)\| \leq K(\|\tilde{A}-A\| + \|\tilde{B}-B\| + \|\tilde{C}-C\|), \tag{4.5}$$

where the positive constant K does not depend on $\tilde{A},\tilde{B},\tilde{C}$. Define $\tilde{W}(\lambda)$ by the equality

$$\tilde{E}(\lambda)^{-1} = \begin{pmatrix} * & * \\ * & \tilde{W}(\lambda) \end{pmatrix}, \quad \lambda \in \Omega .$$

Again by Theorem 4.1 in [20], $(\tilde{A},\tilde{B},\tilde{C};X,Y)$ is a spectral node for $\tilde{W}(\lambda)$ on Ω; the estimate (4.4) follows from (4.5). □

COROLLARY 4.4. *Let (C,A) (resp. (A,B)) be a right (resp. left) spectral pair for some analytic operator function $W \in F(\Omega)$, where $A : X \to X$, $C : X \to Y$ (resp. $A : X \to X$, $B : Y \to X$). Then a pair of operators $(\tilde{C},\tilde{A})$ with $\tilde{A} : X \to X$, $\tilde{C} : X \to Y$ (resp. $(\tilde{A},\tilde{B})$ with $\tilde{A} : X \to X$, $\tilde{B} : Y \to X$) is a right (resp. left) spectral pair for some analytic operator function $\tilde{W} \in F(\Omega)$ whenever $\|\tilde{C} - C\| + \|\tilde{A} - A\|$ (resp. $\|\tilde{A} - A\| + \|\tilde{B} - B\|$) is small enough. In this case $\tilde{W}$ can be chosen so that*

$$|||\tilde{W} - W||| \leq K(\|\tilde{C} - C\| + \|\tilde{A} - A\|)$$

(resp.

$$|||\tilde{W} - W||| \leq K(\|\tilde{B} - B\| + \|\tilde{A} - A\|) \text{) },$$

where the positive constant K is independent of $(\tilde{C},\tilde{A})$ (resp. of $(\tilde{A},\tilde{B})$).

In a certain sense the following result is the converse of Theorem 4.3. It shows that spectral nodes for close analytic functions can be chosen close as well.

THEOREM 4.5. *Let Ω be a bounded Cauchy domain, and let $W_0 \in F(\Omega)$ with spectral node $(A_0,B_0,C_0;X,Y)$. Then there exist positive constants η and K with the following property: for every function $W_1 \in F(\Omega)$ with $|||W_1 - W_0||| < \eta$, there exists a spectral node $(A_1,B_1,C_1;X,Y)$ of W_1 with the same state space X and with the property that*

$$\|A_1 - A_0\| + \|B_1 - B_0\| + \|C_1 - C_0\| \leq K|||W_1 - W_0||| . \tag{4.6}$$

Proof. For simplicity it will be assumed that zero is inside Ω. We shall employ the construction of spectral nodes given in Theorem 1.1 (which holds also for the functions from $F(\Omega)$, when $\partial\Delta$ is replaced by $\partial\Omega$).

Let $W_0, W_1 \in F(\Omega)$. For $k = 0,1$, define the operator $T_k \in L(C(\partial\Omega, Y))$ as follows:

$$(4.7) \qquad (T_k f)(z) = zf(z) + \frac{1}{2\pi i}\int_{\partial\Omega} (W_k(\lambda) - I) f(\lambda) d\lambda , \qquad f \in C(\partial\Omega, Y) .$$

As proved in [5] (Theorem 2.3), $\sigma(T_k) = \Sigma(W_k) \cup \partial\Omega$.

Introduce the Riesz projections

$$P_k = \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - T_k)^{-1} d\lambda , \quad k = 0,1 ,$$

where Γ_0 is a suitable contour in Ω around $\Sigma(W_0)$. Observe that Γ_0 is also a contour around $\Sigma(W_1)$ if $|||W_1 - W_0|||$ is small enough; from these formulas we obtain (for $|||W_1 - W_0|||$ small enough):

$$(4.8) \qquad ||T_0 - T_1|| + ||P_0 - P_1|| \le K_1 |||W_1 - W_0||| ,$$

where the constant K_0 does not depend on W_1. It follows that for $|||W_1 - W_0|||$ sufficiently small

$$(4.9) \qquad C(\partial\Omega, Y) = \operatorname{Ker} P_0 \oplus \operatorname{Im} P_0 = \operatorname{Ker} P_0 \oplus \operatorname{Im} P_1 .$$

Let R be the angular operator (see [2], Section 5.1) of $\operatorname{Im} P_1$ with respect to P_0, that is, R is an operator from $\operatorname{Im} P_0$ into $\operatorname{Ker} P_0$ such that

$$\operatorname{Im} P_1 = \{Rx + x \mid x \in \operatorname{Im} P_0\} .$$

Define $S : \operatorname{Im} P_1 \to \operatorname{Im} P_0$ by $Sx = P_0 x$. From (4.9) we see that for $|||W_1 - W_0|||$ sufficiently small the map S is invertible and its inverse is given by $S^{-1}x = Rx + x$. Note that (cf. Lemma I.3.1 in [10])

$$(4.10a) \qquad ||S^{-1}|| \le (1 - ||P_1 - P_0||)^{-1} ;$$

$$(4.10b) \qquad ||R|| \le ||P_1 - P_0|| (1 - ||P_1 - P_0||)^{-1} .$$

In view of Theorem 1.1 and the similarity of spectral nodes for a given function, we may assume that $X = \operatorname{Im} P_0$ and that the operators $A_0 : \operatorname{Im} P_0 \to \operatorname{Im} P_0$, $B_0 : Y \to \operatorname{Im} P_0$ and $C_0 : \operatorname{Im} P_0 \to Y_0$ are given by

