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TRANSPORT THEORY IN L_p-SPACES

C.V.M. van der Mee

In this article boundary value problems of linear transport theory are studied in L_p-spaces ($1\le p<+\infty$). It is shown that the results valid in L_2-space can also be derived in L_p-space ($1\le p<+\infty$). For a non-multiplying medium formal expressions for the solutions are obtained.

INTRODUCTION

In this article a study is made of the integro-differential equation

$$\mu\frac{d\psi}{dx}(x,\mu)+\psi(x,\mu)=\int_{-1}^{+1}g(\mu,\mu')\psi(x,\mu')d\mu'+f(x,\mu)\qquad(0<x<\tau,\ -1\le\mu\le+1),\tag{0.1}$$

where

$$g(\mu,\mu')=(2\pi)^{-1}\int_0^{2\pi}\hat{g}(\mu\mu'+\sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)d\alpha.\tag{0.2}$$

This equation describes the time-independent transfer of unpolarized radiation through a stellar or planetary atmosphere (cf. [3,23,16]) or the stationary transport of neutrons with uniform speed through a fuel plate of a nuclear reactor (see [4]). In this equation x is a position coordinate, τ is the thickness of the atmosphere or fuel plate (in suitable units) and μ is the cosine of the angle describing the direction of propagation. The real-valued function $\hat{g}$ describes the scattering properties of the medium and is called the phase function. It includes the albedo of collision rate $c=\int_{-1}^{+1}\hat{g}(t)dt$ as a factor. The term $f(x,\mu)$ accounts for internal radiative or neutron sources. The function $\psi(x,\mu)$ represents the (azimuth-averaged) intensity of the radiation or the angular density of the neutrons. Given $\hat{g}$ and f, the problem is to determine the unknown function ψ under suitable boundary conditions.

To treat Eq. (0.1) within an operator-theoretic framework one usually considers the Hilbert space $H = L_2[-1,+1]$ of square integrable functions on $[-1,+1]$ (cf. [14,22], for instance) and introduces the vectors $\psi(x)$ and $f(x)$ in $L_2[-1,+1]$ and the operators T and B on $L_2[-1,+1]$ by

(0.3a) $\psi(x)(\mu) = \psi(x,\mu), \quad f(x)(\mu) = f(x,\mu);$

(0.3b) $(Th)(\mu) = \mu h(\mu), \quad (Bh)(\mu) = \int_{-1}^{+1} g(\mu,\mu')h(\mu')d\mu'. \quad (0<x<\tau,\ -1\le\mu\le+1)$

Now Eq. (0.1) can be restated in the form of the operator differential equation

(0.4) $(T\psi)'(x) = -(I-B)\psi(x) + f(x) \quad (0<x<\tau)$

with suitable boundary conditions.

In many physical and mathematical works instead of $L_2[-1,+1]$ one also considers other spaces of functions on $[-1,+1]$. The asymptotics of the solutions of (an equivalent form of) Eq. (0.1) can be described in $L_p[-1,+1]$ for $1\le p<+\infty$ (cf. [6]). The (incomplete) normed space $T^{-1}L_p[-1,+1]$ has been used in [19] for $1<p<+\infty$ and in [18] for $p=1$. Further, it seems that neutron physicists commonly believe that $L_1[-1,+1]$ is the "natural" space in which to solve Eq. (0.1) (see [18,13,4]).

In this article we shall obtain formal solutions of Eq. (0.1) in $L_p[-1,+1]$ $(1\le p<+\infty)$. If $\hat{g} \in L_r[-1,+1]$ for some $r>1$, a transport theory is developed in $L_p[-1,+1]$ $(1\le p<+\infty)$ by extending the approach of [22], which contains a theory in $L_2[-1,+1]$. Throughout this work we restrict ourselves to non-multiplying media, where $a_n = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}(t)P_n(t)dt \le +1$ $(n = 0,1,2,\ldots)$ and $P_0,P_1,\ldots$ are the usual Legendre polynomials. Basically the same results are obtained for different function spaces.

To impose boundary conditions on Eq. (0.4), in $L_2[-1,+1]$ one defines the complementary projections P_+ and P_- by

(0.5) $(P_\pm h)(\mu) = h(\mu) \quad (\mu \gtrless 0), \quad (P_\pm h)(\mu) = 0 \quad (\mu \lessgtr 0).$

For finite τ one imposes the boundary conditions

(0.6) $\lim_{x\downarrow 0} P_+\psi(x) = P_+\phi, \quad \lim_{x\uparrow\tau} P_-\psi(x) = P_-\phi,$

where $\phi \in L_2[-1,+1]$ is given, and one calls this boundary value problem the *finite-slab problem*. For infinite τ one imposes the boundary conditions

$$(0.7) \qquad \lim_{x\downarrow 0} P_+\psi(x) = \phi_+, \qquad \|\psi(x)\| = O(1) \quad (x\to+\infty),$$

where $\phi_+ \in L_2[0,1]$ is given and the norm is taken in $L_2[-1,+1]$; the boundary value problem (0.4) - (0.7) is called the *half-space problem*. In an analogous way these two boundary value problems can be stated in other function spaces too.

Let us give a description of some of the results on the finite-slab and half-space problems. To process the numerous function spaces together we introduce the notion of a C-admissible Banach space H of functions $h:[-1,+1]\to\mathbb{C}$. On such a space H and for $0\neq x\in\mathbb{R}$ we define $H(x)$ by

$$(0.8) \qquad (H(x)h)(\mu) = \begin{cases} |\mu|^{-1}e^{-x/\mu}h(\mu), & x\mu>0; \\ \\ 0 \qquad , & x\mu<0; \end{cases}$$

the function $H(.)$ is called the *propagator function*. On a C-admissible space H it makes sense to define T, P_+, P_- and $H(x)$ $(0\neq x\in\mathbb{R})$ by (0.3b), (0.5) and (0.8) as bounded linear operators. Further, to process the conditions on $\hat{g}$, we call a C-admissible space compatible with $\hat{g}$ (or with B), if the operator B in (0.3b) is a limit in the operator norm of H of operators of finite rank and $\int_{-\infty}^{+\infty}\|H(x)B\|dx < +\infty$. Now on different function spaces the Transport Equation can be treated from one point of view.