(4.11) $\quad A_0 = T_0|\operatorname{Im} P_0$;

(4.12) $\quad (B_0 y)(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Delta} (z-w)^{-1} W_0(w)^{-1} y\,dw$,

where Δ is a bounded Cauchy domain such that $\Sigma(W_0) \subset \Delta \subset \overline{\Delta} \subset \Omega$;

(4.13) $\quad C_0 f = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} f(w)\,dw$.

Now replace in the right-hand sides of (4.11), (4.12) and (4.13) the index 0 by 1 and denote these new operators by $\tilde{A}_1$, $\tilde{B}_1$ and $\tilde{C}_1$, respectively. Then $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1; \operatorname{Im} P_1, Y)$ is a spectral node for W_1 on Ω (by Theorem 1.1). Next assume that $|||W_1 - W_0|||$ is sufficiently small, and define

$$A_1 = S\tilde{A}_1 S^{-1}, \quad B_1 = S\tilde{B}_1, \quad C_1 = \tilde{C}_1 S^{-1},$$

where S is as in the previous paragraph. By similarity $(A_1, B_1, C_1; \operatorname{Im} P_0, Y)$ is a spectral node for W_1. Since both S and S^{-1} are uniformly bounded for $|||W_1 - W_0|||$ sufficiently small, the inequality (4.6) is easily derived from (4.8), (4.10b) and the particular form the operators have. □

A closer inspection of the proof of Theorem 4.5 yields explicit bounds for the numbers η and K, but we shall not go into this here (cf. [2], Chapter VII).

Proof of Theorem 4.2. Let W_1 be a stable right divisor of W on $\overline{\Omega}$. Choose spectral nodes $\theta = (A, B, C; X, Y)$ and $\theta_1 = (A_1, B_1, C_1; X, Y)$ for W and W_1, respectively. Put

$$S = \frac{1}{2\pi i}\int_{\partial\Omega} (\lambda - A)^{-1} B W(\lambda) C_1 (\lambda - A_1)^{-1} d\lambda .$$

From Proposition 4.1 we know that $N = \operatorname{Im} S$ is the supporting subspace of (C, A) associated with the right divisor W_1. We have to prove that N is a stable A-invariant subspace. Take $\varepsilon > 0$. Let $\tilde{A} : X \to X$ be a bounded linear operator. By Theorem 4.3 there exist positive constants η, K such that $\|A - \tilde{A}\| < \eta$ implies that $(\tilde{A}, B, C; X, Y)$ is a spectral node for some $\tilde{W} \in F(\Omega)$ with

$$|||\tilde{W} - W||| \le K\|\tilde{A} - A\| .$$

Now recall that W_1 is a stable right divisor. So there exists $0 < \delta < \varepsilon$ such that $\|\tilde{A} - A\| < \delta$ implies that the corresponding $\tilde{W}$ has a right divisor $\widetilde{W}_1$ with $|||\widetilde{W}_1 - W_1||| < \varepsilon$. Next we apply Theorem 4.5. Assuming that $\varepsilon > 0$ has been chosen sufficiently small, there exists a spectral node $\widetilde{\theta}_1 = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1; X, Y)$ for $\widetilde{W}_1$ such that

$$||\widetilde{A}_1 - A_1|| + ||\widetilde{B}_1 - B_1|| + ||\widetilde{C}_1 - C_1|| \leq K_1|||\widetilde{W}_1 - W_1|||,$$

where K_1 is a positive constant independent of $\widetilde{W}_1$. Put $\widetilde{N} = \operatorname{Im}\widetilde{S}$, where

$$\widetilde{S} = \frac{1}{2\pi i}\int_{\partial\Omega} (\lambda-\widetilde{A})^{-1}B\widetilde{W}(\lambda)\widetilde{C}_1(\lambda-\widetilde{A}_1)^{-1}d\lambda .$$

Then $\widetilde{N}$ is the supporting subspace of the pair $(C,\widetilde{A})$ corresponding to the right divisor $\widetilde{W}$. In particular, $\widetilde{N}$ is a closed $\widetilde{A}$-invariant subspace. Since S is left invertible, there exists a positive constant γ such that $\operatorname{gap}(N,\widetilde{N}) \leq \gamma\,||S - \widetilde{S}||$ for $||S - \widetilde{S}||$ sufficiently small. So $\operatorname{gap}(N,\widetilde{N})$ can be made as small as we want, and thus N is a stable A-invariant subspace.