If a C-admissible space H is compatible with $\hat{g}$ and T[H] is dense in H, the finite-slab problem (0.4) - (0.6) is proved to be equivalent to the vector-valued convolution equation

$$(0.9) \qquad \psi(x) - \int_0^\tau H(x-y)B\psi(y)dy = \omega(x) \quad (0<x<\tau),$$

where $\omega(x,\mu) = e^{-x/\mu}\phi(\mu) + \int_0^x \mu^{-1}e^{-(x-y)/\mu}f(y,\mu)dy$ $(0<\mu\le 1)$ and $\omega(x,\mu) = e^{(\tau-x)/\mu} - \int_x^\tau \mu^{-1}e^{-(x-y)/\mu}f(y,\mu)dy$ $(-1\le\mu<0)$. Here ϕ is an arbitrary function in H. For every right-hand side $\omega(x)$ the convolution equation (0.9) is shown to be uniquely solvable by comparison to the analogous equation in $L_2[-1,+1]$. In particular, for phase functions $\hat{g}\in L_r[-1,+1]$ with $r>1$ and for non-multiplying media the finite-slab problem (0.4) - (0.6) is proved to be well-posed in the function spaces $L_p[-1,+1]$ $(1\le p<+\infty)$. Analogous results are established in a similar way for the half-space problem (0.4) - (0.7) in non-conservative media, but also in conservative cases the existence (and sometimes the uniqueness) of a solution is proved.

Now that the finite-slab and half-space problems in non-multiplying media have been investigated as to their well-posedness, on a C-admissible space compatible with $\hat{g}$ we define certain analytic semigroups, construct spectral subspaces and projections and prove intertwining properties between pairs of corresponding operators. With the help of these entities formal expressions for the solutions are obtained. In fact, for $1\le p<+\infty$ it is shown that for every $\chi\in L_p[-1,+1]$ and $0<\tau<+\infty$ the operator differential equation

$$(0.10a)\quad (T\psi)'(x) = -(I-B)\psi(x) \quad (0<x<\tau)$$

with boundary conditions

$$(0.10b)\quad \lim_{x\downarrow 0} \|TP_+\psi(x) - P_+\chi\|_p = 0, \quad \lim_{x\uparrow\tau} \|TP_-\psi(x) - P_-\chi\|_p = 0$$

has a unique solution $\psi:(0,\tau)\to L_p[-1,+1]$, provided $\hat{g}\in L_r[-1,+1]$ for some $r>1$. The analogous half-space result is established too. Note that the boundary value problem (0.10) is more general than the problem (0.4) - (0.6) (with $f(x)\equiv 0$).

In this article mathematical methods of a diverse nature are employed. On the one hand, the theory of vector-valued convolution equations (cf. [9,6,7,10,11,8,1]) enables us to treat equations of the form (0.9) on different function spaces at the same time. On the other hand, semigroup theory and especially the use of analytic semigroups (cf. [17], Section IX.1; see also Ch. VIII of [5]) enable us to write down formal expressions for the solution. The theory of the Transport Equation developed in [22] is heavily used.

Let us shortly describe the contents of this paper. After the preliminary first section in which the conditions on the phase function $\hat{g}$ are translated into the compatibility with $\hat{g}$ of certain C-admissible spaces, the equivalence of the finite-slab and half-space problems to a convolution equation of the form (0.9) is proved. In Section 3 we show the finite-slab problem in non-multiplying media and the half-space problem in non-conservative media to be well-posed. In Section 4 this result is extended to deal with the half-space problem in the conservative case. Using the unique solutions of the convolution equation on the full real line, in Section 5 we construct spectral subspaces and projections, analytic semigroups and intertwining operator pairs. These entities enable us to write down formal expressions for the solutions, also for the more general problem (0.10) (see Section 6).

In a future publication we shall show how the present approach extends to multigroup Transport Theory (see [4] for the physical aspects). Among other things we shall construct spectral subspaces and projections, analytic semigroups and intertwining operator pairs analogously to the present sixth section, and derive statements about the partial indices of the symbol of the Wiener-Hopf version of the multigroup half-space problem.

Throughout this article all Banach spaces are complex and $\langle .,. \rangle$ denotes the inner product of a Hilbert space. By Ker T, Im T and $\sigma(T)$ we mean the null space, range and spectrum of an operator T. The symbol $L(H_1,H_2)$ stands for the set of bounded linear operators from H_1 into H_2. We write $L(H)$ for $L(H,H)$. By I_H (or I) we mean the identity operator on H. The Riemann sphere $\mathbb{C}\cup\{\infty\}$is denoted by $\mathbb{C}_\infty$.

1. PRELIMINARIES

Let C be the algebra over $\mathbb{C}$ of functions $\phi = \phi_- \oplus \phi_+$ from $C[-1,0] \oplus C[0,1]$ generated by the functions $\phi_+(\mu) = \phi_-(-\mu) = |\mu|^\alpha$ $(0\le\mu\le1,\ \alpha\ge0)$, the step function $\phi_+(\mu) = \phi_-(-\mu) = \frac{1}{2}(\text{sign}(\mu) + 1)$ $(0<\mu\le1)$ and the functions $\phi_+(\mu) = \phi_-(-\mu) = |\mu|^{\gamma-1}e^{-|x/\mu|}$ $(0<\mu\le1,$ $\gamma\in\mathbb{R},\ 0\ne x\in\mathbb{R})$. Let H be a Banach space of functions $h:[-1,+1]\to\mathbb{C}$ with the property that every two functions $h_1,h_2\in H$ with $h_1(\mu) = h_2(\mu)$ for $0\ne\mu\in[-1,+1]$ are to be identified. Examples of such spaces are $L_2[-1,+1]$ and $C[-1,0]\oplus C[0,1]$. Then H is called *C-admissible* if for every $\phi\in C$ the operator T_ϕ given by

$$(T_\phi h)(\mu) = \phi(\mu)h(\mu) \quad (-1\le\mu\le+1),$$

is bounded on H and $\|T_\phi\| \leq M_H \sup\{|\phi(\mu)|: -1\leq\mu\leq+1\}$, where the constant M_H only depends on H. Clearly, $\sigma(T_\phi) \subset \overline{\{\phi(\mu): -1\leq\mu\leq+1\}}$. Define the operators T, P_+ and P_- by

$$(1.1)\quad (Th)(\mu) = \mu h(\mu);\quad (P_+h)(\mu) = \begin{cases} h(\mu), & \mu\geq 0; \\ 0, & \mu<0; \end{cases} \quad (P_-h)(\mu) = \begin{cases} 0 & \mu\geq 0; \\ h(\mu), & \mu<0. \end{cases}$$

Then T is a bounded operator with $\sigma(T)\subset[-1,+1]$ and P_+ and P_- are complementary projections that commute with T. Their ranges are denoted by H_+ and H_-, respectively. Note that

$$(1.2)\quad H_+ \oplus H_- = H,\quad \sigma(T|_{H_+}) \subset [0,1],\quad \sigma(T|_{H_-}) \subset [-1,0].$$