To prove the converse, assume that N is a stable A-invariant subspace. Let $(A,B,C;X,Y)$ be a spectral node for W. By Theorem 4.5 there exist $\eta > 0$ and $K > 0$ such that $|||\widetilde{W} - W||| < \eta$ implies that $\widetilde{W}$ has a spectral node $\widetilde{\theta} = (\widetilde{A},\widetilde{B},\widetilde{C};X,Y)$ with

$$(4.14)\qquad ||A - \widetilde{A}|| + ||B - \widetilde{B}|| + ||C - \widetilde{C}|| \leq K|||W - \widetilde{W}||| .$$

We assume that $|||W - \widetilde{W}||| < \eta$, and we take $\widetilde{\theta}$ as above. Take $\varepsilon > 0$. Since N is stable, we can choose η sufficiently small such that the operator $\widetilde{A}$ has a closed invariant subspace $\widetilde{N}$ with

$$(4.15)\qquad \operatorname{gap}(N,\widetilde{N}) < \varepsilon .$$

Recall that N is complemented in X. So there exists a projection P of X onto N. Assuming ε to be sufficiently small we may conclude from (4.15) that

$$(4.16)\qquad X = \operatorname{Ker}P \oplus N = \operatorname{Ker}P \oplus \widetilde{N} .$$

In particular, $\widetilde{N}$ is complemented in X, and hence the pair $(\widetilde{C}|_{\widetilde{N}},\widetilde{A}|_{\widetilde{N}})$ is a right restriction of the pair $(\widetilde{C},\widetilde{A})$. Define $S : \widetilde{N} \to N$ by setting $Sx = Px$. According to (4.16) the operator S is invertible. For η and ε sufficiently small the pair $(\widetilde{C}S^{-1},S\widetilde{A}S^{-1})$ is close to the pair $(C|_N,A|_N)$. To see this one uses (4.14) and similar arguments as in the proof of Theorem 4.5. By Corollary 4.4, there exists $\widetilde{W}_1 \in F(\Omega)$ so that $|||\widetilde{W}_1 - W_1|||$ is small and the pair $(\tilde{C}S^{-1}, S\tilde{A}S^{-1})$ is a right spectral pair of $\tilde{W}_1$. But then $\tilde{W}_1$ must be a right divisor of W and the proof is complete. □

There is a somewhat other version of Theorem 4.2, which is useful in connection with spectral divisors. Let $W \in F(\Omega)$, and let $W_1 \in F(\Omega)$ be a right divisor of W on $\bar{\Omega}$. We call W_1 *Lipschitz-stable* if there exist positive constants η and K with the following property: every operator function $\tilde{W} \in F(\Omega)$ with $|||\tilde{W} - W||| < \eta$ has a right divisor $\tilde{W}_1 \in F(\Omega)$ such that

$$|||\tilde{W}_1 - W_1||| \leq K|||\tilde{W} - W||| .$$

Similarly, a closed invariant subspace $\tilde{N}$ of the operator $A : X \to X$ is called *Lipschitz-stable* if there exist positive constants η and K with the following property: for every operator $\tilde{A}$ on X with $||\tilde{A} - A|| < \eta$ there exists a closed $\tilde{A}$-invariant subspace N of X such that

$$\mathrm{gap}(\tilde{N}, N) \leq K\,||\tilde{A} - A|| .$$

Using the same arguments as in the proof of Theorem 4.2 one can show that Theorem 4.2 remains true if in the statement of this theorem the words "stable" are replaced by "Lipschitz-stable".

The remark made in the previous paragraph is useful in applications to spectral divisors. Let (C,A) be a right spectral pair of $W \in F(\Omega)$, and let $W_1 \in F(\Omega)$ be a right divisor of W on Ω. According to Proposition 2.4 the function W_1 is a spectral divisor if and only if the supporting subspace N of (C,A) associated with the divisor W_1 is a spectral subspace of A, i.e.,

$$(4.17) \qquad N = \mathrm{Im}\left(\frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} d\lambda\right) ,$$

where Γ is a suitable contour in Ω such that $\Gamma \cap \sigma(A) = \emptyset$. From (4.17) it is clear that spectral subspaces are Lipschitz-stable. So we have the following corollary.

COROLLARY 4.6. *A right spectral divisor of $W \in F(\Omega)$ is Lipschitz-stable.*

It turns out that in the finite dimensional case (i.e., $\dim Y < \infty$) the only Lipschitz-stable right divisors of $W \in F(\Omega)$ are the spectral divisors. To prove this fact note that in this case one can make a spectral node $(A,B,C;X,Y)$ for W on Ω such that dim $X < \infty$. But then one can apply the next theorem to get the desired result.