For $\gamma\in\mathbb{R}$ and $0\neq x\in\mathbb{R}$ we define the operator $|T|^\gamma H(x)$ by

$$(1.3a)\quad (|T|^\gamma H(x)h)(\mu) = \begin{cases} |\mu|^{\gamma-1}e^{-x/\mu}h(\mu), & x\mu>0; \\ 0, & x\mu<0; \end{cases}$$

then $H(x)$ is bounded on every C-admissible space H and for $\gamma<1$ we have

$$(1.3b)\quad \||T|^\gamma H(x)\| \leq M_H \sup_{0<\mu\leq 1} |\mu|^{\gamma-1}e^{-|x|/\mu} = \begin{cases} M_H(|x|/(1-\gamma))^{\gamma-1}e^{\gamma-1}, & 0<|x|\leq 1; \\ 0, & |x|\geq 1. \end{cases}$$

The function $H(.)$ is called the *propagator function.*

PROPOSITION 1.1. *Let* H *be a C-admissible Banach space, and assume that on* H *the operator* T *has a dense range. Then the expressions*

$$(1.4a)\quad (U_+(x)h)(\mu) = \begin{cases} e^{-x/\mu}h(\mu), & \mu\geq 0; \\ 0, & \mu<0; \end{cases} \quad (U_-(x)h)(\mu) = \begin{cases} 0, & \mu\geq 0; \\ e^{+x/\mu}h(\mu), & \mu<0, \end{cases}$$

define bounded analytic semigroups on H_+ *and* H_- *with infinitesimal generators* $(-T|_{H_+})^{-1}$ *and* $(+T|_{H_-})^{-1}$*, respectively. Moreover, for* $h\in H$ *we have*

$$(1.4b)\quad \lim_{\lambda\to 0,\, \mathrm{Re}\lambda\leq 0} \|T(T-\lambda)^{-1}P_+h - P_+h\| = 0,$$

$$\lim_{\lambda\to 0,\mathrm{Re}\,\lambda\geq 0} \|T(T-\lambda)^{-1}P_-h-P_-h\| = 0.$$

Proof. Observe that the propagator function $H(.)$ has an analytic continuation to the non-imaginary part of $\mathbb{C}$, given by

$$(1.5)\qquad (H(x)h)(\mu) = \begin{cases} |\mu|^{-1}e^{-x/\mu}h(\mu), & \mu(\mathrm{Re}\ x) > 0; \\ 0 & , \ \mu(\mathrm{Re}\ x) < 0. \end{cases}$$

In fact, by the C-admissibility of H we have for Re x, Re y > 0 and $h \in H$:

$$\left\|\left[\frac{U_+(x)-U_+(y)}{x-y} - H(x)\right]h\right\|_H \leq \|h\|_H\cdot M_H \sup_{0<\mu\leq 1}\left|\frac{e^{-x/\mu}-e^{-y/\mu}}{x-y} - \frac{1}{\mu}e^{-x/\mu}\right|,$$

and the right-hand side vanishes as $y \to x$. So U_+ is analytic on the open right half-plane with derivative $-H(x)$. Similarly U_- is analytic on the open right half-plane with derivative $-H(-x)$.

By the C-admissibility on H, for Re $x > 0$ and $h \in H$ we have

$$\|T[U_+(x) - I]h\|_H \leq \|h\|_H\cdot M_H \sup_{0<\mu\leq 1}|\mu(1-e^{-x/\mu})|,$$

and for $0<\phi\leq\frac{1}{2}\pi$ the right-hand side vanishes as $x \to 0$ with $|\arg x| \leq \phi$. Since on H the operator T has a dense range and $\|U_+(x) - P_+\| \leq 2M_H$, for $0\leq\phi<\frac{1}{2}\pi$ one gets

$$\lim_{x\to 0,|\arg x|\leq\phi} \|[U_+(x) - P_+]h\|_H = 0, \quad h \in H.$$

Hence, $(U_+(x))_{x\geq 0}$ is a bounded analytic semigroup on H_+ with infinitesimal generator $(-T|_{H_+})^{-1}$.

To prove the identities (1.4b) we employ the C-admissibility of H and derive the estimates

$$\|[T(T-\lambda)^{-1}P_+ - P_+]T\| = \|\lambda T(\lambda-T)^{-1}P_+\| \leq M_H\cdot \sup_{0<\mu\leq 1}|\lambda\mu(\mu-\lambda)^{-1}| \leq M_H\cdot|\lambda|;$$
$$\|\lambda(\lambda-T)^{-1}P_+\| \leq M_H \sup_{0<\mu\leq 1}|\lambda(\lambda-\mu)^{-1}| \leq M_H.$$

From these estimates and the density of the range of T the first one of the identities (1.4b) is clear. The second one is proved analogously. □

For specific C-admissible spaces H such as $L_p[-1,+1]$ and $C[-1,0] \oplus C[0,1]$ this proposition is well-known. Here we organize these properties by one statement.

Let $\hat{g} \in L_1[-1,+1]$, and define an operator B by

$$(1.6) \qquad (Bh)(\mu) = \int_{-1}^{+1} \left[(2\pi)^{-1} \int_0^{2} \hat{g}(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\alpha)d\alpha \right] h(\mu')d\mu', \quad -1 \le \mu \le +1.$$

Then a C-admissible Banach space H is called *compatible* with $\hat{g}$ (or with B) if the operator B is a limit in the norm of $L(H)$ of operators of finite rank and for some (and hence every) $0<\tau\le+\infty$ one has

$$\int_{-\tau}^{+\tau} \|H(x)B\|_{L(H)} dx < +\infty.$$

Similarly one defines the compatibility of H with an arbitrary operator $B \in L(H)$.

PROPOSITION 1.2. *Let* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Then for every* $1\le p<+\infty$ *the Banach space* $L_p[-1,+1]$ *is compatible with* $\hat{g}$.

This proposition is due to Feldman (cf. [6], Theorem 1). In [22] (Th. VI 1.1) this result has been obtained for $p = 2$ by showing the existence of $0<\gamma<(r-1)/(2r)$ and a bounded operator D such that $B = |T|^{\gamma}D$, and by employing (1.3b); this proof can be repeated for $1\le p<+\infty$, provided one takes $0<\alpha<(r-1)/(pr)$.

2. EQUIVALENCE THEOREMS

Throughout this section H will be a C-admissible Banach space on which the operator T has a dense range. In such a space we study the equivalence of the operator differential equation (0.4) and a convolution equation of the form (0.9).

THEOREM 2.1. *Let* $0<\tau<+\infty$. *Let* H *be compatible with the operator* $B \in L(H)$. *Suppose that* $\omega:[0,\tau]\to H$ *is a continuous function such that* $T\omega$ *is differentiable. Then an essentially bounded (strongly measurable) vector function* $\psi:(0,\tau)\to H$ *is a solution of the convolution equation*

$$(2.1) \qquad \psi(x) - \int_0^{\tau} H(x-y)B\psi(y)dy = \omega(x), \qquad 0<x<\tau,$$

if and only if ψ *is a solution of the operator differential equation*

(2.2a) $(T\psi)'(x) = -(I-B)\psi(x) + (T\omega)'(x) + \omega(x) \quad (0<x<\tau)$

with boundary conditions

(2.2b) $\lim_{x\downarrow 0} P_+\psi(x) = P_+\omega(0), \quad \lim_{x\uparrow\tau} P_-\psi(x) = P_-\omega(\tau).$

THEOREM 2.2. *Let* $\tau = +\infty$. *Let* H *be compatible with the operator* $B \in L(H)$. *Suppose that* $\omega:[0,+\infty) \to H$ *is a bounded continuous function such that* $T\omega$ *is differentiable. Then an essentially bounded (strongly measurable) vector function* $\psi:(0,+\infty) \to H$ *is a solution of the convolution equation*

(2.3) $$\psi(x) - \int_0^{+\infty} H(x-y)B\psi(y)dy = \omega(x), \quad 0<x<+\infty,$$

if and only if ψ *is a solution of the operator differential equation*

(2.4a) $(T\psi)'(x) = -(I-B)\psi(x) + (T\omega)'(x) + \omega(x) \quad (0<x<+\infty)$

with boundary condition

(2.4b) $\lim_{x\downarrow 0} P_+\psi(x) = P_+\omega(0).$

For $H = L_2[-1,+1]$ such theorems have been derived in [22]. Here in the present context we only prove Theorem 2.1 and conclude this section with some remarks.

Proof of Theorem 2.1. Let $\psi:(0,\tau) \to H$ be an essentially bounded solution of the boundary value problem (2.2). As H is compatible with B, we have $\int_{-\tau}^{+\tau}\|H(x)B\|dx < +\infty$, and thus the function $H(.)B$ is Bochner integrable on $(-\tau,+\tau)$. Since ψ is essentially bounded, it follows that the integral

(2.5) $$g(x) = \int_0^{\tau} H(x-y)B\psi(y)dy \quad (0\le x\le\tau)$$

is an absolutely convergent Bochner integral. Choose $x \in (0,\tau)$ and put $\chi = \psi-\omega$. Take $0<\tau_1<x<\tau_2<\tau$. A simple partial integration yields

$$\left(\int_0^{\tau_1} + \int_{\tau_2}^{\tau}\right)H(x-y)[(T\chi)'(y) + \chi(y)]dy =$$
$$= [U_+(x-y)\chi(y)]_{y=0}^{\tau_1} - [U_-(y-x)\chi(y)]_{y=\tau_2}^{\tau},$$

where U_+ and U_- are defined by (1.4a). Employing the boundary conditions (2.2b), the semigroup properties following from Proposition 1.1 and the continuity of $T\psi$ we get

$$\int_0^\tau H(x-y)[(T\chi)'(y) + \chi(y)]dy = \psi(x) - \omega(x), \quad 0<x<\tau.$$

Inserting Eq. (2.2a) we obtain the convolution equation (2.1).

Conversely, let $\psi:(0,\tau)\to H$ be an essentially bounded solution of the convolution equation (2.1). As $H(.)B\in L_1((-\tau,+\tau);L(H))$ and $\psi\in L_\infty((0,\tau);H)$ and as the function g is the convolution product of $H(.)B$ and ψ, it follows that g is continuous on $[0,\tau]$ (see [27]; 30.17, 30.18, 31.7 and 31.9, where a scalar analogue is studied). For $0<x<\tau$ and $0<\varepsilon<\tau-x$ we have $T\{g(x+\varepsilon)-g(x)\}/\varepsilon = h_1 + h_2 + h_3 + h_4$, where

$$h_1 = \varepsilon^{-1}[U_+(\varepsilon) - P_+]T\int_0^x H(x-y)B\psi(y)dy;$$

$$h_2 = -\varepsilon^{-1}[U_-(\varepsilon) - P_-]T\int_x^\tau H(x-y)B\psi(y+\varepsilon)dy;$$

$$h_3 = \varepsilon^{-1}\int_x^{x+\varepsilon} U_+(x+\varepsilon-y)B\psi(y)dy,$$

$$h_4 = -\varepsilon^{-1}\int_x^{x+\varepsilon} U_-(y-x)B\psi(y)dy.$$

Since $-(T|_{H_+})^{-1}$ is the generator of the semigroup $U_+(.)$, we have $h_1\to {}_0\int^x H(x-y)B\psi(y)dy$ as $\varepsilon\downarrow 0$. Using the continuity of ψ, the boundedness of ψ on $[0,\tau]$ and the Bochner integrability of $H(.)B$ on $(-\tau,+\tau)$, we apply the theorem of dominated convergence for Bochner integrals (cf. [27]) and get ${}_x\int^\tau H(x-y)B\psi(y+\varepsilon)dy \to {}_x\int^\tau H(x-y)B\psi(y)dy$ as $\varepsilon\downarrow 0$. So $h_2\to -{}_x\int^\tau H(x-y)B\psi(y)dy$. By the continuity of ψ we have $h_3\to P_+B\psi(x)$ and $h_4\to P_-B\psi(x)$. Hence, for $0<x<\tau$ one gets

$$\lim_{\varepsilon\downarrow 0}\frac{T\{g(x+\varepsilon)-g(x)\}}{\varepsilon} = -\int_0^\tau H(x-y)B\psi(y)dy + B\psi(x).$$

Similarly, $\lim_{\varepsilon\downarrow 0} T\{g(x+\varepsilon)-g(x)\}/\varepsilon = -{}_0\int^\tau H(x-y)B\psi(y)dy + B\psi(x)$ for $0<x<\tau$. Thus Tg is differentiable on $(0,\tau)$ with derivative $-g+B\psi$.

Recall that Tg and $T\omega$ are differentiable on $(0,\tau)$ and $\psi = g+\omega$ (cf. Eq. (2.1)). Therefore, $T\psi$ is differentiable on $(0,\tau)$ and

$$(T\psi)'(x) = (Tg)'(x) + (T\omega)'(x) = -g(x) + B\psi(x) + (T\omega)'(x).$$

Inserting $g = \psi - \omega$ we get Eq. (2.2a). The boundary conditions (2.2b) follow from the continuity of g on $[0,\tau]$ and $g(0) \in H_-$ and $g(\tau) \in H_+$. □

Let us consider the operator differential equation

$$(2.6) \qquad (T\psi)'(x) = -(I-B)\psi(x) + f(x), \qquad 0<x<\tau.$$

For finite τ we impose the boundary conditions (0.6) and for infinite τ the boundary conditions (0.7). To apply Theorem 2.1 (for finite τ) we have to find a continuous function $\omega:[0,\tau]\to H$ such that $T\omega$ is differentiable on $(0,\tau)$ and

$$(2.7a) \quad (T\omega)'(x) + \omega(x) = f(x) \qquad (0<x<\tau);$$

$$(2.7b) \quad P_+\omega(0) = P_+\phi, \qquad P_-\omega(\tau) = P_-\phi.$$

For infinite τ we have to find a bounded continuous function $\omega:[0,+\infty)\to H$ such that $T\omega$ is differentiable, Eq. (2.7a) (with $\tau=+\infty$) holds and

$$(2.7c) \quad P_+\omega(0) = \phi_+,$$

in order to apply Theorem 2.2. For finite (resp. infinite) τ Eq. (2.1) with boundary conditions (0.6) (resp. (0.7)) has the same essentially bounded solutions as the convolution equation (2.1) (resp. (2.3)).