THEOREM 4.7. *Let $A : X \to X$ be an operator acting on a finite dimensional space X. An A-invariant subspace of X is Lipschitz-stable if and only if it is a spectral subspace of A.*

Proof. In view of the description of the stable invariant subspaces in the finite dimensional case (given before Theorem 4.2), we have to check that the only Lipschitz-stable invariant subspaces of the Jordan block

$$J = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad : \quad \mathbb{C}^n \to \mathbb{C}^n$$

are the trivial spaces (0) and $\mathbb{C}^n$. Here $\mathbb{C}^n$ is considered as a Hilbert space with the usual inner product. For $\varepsilon > 0$ let

$$J_\varepsilon = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \\ \varepsilon & 0 & 0 & \cdots & 0 \end{bmatrix} .$$

Then the operator J_ε has n different eigenvalues $\varepsilon_1, \ldots, \varepsilon_n$, which are the n different roots of the equation $x^n - \varepsilon = 0$. The corresponding eigenvectors are $y_i = (1, \varepsilon_i, \ldots, \varepsilon_i^{n-1})^T$, $i = 1, \ldots, n$. The only J_ε-invariant subspaces are those spanned by any subset of $\{y_1, \ldots, y_n\}$. For $k = 1, \ldots, n-1$ the only k-dimensional J-invariant subspace N_k is spanned by the first k unit coordinate vectors. Denote by P_k the orthogonal projection onto N_k, and let $P_{k,\varepsilon}$ denote the orthogonal projection onto a k-dimensional J_ε-invariant subspace $N_{k,\varepsilon}$ $(1 \le k \le n-1)$. For some i we have $y_i \in N_{k,\varepsilon}$. So

$$\text{gap } (N_k, N_{k,\varepsilon}) = \|P_k - P_{k,\varepsilon}\| \ge \frac{1}{\|y_i\|} \|P_k y_i - P_{k,\varepsilon} y_i\| = \\ = \left\{ \left(\sum_{j=k}^{n-1} |\varepsilon_i^j|^2 \right) \Big/ \left(\sum_{j=0}^{n-1} |\varepsilon_i^j|^2 \right) \right\}^{\frac{1}{2}} .$$

Now use that $|\varepsilon_i| = \sqrt[n]{\varepsilon}$. One finds that for ε sufficiently small

$$\operatorname{gap}(N_k, N_{k,\varepsilon}) \geq \tfrac{1}{2}\varepsilon^{k/n} .$$

On the other hand, $\|J - J_\varepsilon\| = \varepsilon$. But then it is clear that for $1 \leq k \leq n-1$ the space N_k is not a Lipschitz-stable invariant subspace of J, and thus J does not have any non-trivial Lipschitz-stable invariant subspaces. □

5. EXTENSION OF DOMAINS

Let $W : \Omega \to L(Y)$ be an analytic operator function with compact spectrum. Assume Ω_0 is an open set containing Ω as a subset. For example, if Ω is not simply connected, one could take Ω_0 to be the union of Ω and all its "holes". In this section we deal with the following question: Is it true that a spectral pair (right or left) for W on Ω is also a spectral pair for some analytic operator function on the larger set Ω_0?
The next theorem shows that the answer is yes.

THEOREM 5.1. *If (C,A) (resp. (A,B)) is a right (resp. left) spectral pair on Ω for an analytic function $W : \Omega \to L(Y)$ with compact spectrum, then (C,A) (resp. (A,B)) is also a right (resp. left) spectral pair on $\mathbb{C}$ for an entire function $Z : \mathbb{C} \to L(Y)$ (also with compact spectrum).*

We have no explicit formula for the entire operator function appearing in the above theorem. However, in the Hilbert space case for extensions to bounded and certain unbounded domains Ω_0 explicit formulas for the new operator function with the same spectral pair may be given. This will be shown in Part III of the paper. In its present form Theorem 5.1 is proved by using a result concerning the triviality of certain analytic cocycles. For the sake of completeness we shall state this result (without proof) in the form it will be needed.

Let $GL(Y)$ be the group of all invertible operators acting in Y, and let $U = \{U_j\}_{j\in J}$ be an open cover of the complex plane $\mathbb{C}$. An *analytic U-cocycle* is a collection of functions $f = (f_{jk})_{j,k\in J}$, where $f_{jk} : U_j \cap U_k \to GL(Y)$ is analytic and satisfies the following conditions:

$$f_{jk}(\lambda)f_{kl}(\lambda) = f_{jl}(\lambda) \qquad (\lambda \in U_j \cap U_k \cap U_l) .$$

The analytic U-cocycle f is called *trivial* if there exist analytic functions $g_j : U_j \to GL(Y)$ such that $f_{jk}(\lambda) = g_j(\lambda)(g_k(\lambda))^{-1}$, $\lambda \in U_j \cap U_k$.

THEOREM 5.2. *Let U be an open cover of $\mathbb{C}$. Then every analytic U-cocycle is trivial.*

More general versions of Theorem 5.2 are stated and proved in [15]; they can be deduced also from general results on infinite dimensional fibre bundles (see [4]).