For finite τ consider

$$(2.8) \qquad \omega(x) = [U_+(x) + U_-(\tau-x)]\phi + \int_0^\tau \mathcal{H}(x-y)f(y)dy, \qquad 0\le x\le\tau.$$

If $H = L_p[-1,+1]$ for some $1\le p<+\infty$ and f acts as a bounded continuous function from $(0,\tau)$ into $L_{pr}[-1,+1]$ for some $r>1$, the integral at the right-hand side of (2.8) is an absolutely convergent Bochner integral and ω is a continuous function satisfying (2.7a) - (2.7b). To see this, note that there exists $0<\gamma<1$ such that $\|\mathcal{H}(x-y)\| = O(|x-y|^{\gamma-1})$ $(y\to x)$ as an operator from $L_{pr}[-1,+1]$ into $L_p[-1,+1]$ (see the proof of Theorem 1 of [6]). So $x \to \int_0^\tau \mathcal{H}(x-y)f(y)dy$ is the convolution product of an L_1-function and an L_∞-function and therefore continuous on $[0,\tau]$. Thus ω is continuous on $[0,\tau]$ indeed, and $TP_+\omega(.)$ and $TP_-\omega(.)$ have the form

$$\chi_+(x) = TP_+\omega(x) = U_+(x)(T\phi) + \int_0^x U_+(x-y)f(y)dy; \qquad (0\le x\le\tau)$$

$$\chi_-(x) = TP_-\omega(x) = U_-(\tau-x)(T\phi) - \int_x^\tau U_-(x-y)f(y)dy. \quad (0\le x\le\tau)$$

Using Proposition 1.1 and standard semigroup theory we see that χ_+ and χ_- have to be solutions of certain Cauchy problems (cf. [17], Section IX 1.5), and therefore ω is a solution of the boundary value problem (2.7a) - (2.7b). For infinite τ analogous arguments can be given; now the formula for ω is

$$(2.9) \qquad \omega(x) = U_+(x)\phi_+ + \int_0^{+\infty} H(x-y)f(y)dy, \quad 0\le x<+\infty.$$

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

Throughout this section H will be a C-admissible Banach space that is continuously embedded in $L_1[-1,+1]$ and on which T has a dense range. On such function spaces and for a phase function $\hat{g}$ to which H is compatible and for which the expansion coefficients $a_n = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}(t)P_n(t)dt \le 1$ $(n = 0,1,2,\ldots)$, we prove that the boundary value problems (0.4) - (0.6) and (0.4) - (0.7) have a unique solution. Here $P_n(\mu) = (2^n\cdot n!)^{-1}(\frac{d}{d\mu})^n(\mu^2-1)^n$ denotes the usual Legendre polynomial of degree n.

THEOREM 3.1. *Let* $0<\tau<+\infty$, *and for some* $r>1$ *let* $\hat{g}\in L_r[-1,+1]$ *have the property that* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}(t)P_n(t)dt \le 1$ $(n = 0,1,2,\ldots)$. *Then on all C-admissible Banach spaces* H *continuously embedded in* $L_1[-1,+1]$ *and compatible with* $\hat{g}$ *on which* T *has a dense range, the boundary value problem*

$$(3.1a) \qquad (T\psi)'(x) = -(I-B)\psi(x) \quad (0<x<\tau);$$

$$(3.1b) \qquad \lim_{x\downarrow 0}\|P_+\psi(x) - P_+\phi\|_H = 0, \qquad \lim_{x\uparrow\tau}\|P_-\psi(x) - P_-\phi\|_H = 0$$

has a unique bounded solution $\psi:(0,\tau)\to H$.

As we shall prove in Section 6, we may drop the assumption that ψ is bounded, and show that (without this assumption) the boundary value problem (3.1) has a unique solution. Also a more general type of boundary conditions will be considered.

THEOREM 3.2. *Let* $\tau=+\infty$, *and for some* $r>1$ *let* $\hat{g}\in L_r[-1,+1]$ *have the property that* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}(t)P_n(t)dt < 1$ $(n = 0,1,2,\ldots)$.

Then on all C-admissible Banach spaces H *that are continuously embedded in* $L_1[-1,+1]$ *as a dense linear subspace, are compatible with* $\hat{g}$ *and on which* T *has a dense range, the boundary value problem*

(3.2a) $(T\psi)'(x) = -(I-B)\psi(x) \quad (0<x<+\infty);$

(3.2b) $\lim_{x\downarrow 0} \|P_+\psi(x) - \phi_+\|_H = 0$

has a unique bounded solution $\psi:(0,+\infty)\to H$.

In Section 6 we shall consider a more general type of boundary conditions. In the next section we generalize this result to the case $a_n \le +1$ $(n = 0,1,2,\dots)$.

Proof of Theorem 3.1. Let H satisfy the conditions of this theorem. According to Theorem 2.1 the boundary value problem (3.1) has the same essentially bounded solutions $\psi:(0,\tau)\to H$ as the convolution equation

$$(3.3) \qquad \psi(x) - \int_0^\tau H(x-y)B\psi(y)dy = \omega(x) \qquad (0<x<\tau)$$

with right-hand side $\omega(x) = [U_+(x) + U_-(\tau-x)]\phi$ (see also formula (2.8)).

For $1\le p\le+\infty$ the operator $K^{H,p}$ defined by

$$(K^{H,p}\psi)(x) = \int_0^\tau H(x-y)B\psi(y)dy \qquad (0<x<\tau),$$

is a compact operator on $L_p((0,\tau);H)$. This can be proved in the same way as Lemma 1.1 in [8], because B is the limit in the norm of operators of finite rank. Moreover, $I-K^{L_2[-1,+1],p}$ is invertible for every $1\le p\le+\infty$ (see [22], Section V.4). Since $L_1[-1,+1]$ is compatible with $\hat{g}$ (cf. Proposition 1.2), and $L_2[-1,+1]$ is a dense subspace of $L_1[-1,+1]$, it follows that for $1\le p\le+\infty$ the operator $I-K^{L_1[-1,+1],p}$ has a dense range. But $K^{L_1[-1,+1],p}$ is a compact operator. So for $1\le p\le+\infty$ the operator $I-K^{L_1[-1,+1],p}$ is invertible. However, $\mathrm{Ker}\,(I-K^{H,p}) \subset \mathrm{Ker}\,(I-K^{L_1[-1,+1],p}) = \{0\}$ for $1\le p\le+\infty$. Thus for $1\le p\le+\infty$ the operator $I-K^{H,p}$ is invertible.