Proof of Theorem 5.1. Let (C,A) be a right spectral pair for W on Ω. Consider the open cover $U = \{U_1,U_2\}$ of $\mathbb{C}$, where $U_1 = \Omega$ and $U_2 = \mathbb{C} \setminus \Sigma(W)$. For $i,j = 1,2$ define an analytic operator function $f_{i,j} : U_i \cap U_j \to L(Y)$ by setting $f_{11}(\lambda) = I$, $f_{12}(\lambda) = W(\lambda)$, $f_{21}(\lambda) = W(\lambda)^{-1}$ and $f_{22}(\lambda) = I$. For $\lambda \in U_i \cap U_j$ the value of $f_{ij}(\lambda) \in GL(Y)$. Obviously, $(f_{ij})_{i,j=1,2}$ is an analytic cocycle. By Theorem 5.2 this cocycle is trivial. So there exist analytic functions $g_1 : U_1 \to GL(Y)$ and $g_2 : U_2 \to GL(Y)$ such that $f_{12} = g_1g_2^{-1}$ and $f_{21} = g_2g_1^{-1}$. Define the entire operator function $Z : \mathbb{C} \to L(Y)$ by setting $Z(\lambda) = g_2^{-1}(\lambda)$ for $\lambda \in U_2$ and $Z(\lambda) = g_1^{-1}(\lambda)W(\lambda)$ for $\lambda \in \Omega$. Then

$$W(\lambda) = g_1(\lambda)Z(\lambda) , \qquad \lambda \in \Omega . \tag{5.1}$$

Since the operator function $g_1(\lambda)$ has invertible values for $\lambda \in \Omega$, we may conclude from (5.1) (see Corollary 2.3) that (C,A) is a right spectral pair for Z on Ω. But since $Z(\lambda)$ is invertible for all $\lambda \in \mathbb{C} \setminus \Sigma(W)$, it is easily seen that (C,A) is a right spectral pair for Z on the entire complex plane. The statement about left spectral pairs is proved analogously. □

COROLLARY 5.3. *Let (C,A) be a right spectral pair for an analytic operator function $W : \Omega \to L(Y)$ with compact spectrum. Then any spectral subspace of A is a supporting subspace of (C,A) on Ω.*

Proof. Let σ be a compact and relatively open subset of the spectrum of A, and let N be the corresponding spectral subspace. Let Ω_0 be an open subset of Ω such that $\sigma \subset \Omega_0$ and $(\Sigma(W)\setminus\sigma) \cap \Omega_0 = \emptyset$. Then $(C|_N,A|_N)$ is a right spectral pair for W on Ω_0. Now apply Theorem 5.1. So there exists an entire operator function $Z : \mathbb{C} \to L(Y)$ such that $(C|_N, A|_N)$ is a right spectral pair for Z on $\mathbb{C}$. Put $W_1(\lambda) = Z(\lambda)$ for $\lambda \in \Omega$. Then $(C|_N, A|_N)$ is also a spectral pair for W_1 on Ω, and the corollary is proved. □

We do not know whether Corollary 5.3 holds for any complemented A-invariant subspace N such that $\sigma(A|_N) \subset \Omega$ (and not only for spectral subspaces, as asserted in Corollary 5.3). However, in case Y is a Hilbert space and Ω is bounded, this more general statement holds true indeed. This we shall prove in Part III without using the cocycle theory.

Let $W : \Omega \to L(Y)$ be an analytic operator fucntion with compact spectrum. Assume that σ is a compact and relatively open subset of $\Sigma(W)$. Using Theorems 5.1 and 5.2 one can show that W has a spectral right divisor W_1 on Ω such that $\Sigma(W_1) = \sigma$. Indeed, according to Theorem 5.1 and Corollary 2.3 the function W admits the following factorization:

$$W(\lambda) = E(\lambda)Z(\lambda), \qquad \lambda \in \Omega,$$

where $E(\lambda)$ is an invertible operator depending analytically on the parameter $\lambda \in \Omega$ and $Z : \mathbb{C} \to L(Y)$ is an entire operator function with $\Sigma(W) = \Sigma(Z)$. Put $\sigma_1 = \sigma$ and $\sigma_2 = \Sigma(W) \setminus \sigma_1$. Then $\Sigma(Z)$ is the disjoint union of the two compact sets σ_1 and σ_2. Since Z is entire, Theorem 5.2 implies that there exists a factorization $Z(\lambda) = Z_2(\lambda)Z_1(\lambda)$ for $\lambda \in \Omega$ such that for $i = 1,2$ the function $Z_i : \mathbb{C} \to L(Y)$ is an entire operator function with $\Sigma(Z_i) = \sigma_i$. It follows that for W we have the following factorization:

$$W(\lambda) = E(\lambda)Z_2(\lambda)Z_1(\lambda), \qquad \lambda \in \Omega, \tag{5.2}$$

where E, Z_1 and Z_2 are as above. For $\lambda \in \Omega$ put $W_1(\lambda) = Z_1(\lambda)$. Then, obviously, the function W_1 is a spectral right divisor of on Ω and $\Sigma(W_1) = \sigma$. On the basis of Theorems 5.1 and 5.2 it is also possible to obtain factorizations of $W(\lambda)$ that are analogous to the ones of (5.2), but have $E(\lambda)$ as the middle factor or as the third factor.

6. THE FINITE COLUMN CONDITION

The following theorem is one of the main results of the present paper.

THEOREM 6.1. *Let* C, A *(resp.* (A,B)*) be a right (resp. left) spectral pair of an analytic function* $W : \Omega \to L(Y)$ *with compact spectrum, where* $A : X \to X$*. Then for* m *large enough the operator*

$$(6.1) \qquad \begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{m-1} \end{pmatrix} : X \to Y^m$$

is left invertible and the operator

$$(6.2) \qquad \begin{pmatrix} B & AB & \dots & A^{m-1}B \end{pmatrix} : Y^m \to X$$

is right invertible.

Proof. By Theorem 5.1 we may assume that for some $r > 0$

$$(6.3) \qquad \sigma(A) \subset \{\lambda \mid |\lambda| < r\} \subset \{\lambda \mid |\lambda| \le r\} \subset \Omega .$$

Choose B such that $(A,B,C;X,Y)$ is a spectral node for W on Ω. Put $U(\lambda) = (\lambda - A)^{-1}BW(\lambda)$. We know that $U(\lambda)$ is analytic on Ω, and so $U(\lambda) = \Sigma_{j=0}^{\infty} \lambda^n U_n$ for $|\lambda| \le r$. By Corollary 1.3 in [20]

$$I = \frac{1}{2\pi i} \int_{|\lambda|=r} (\lambda - A)^{-1}BW(\lambda)C(\lambda - A)^{-1}d\lambda = \Sigma_{j=0}^{\infty} U_j CA^j .$$

It follows that $\Sigma_{j=0}^{m} U_j CA^j$ is invertible for m sufficiently large. So the operator (6.1) is left invertible for m large enough. The right invertibility of (6.2) for some $m > 0$ can be proved in a similar way. □

In part III we shall prove that for X and Y Hilbert spaces the converse of Theorem 6.1 holds true.

Let m_1 be the smallest non-negative integer m such that the operator (6.1) is left invertible, and let m_2 be the smallest non-negative integer m such that the operator (6.2) is right invertible. The numbers m_1 and m_2 are uniquely determined by the operator function W and they do not depend on the particular choice of the pairs (C,A) and (A,B). We call m_1 (resp. m_2) the *right* (resp. *left*) *degree* of W on Ω. If $W(\lambda) = \Sigma_{j=0}^{l} \lambda^j A_j$ is an operator polynomial, then both m_1 and m_2 are less than or equal to l. The numbers m_1 and m_2 will play a role in Section 7.

Theorem 6.1 can be used to give a somewhat simplified version of Theorem 3.1. For brevity let us use the symbol $Q_m(C,A)$ to denote the operator defined by (6.1). If for some m the operator $Q_m(C,A)$ is left invertible, then the same is true for the operator S appearing in the statement (Q_3) of Theorem 3.1. More precisely, let $C : X \to Y$ and $A : X \to X$ be operators, and assume that $\sigma(A) \subset \Omega$, where Ω is an open set in $\mathbb{C}$. Consider the operator $S : X \to C(\partial\Delta, Y)$ defined by

$$(Sx)(z) = C(z-A)^{-1}x \ , \qquad z \in \partial\Delta \ .$$

Here Δ is a bounded Cauchy domain such that $\sigma(A) \subset \Delta \subset \overline{\Delta} \subset \Omega$. Now assume that for some $m > 0$ the operator $Q_m(C,A)$ is left invertible. Choose $S_j^+ : Y \to X$, $j=0, \ldots, m-1$, such that $(S_0^+ \ \ldots \ S_{m-1}^+)$ is a left inverse of the operator $Q_m(C,A)$. Define $S^+ : C(\partial\Delta, Y) \to X$ by setting

$$S^+f = \sum_{j=0}^{m-1} S_j^+ \left(\frac{1}{2\pi i} \int_{\partial\Delta} \lambda^j f(\lambda) d\lambda \right) .$$

Then S^+ is a well-defined bounded operator and S^+ is a left inverse of S.

The observation made above can be employed to combine Theorems 6.1 and 3.1 into the following characterization of right spectral pairs.

THEOREM 6.2. *A pair of operators $C : X \to Y$ and $A : X \to X$ is a right spectral pair on Ω for an analytic function $W : \Omega \to L(Y)$ with compact spectrum if and only if the following four conditions are fulfilled:*

(α_1) $\sigma(A) \subset \Omega$;

(α_2) *$W(\lambda)C(\lambda-A)^{-1}$ has an analytic continuation to Ω;*

(α_3) *for some $m > 0$ the operator $Q_m(C,A)$ is left invertible;*

(α_4) *every other pair of operators satisfying (α_1), (α_2) and (α_3) is a right restriction of (C,A).*

Proof. Let (C,A) be a spectral pair for W on Ω. Then (α_1) and (α_2) hold by definition. Further, because of Theorem 6.1, statement (α_3) is true. To prove (α_4), let (C_0,A_0) be a pair of operators with the properties (α_1), (α_2) and (α_3). Then, by the observations made in the paragraphs preceding the present theorem, the pair (C_0, A_0) satisfies the conditions (Q_1), (Q_2) and (Q_3) of Theorem 3.1. So (C_0, A_0) is a right restriction of the pair (C, A).