Now it is clear that for $1\le p\le+\infty$ and every $\omega\in L_p((0,\tau);H)$ the convolution equation (3.3) has a unique solution ψ in $L_p((0,\tau);H)$. If ω is continuous on $[0,\tau]$, then $\psi\in L_\infty((0,\tau);H)$ and hence $\psi:(0,\tau)\to H$ is continuous on $[0,\tau]$ too. Observe that

$\omega(x) = [U_+(x) + U_-(\tau-x)]\phi$ depends continuously on x in $[0,\tau]$. Thus the boundary value problem (3.1) has a unique bounded solution $\psi:(0,\tau)\to H$. □

Proof of Theorem 3.2. Let H satisfy the conditions of this theorem. According to Theorem 2.2 the boundary value problem (3.2) has the same essentially bounded solutions $\psi:(0,+\infty)\to H$ as the convolution equation

$$(3.4)\qquad \psi(x) - \int_0^{+\infty} H(x-y)B\psi(y)dy = \omega(x) \qquad (0<x<+\infty)$$

with right-hand side $\omega(x) = U_+(x)\phi_+$ (see also formula (2.9)).

As $\int_{-\infty}^{+\infty}\|H(x)B\|_{L(H)}dx < +\infty$, the symbol of the Wiener-Hopf operator integral equation is continuous in the norm; up to a trivial change of variable it is given by

$$(3.5)\qquad W(\lambda) = I - \int_{-\infty}^{+\infty} e^{x/\lambda}H(x)Bdx = I-\lambda(\lambda-T)^{-1}B, \qquad \mathrm{Re}\ \lambda = 0.$$

Under the conditions of this theorem, on $L_2[-1,+1]$ the symbol $W(\lambda)$ has a left and a right canonical Wiener-Hopf factorization with respect to the imaginary axis (cf. Theorems V 7.2 and 7.3 of [22]; see also Chapter VI of [1]). Since B is a compact operator on $L_1[-1,+1]$ and $L_2[-1,+1]$ is a dense linear subspace of $L_1[-1,+1]$, for every extended imaginary λ the operator $W(\lambda)$ is invertible on $L_1[-1,+1]$ and $L_2[-1,+1]$. Because B is a compact operator on H and H is a subspace of $L_1[-1,+1]$, for all extended imaginary λ the operator $W(\lambda)$ is invertible on H.

Observe that on H the symbol W has the following properties: (1) W belongs to the Wiener algebra of functions of the form $\{cI + \int_{-\infty}^{+\infty}e^{x/\lambda}k(x)dx : c\in\mathbb{C},\ k\in L_1((-\infty,+\infty);L(H))\}$; (2) W can be written in the form

$$W(\lambda) = I - \lambda(\lambda-T)^{-1}P_+B - \lambda(\lambda-T)^{-1}P_-B \qquad (\mathrm{Re}\ \lambda = 0),$$

where B is the limit in the norm of $L(H)$ of operators of finite rank; (3) $W(\lambda)$ is invertible in $L(H)$ for all extended imaginary λ. By a result of Gohberg and Leiterer ([11], Theorems 4.3 and 4.4) on H the symbol W has a left and a right Wiener-Hopf factorization with respect to the imaginary line with factors in the above Wiener algebra.

For $1\le p\le+\infty$ let $K^{H,p}$ be the operator on $L_p((0,+\infty);H)$ defined by

$$(K^{H,p}\psi)(x) = \int_0^{+\infty} H(x-y)B\psi(y)dy \quad (0<x<+\infty).$$

From the existence of a Wiener-Hopf factorization of W one derives that for $1\le p\le+\infty$ the operator $I - K^{H,p}$ is a Fredholm operator. The derivation is standard (see [9]). By the proof of Theorem V 5.1 of [22] for $1\le p\le+\infty$ the operator $I - K^{L_2[-1,+1],p}$ is invertible. Since $L_2[-1,+1]$ is continuously embedded in $L_1[-1,+1]$ as a dense linear subspace and $I - K^{L_1[-1,+1],p}$ is a Fredholm operator, it follows that for $1\le p\le+\infty$ the operator $I - K^{L_1[-1,+1],p}$ is invertible. Because H is continuously embedded in $L_1[-1,+1]$ as a dense linear subspace and $I - K^{H,p}$ is a Fredholm operator, it follows that for $1\le p\le+\infty$ the operator $I - K^{H,p}$ is invertible.

Now it is clear that for $1\le p\le+\infty$ and every $\omega\in L_p((0,+\infty);H)$ the Wiener-Hopf equation(3.4) has a unique solution ψ in $L_p((0,+\infty);H)$. If ω is bounded and continuous on $[0,+\infty)$, then $\psi\in L_\infty((0,+\infty);H)$ and hence $\psi:(0,+\infty)\to H$ is bounded and continuous too. Observe that $\omega(x) = U_+(x)\phi_+$ depends continuously on x in $[0,+\infty)$ and is bounded. Thus the boundary value problem (3.2) has a unique bounded solution $\psi:(0,+\infty)\to H$. □

For $H = L_p[-1,+1]$ ($1\le p<+\infty$) Theorems 3.1 and 3.2 apply to phase functions $\hat{g}\in L_r[-1,+1]$ for some $r>1$ (cf. Proposition 1.2). If H is a C-admissible Banach space continuously embedded in $L_1[-1,+1]$, H contains the function $e(\mu)\equiv 1$ (and thus all polynomials) and if $\int_{-\infty}^{+\infty}\| H(x)e\|_H dx < +\infty$, then these theorems apply to degenerate phase functions of the form $\hat{g}(\mu) = \sum_{n=0}^{N} a_n(n+\frac{1}{2})P_n(\mu)$ (N finite).

In the statement of Theorem 3.1 (resp. 3.2) the finite-slab (resp. half-space) problem is considered for a non-multiplying (resp. non-conservative) medium. In Eq. (3.1a) (resp. (3.2a)) we did not allow an inhomogeneous term f(x) (as in Eq. (0.4)). For specific spaces H Theorems 3.1 and 3.2 are easily adapted to cover the occurrence of an inhomogeneous term. If $H = L_p[-1,+1]$ ($1\le p<+\infty$), we require that for some $r'>1$ the function f acts as a bounded continuous function from $(0,\tau)$ (resp. $(0,+\infty)$) into $L_{pr'}[-1,+1]$.

For such f a suitable $\omega \in L_\infty((0,\tau);H)$ (resp. $\omega \in L_\infty((0,+\infty);H)$) can be constructed, which is continuous on $[0,\tau]$ (resp. bounded and continuous on $[0,+\infty)$). This right-hand side ω of Eq. (3.3) (resp. Eq. (3.4)) is given by (2.8) (resp. (2.9)). If one adds such an inhomogeneous term f(x) to Eq. (3.1a) (resp. (3.2a)), then Theorem 3.1 (resp. 3.2) can be derived as before.

4. SPECTRAL ANALYSIS OF $L(\lambda) = I - B - \lambda T$

Throughout this section H stands for a C-admissible Banach space that is densily and continuously embedded in $L_1[-1,+1]$; it is supposed that T has a dense range on H. If for some $r > 1$ the phase function $\hat{g} \in L_r[-1,+1]$ and H is compatible with $\hat{g}$, we study on H the operator pencil

$$(4.1) \qquad L(\lambda) = I - B - \lambda T.$$

In [22], Section III.3, the pencil $L(\lambda)$ has been studied in detail on $L_2[-1,+1]$, which is compatible to every $\hat{g} \in L_r[-1,+1]$ with $r > 1$ (cf. Proposition 1.2). On $L_1[-1,+1]$ we have $\{\lambda \in \mathbb{C} : I - \lambda T \text{ is not invertible}\} = (-\infty,-1] \cup [1,+\infty)$, just as on $L_p[-1,+1]$ for $1 \le p \le +\infty$. For general H the pencil $L(\lambda)$ is a compact perturbation of $L_0(\lambda) = I - \lambda T$ and the spectrum of $L_0(\lambda)$ is contained in $(-\infty,-1] \cup [1,+\infty)$. By a result of Gohberg and Sigal [12] the set

$$\{\lambda \in \mathbb{C} : \lambda \notin (-\infty,-1] \cup [1,+\infty);\ L(\lambda) \text{ is not invertible on } H\}$$

consists of normal points only and these points can only accumulate at points of $(-\infty,-1] \cup [1,+\infty)$. Hence, if $L(0) = I - B$ is not invertible, then $\lambda = 0$ is an isolated normal point of the spectrum of $L(\lambda)$.

To state the next proposition let $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t) P_n(t)\,dt$ $(n = 0,1,2,\ldots)$; these numbers will be called the *expansion coefficients* of $\hat{g}$. On $L_p[-1,+1]$ $(1 \le p \le +\infty)$ one has $BP_n = a_n P_n$ $(n = 0,1,2,\ldots)$ and $\sigma(B) = \{a_n : n \ge 0\} \cup \{0\}$ (see Appendix XII.8 of [26]). Since H is continuously and densily embedded in $L_1[-1,+1]$ and B is compact on H, the spectrum of B on H is the same set, provided, of course, H is compatible with $\hat{g}$.

PROPOSITION 4.1. *Let* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Then outside* $(-\infty,-1] \cup [1,+\infty)$ *the spectrum of the pencil* $L(\lambda)$ *on* H *does not depend on* H *nor do the partial multiplicities of the normal points of* $L(\lambda)$. *Put* $\sigma = \{a_n : n \geq 0\}$. *If* $\sigma \subset (-\infty,1]$, *then the spectrum* $\Sigma(L)$ *of* $L(\lambda)$ *is a part of the real line and the order of any pole of* $L(\lambda)^{-1}$ *at a non-zero (resp. zero) normal point equals* +1 *(resp.* +2 *at most).*

Proof. Put $\Omega = \mathbb{C}\backslash\{(-\infty,-1] \cup [1,+\infty)\}$. Then on Ω the operator $L(\lambda)$ is a Fredholm operator of index 0. Since H and $L_2[-1,+1]$ are both of them densily and continuously embedded in $L_1[-1,+1]$, on these three spaces the part of the spectrum of $L(\lambda)$ on Ω is the same and for every $\lambda_0 \in \Omega$ in the spectrum of $L(\lambda)$ the partial multiplicities and Jordan chains coincide (cf. [1] for these notions). The rest of this proposition is clear from the properties of $L(\lambda)$ on $L_2[-1,+1]$ (see Section III.3 of [22]). □

If $A = I - B$ is not invertible and $\sigma = \{a_n : n \geq 0\} \subset (-\infty,1]$, then $\lambda = 0$ is a pole of $L(\lambda)^{-1}$ of order at most 2. Put

$$P_0 = -(2\pi i)^{-1}\int_\Gamma L(\lambda)^{-1}T d\lambda, \qquad P_0^+ = -(2\pi i)^{-1}\int_\Gamma TL(\lambda)^{-1}d\lambda,$$

where Γ is a positively oriented circle separating $\lambda = 0$ from the rest of the spectrum of $L(\lambda)$. It appears that P_0 and P_0^+ are projections of the same finite rank, $\operatorname{Ker} A \subset \operatorname{Im} P_0$ and $TP_0 = P_0^+T$ (see Section 1.3 of [24]; see also Section III.3 of [22]). The finite-dimensial subspace $H_0 = \operatorname{Im} P_0$ of H is called the *singular subspace*. We also put $H_0^+ = \operatorname{Im} P_0^+$. Note that H_0 and H_0^+ have the same form as the corresponding subspaces in $L_2[-1,+1]$.

It is easy to see that A acts as an invertible operator from $H_1 = \operatorname{Ker} P_0$ onto $H_1^+ = \operatorname{Ker} P_0^+$. So there exist bounded operators S on H_1 and S^+ on H_1^+ such that $ASx = S^+Ax = Tx$, $x \in H_1$. The operator S we call the *associate operator*. If $\sigma = \{a_n : n \geq 0\} \subset (-\infty,1]$, then $\sigma(S) = \sigma(S^+) \subset \mathbb{R}$. For the conservative isotropic case, where $a_0 = 1$ and $a_n = 0$ for $n \geq 1$, and on $L_2[-1,+1]$ the singular subspace and the associate operator have been constucted by Lekkerkerker [20].

Next for the case $\sigma = \{a_n : n \geq 0\} \subset (-\infty,1]$ we analyze on H the operator differential equation

$$(4.2) \qquad (T\psi)'(x) = -(I-B)\psi(x) + f(x) \qquad (0<x<\tau)$$

with boundary conditions (0.6) (for finite τ) or (0.7) (for infinite τ). First recall that the singular subspace H_0 does not depend on the particular choice of H. So as in $L_2[-1,+1]$ this subspace H_0 consists of polynomials only (cf. [22], Theorem VI 4.1). On H_0 one defines the indefinite inner product

$$(4.3) \qquad \langle x,y\rangle_T = \int_{-1}^{+1} \mu x(\mu)\overline{y(\mu)}d\mu; \qquad x,y \in H,$$

which makes H_0 a Krein space (see [2] for the terminology). On H_0 there exist subspaces M_+ and M_- that are maximal strictly positive and maximal strictly negative in (4.3) and have the property $M_+ \dot{\oplus} M_- = H_0$ (cf. [2]). Clearly these subspaces M_+ and M_- can be chosen independently of the choice of H.