Now conversely, let (C, A) be a pair of operators $C : X \to Y$ and $A : X \to X$ which satisfies the conditions (α_1), (α_2), (α_3) and (α_4). Consider the operators Q and V defined by formulas (1.3) and (1.1), respectively. We know that the pair (Q, V) is a right spectral pair for W on Ω. So, by the first part of the proof, the pair (Q, V) has the properties (α_1), (α_2), (α_3) and (α_4). So, using (α_4) for (C, A), the pair (Q, V) is a right restriction of the pair (C, A). On the other hand, by (α_4) for (Q, V), the pair (C, A) is a right restriction of (Q, V). But then the two pairs are similar, and hence (C, A) is a right spectral pair for W on Ω. □

In an analogous way one can prove that Theorem 3.2 remains true if in Theorem 3.2 the condition (Q_3') is replaced by : for some $m > 0$ the operator

$$\begin{pmatrix} B & AB & \dots & A^m B \end{pmatrix} \quad : \quad Y^m \to X$$

is right invertible.

The left invertibility of the finite column operator $Q_m(C, A)$ has as a consequence that *any analytic operator function* $W : \Omega \to L(Y)$ *with compact spectrum appears as a right divisor of a monic operator polynomial,* in the following sense: there exists an analytic operator function $V : \Omega \to L(Y)$ (not necessarily with compact spectrum) such that the product $V(\lambda)W(\lambda)$ is an operator polynomial with leading coefficient I. To prove this assertion, first observe that in view of Theorem 5.1 and Corollary 2.3 it is sufficient to consider the case when W is entire (i.e., $\Omega = \mathbb{C}$). Let (C, A) be a be a right spectral pair of W on $\mathbb{C}$. Choose $U_j : Y \to X$, $j = 0, \dots, m-1$, such that the operator $\begin{pmatrix} U_0 & \dots & U_{m-1} \end{pmatrix}$ is a left inverse of the operator $Q_m(C, A)$. Put

$$L(\lambda) = \lambda^m I - CA^m(U_0 + \lambda U_1 + \dots + \lambda^{m-1} U_{m-1}) \ .$$

By [8], Theorem 7.1, the pair (C, A) is a right restriction of a right spectral pair $(\tilde{C}, \tilde{A})$ of L on $\mathbb{C}$. By Theorem 2.1 this implies that W is a right divisor of L on $\mathbb{C}$.

7. APPLICATIONS TO WIENER-HOPF FACTORIZATION

In this section the left invertibility of the finite column $Q_m(C,A)$ is used to derive necessary and sufficient conditions in order that an analytic operator function with compact spectrum admits a generalized Wiener-Hopf factorization (cf. [16,24]).

Let Γ be a domain bounded by the simple closed rectifiable contour Δ, and such that $0 \in \Delta$. An operator valued function $W : \Gamma \to L(Y)$ with invertible values is said to admit a *left (generalized) Wiener-Hopf factorization* (with respect to Γ) if the following representation holds:

$$(7.1) \qquad W(\lambda) = E_-(\lambda)\left(\sum_{i=1}^{r} \lambda^{\nu_i} P_i\right)E_+(\lambda) , \qquad \lambda \in \Gamma ,$$

where the continuous operator function $E_- : \mathbb{C}_\infty\backslash\Delta \to L(Y)$ is analytic on $\mathbb{C}_\infty\backslash\Delta$ and all its values are invertible, the continuous operator function $E_+ : \overline{\Delta} \to L(Y)$ is analytic in Δ and all its values are invertible, $P_1, \ldots ,P_r$ are projections with $P_iP_j = P_jP_i = 0$ for $i \neq j$ and $P_1 + \ldots + P_r = I$, the numbers $\nu_1 < \nu_2 < \ldots < \nu_r$ are integers (positive, negative or zero). Interchanging E_+ and E_- in (7.1), we obtain a *right (generalized) Wiener-Hopf factorization.*

An analytic operator function W on Δ with continuous and invertible boundary values does not always admit a Wiener-Hopf factorization with respect to $\Gamma = \partial\Delta$ (see [16] and the references given there). Here we present criteria (cf.[16,24]) for the possibility of Wiener-Hopf factorization in terms of the moments of $W(\lambda)^{-1}$ with respect to Γ. For a continuous function $V : \Gamma \to L(Y)$ we define the operators of moments to be:

$$M_{pq}(V) = \frac{1}{2\pi i}\int_\Gamma \begin{pmatrix} V(\lambda) & \lambda V(\lambda) & \ldots & \lambda^{q-1}V(\lambda) \\ \lambda V(\lambda) & \lambda^2 V(\lambda) & \ldots & \lambda^q V(\lambda) \\ \vdots & \vdots & & \vdots \\ \lambda^{p-1}V(\lambda) & \lambda^p V(\lambda) & \ldots & \lambda^{p+q-2}V(\lambda) \end{pmatrix} d\lambda \; : Y^q \to Y^p.$$

Let $W : \overline{\Delta} \to L(Y)$ be a continuous operator function which is analytic on Δ and has invertible values on $\Gamma = \partial\Delta$, and let $(A,B,C;X,Y)$ be a spectral node for W on Δ. Then the operators of moments can be expressed in terms of A, B and C in the following way:

$$(7.2)\qquad M_{pq}(W^{-1}) = Q_p(C,A)\begin{pmatrix}B & AB & \dots & A^{q-1}B\end{pmatrix}.$$

To see this, note that $W(\lambda)^{-1} - C(\lambda-A)^{-1}B$ has an analytic extension to Δ, and hence

$$CA^jB = \frac{1}{2\pi i}\int_\Gamma \lambda^j W(\lambda)^{-1}d\lambda\ .$$

Recall that $Q_p(C,A)$ is left invertible whenever $p \geq m_1$, where m_1 is the right degree of W on Ω (see the previous section). Similarly, $\begin{pmatrix}B & AB & \dots & A^{q-1}B\end{pmatrix}$ is right invertible whenever $q \geq m_2$, where m_2 is the left degree of W on Ω. It follows that $M_{pq}(W^{-1})$ has a generalized inverse (i.e., $M_{pq}(W^{-1})$ has a closed and complemented range and a complemented kernel) for $p \geq m_1$ and $q \geq m_2$.

THEOREM 7.1. *Let $W : \overline{\Delta} \to L(Y)$ be a continuous operator function which is analytic on Δ and has invertible values on $\Gamma = \partial\Delta$. Denote by m_1 and m_2 the right and left degree of W on Δ, respectively. Then W admits a left Wiener-Hopf factorization with respect to Γ if and only if the operators*

$$(7.3)\qquad M_{m_1,1}(W^{-1})\ ,\ \dots\ ,\ M_{m_1,m_2-1}(W^{-1})$$

have generalized inverses, and W admits a right Wiener-Hopf factorization with respect to Γ if and only of the operators

$$(7.4)\qquad M_{1,m_2}(W^{-1})\ ,\ \dots\ ,\ M_{m_1-1,m_2}(W^{-1})$$

have generalized inverses.

For the case when W is a polynomial an analogous version of Theorem 7.1 has been proved in [24]. The proof of Theorem 7.1 will be based on the following result obtained in [9], Section III.3.3 .

THEOREM 7.2. *Let $W : \overline{\Delta} \to L(Y)$ be a continuous operator function which is analytic on Δ and has invertible values on $\Gamma = \partial\Delta$, and let $(A,B,C;X,Y)$ be a spectral node for W on Δ. Then W admits a left Wiener-Hopf factorization with respect to Γ if and only if there exists a positive integer m such that the operator $\begin{pmatrix}B & AB & \dots & A^{j-1}B\end{pmatrix} : Y^j \to X$ has a generalized inverse for $j = 1, \dots, m-1$ and is right invertible for $j = m$. The function W admits a right Wiener-Hopf factorization with respect to Γ if and* *only if there exists a positive integer n such that the operator*

$$Q_j(C,A) = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{j-1} \end{bmatrix} : X \to Y^j$$

has a generalized inverse for $j = 1, \ldots, n-1$ *and is left invertible for* $j = n$.

Proof of Theorem 7.1. Assume that W admits a left Wiener-Hopf factorization. Let $(A,B,C;X,Y)$ be a spectral node for W on Δ. By Theorem 7.2 all operators $(B \quad AB \quad \ldots \quad A^{j-1}B)$, $j = 1,2,\ldots,m-1$, have generalized inverses. Since $Q_m(C,A)$ is left invertible, it follows from (7.2) that all operators $M_{m_1,j}(W^{-1})$, $j = 1,2,\ldots$ have generalized inverses.

Conversely, assume that the operators (7.3) have generalized inverses. Since $Q_m(C,A)$ is left invertible, we may conclude from (7.2) that the operators $(B \quad AB \quad \ldots \quad A^{j-1}B)$, $j = 1, \ldots, m_2-1$, have generalized inverses. We know already that the operator $(B \quad AB \quad \ldots \quad A^{m_2-1}B)$ is right invertible. So Theorem 7.2 implies that W admits a left Wiener-Hopf factorization.

For the right Wiener-Hopf factorization the proof is analogous.

We conclude with a few remarks about Theorem 7.1. Let W be as in Theorem 7.1. From (7.2) and the left invertibility of $Q_{m_1}(C,A)$ it is clear that for $l \geq m_1$ the operator $M_{li}(W^{-1})$ has a generalized inverse if and only if $M_{m_1 i}(W^{-1})$ has a generalized inverse ($i = 1,2,\ldots$). Analogously for $l \geq m_2$ the operator $M_{il}(W^{-1})$ has a generalized inverse if and only if $M_{im_2}(W^{-1})$ has a generalized inverse. It follows that the necessary and sufficient condition for W to admit a left Wiener-Hopf factorization can be stated also in the following way: for some $l \geq \max\{m_1,m_2\}$ all the operators

$$M_{l1}(W^{-1}), \ \ldots, \ M_{l\,l-1}(W^{-1}) \tag{7.5}$$

have generalized inverses. Similarly, W admits a right Wiener-Hopf factorization if and only if for some $l \geq \max\{m_1,m_2\}$ all the operators

$$(7.6) \qquad M_{1l}(W^{-1}) \ , \ \ldots \ , \ M_{l-1\,l}(W^{-1})$$

have generalized inverses. For the case when W is an operator polynomial one can take the number l in (7.5) and (7.6) to be equal to the degree of the polynomial (cf.[24]).

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