PROPOSITION 4.2. *Let* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r>1$, *and let* $\sigma = \{a_n : n \geq 0\} \subset (-\infty,1]$. *Then the space* H *is also compatible with the operator*

$$B_u = B(I-P_0) + (I+u^{-1}T)(I-\rho)P_0 + (I-u^{-1}T)\rho P_0,$$

where $u>0$ *and* ρ *is the projection of* H_0 *onto* M_+ *and* M_-. *Further, the spectrum of* B_u *is contained in* $(-\infty,1)$.

Proof. Recall that $L_2[-1,+1]$ is compatible with $\hat{g}$. Further, the operators T, B and P_0 have the same form on H and on $L_2[-1,+1]$ and the subspaces H_0, M_+ and M_- do not depend on the particular choice of H. Since $(I+u^{-1}T)(I-\rho)P_0 + (I-u^{-1}T)\rho P_0$ is an operator of finite rank, the operator B_u is a limit in the norm of H of operators of finite rank. Since $x = Bx + TBT^{-1}(I-B)x$ $(x \in H_0)$ and $\int_{-\infty}^{+\infty}\|H(x)B\|dx < +\infty$, the space H is easily proved to be compatible with B_u. Because H and $L_2[-1,+1]$ are continuously and densily embedded in $L_1[-1,+1]$, the set $\sigma(B_u)$ does not depend on the particular choice of H. The proposition is now immediate from Theorem III 6.3 of [22], which is the analogue in $L_2[-1,+1]$. □

In the statement of Theorem III 6.3 of [22] (applied to the pair (T,B) on $L_2[-1,+1]$) the conclusion is drawn that for $u>0$ the pair (T,B_u) is a positive definite admissible pair on $L_2[-1,+1]$ (see Section III.2 of [22] for the terminology). From this it follows that for every $1 \le p \le +\infty$ and every $\omega \in L_p((0,+\infty);L_2[-1,+1])$ the Wiener-Hopf operator integral equation

$$(4.4) \qquad \phi(x) - \int_0^{+\infty} H(x-y)B_u\phi(y)dy = \omega(x) \qquad (0<x<+\infty)$$

has a unique solution in $L_p((0,+\infty);L_2[-1,+1])$ (cf. [22], Theorem V 5.1). It is also clear that for every $\phi_+ \in L_2(0,1)$ the boundary value problem

$$(4.5a) \qquad (T\tilde{\psi})'(x) = -(I-B_u)\tilde{\psi}(x) \qquad (0<x<+\infty);$$

$$(4.5b) \qquad \lim_{x\downarrow 0} P_+\tilde{\psi}(x) = \phi_+, \qquad \|\tilde{\psi}(x)\|_{L_2[-1,+1]} = O(1) \qquad (x\to+\infty),$$

has a unique solution $\tilde{\psi} : (0,+\infty) \to L_2[-1,+1]$ (cf. [22], Theorem 3.1). We have

THEOREM 4.3. *For some* $r>1$ *let* $\hat{g} \in L_r[-1,+1]$ *have the property that* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1}\hat{g}(t)P_n(t)dt \le 1$ $(n = 0,1,2,\ldots)$. *Then on all C-admissible Banach spaces* H *that are continuously embedded in* $L_1[-1,+1]$ *as a dense linear subspace, are compatible with* $\hat{g}$ *and on which* T *has a dense range, the boundary value problem*

$$(4.6a) \qquad (T\psi)'(x) = -(I-B)\psi(x) \qquad (0<x<+\infty);$$

$$(4.6b) \qquad \lim_{x\downarrow 0} \|P_+\psi(x) - \phi_+\|_H = 0$$

has at least one bounded solution $\psi:(0,+\infty)\to H$. *The number of linearly independent bounded solutions of the corresponding homogeneous problem (where* $\phi_+ = 0$*) is finite and equals* dim Ker $(I-B)$ $- \frac{1}{2}$dim H_0, *where* H_0 *is the singular subspace. In particular, there is a unique bounded solution for every* ϕ_+ *if and only if all partial multiplicities of* $L(\lambda)$ *at* $\lambda = 0$ *equal* +2.

It is possible to give a more direct description of the number of linearly independent bounded solutions. Let $M = \{n \ge 0 : a_n = +1\}$, and call a subset $\{k+1,\ldots,k+m\}$ a *cycle* of M of length m, if and only if $k \notin M$ and $k+m+1 \notin M$. Then M is decomposed into a disjoint union of cycles. Note that Ker $(I-B)$ equals span $\{P_n : n \in M\}$. By Theorem VI 4.1 of [22] we have dim H_0 - dim Ker $(I-B) = \alpha(M)$, where $\alpha(M)$ is the number of cycles of odd length. Hence, dim Ker $(I-B)$ - $\frac{1}{2}$dim H_0 is one half of the difference of the cardinality of M and of $\alpha(M)$. Thus there is a unique bounded solution for every ϕ_+ if and only if all cycles of $M = \{n \geq 0 : a_n = +1\}$ have length one (i.e., if $\{a_n, a_{n+1}\} \neq \{1\}$ for $n \geq 0$).

Proof of Theorem 4.3. Let S be the associate operator (i.e, $ASx = Tx$, $x \in \operatorname{Ker} P_0$). Putting $\psi_0(x) = P_0\psi(x)$ and $\psi_1(x) = (I - P_0)\psi(x)$, Eq. (4.5a) can be decomposed as follows:

$$(4.7a) \quad (S\psi_1)'(x) = -\psi_1(x) \quad (0<x<+\infty);$$

$$(4.7b) \quad (T\psi_0)'(x) = -(I-B)\psi_0(x) \quad (0<x<+\infty).$$

Since $\dim H_0 < +\infty$, T acts as an invertible operator from H_0 onto $H_0^+ = \operatorname{Im} P_0^+$ and the order of a (possible) pole of $L(\lambda)^{-1}$ at $\lambda = 0$ is at most 2, one easily computes that

$$(4.7c) \quad \psi_0(x) = (I - xT^{-1}A)h_0 \quad (0<x<+\infty),$$

where $A = I - B$ and $h_0 \in H_0$. Since ψ_0 is bounded, we see that $Ah_0 = 0$ and $\psi_0(x) \equiv h_0$.

Putting $\tilde{\psi}_0(x) = P_0\tilde{\psi}(x)$ and $\tilde{\psi}_1(x) = (I - P_0)\tilde{\psi}(x)$, Eq. (4.5a) can be decomposed as follows:

$$(4.8a) \quad (S\tilde{\psi}_1)'(x) = -\tilde{\psi}_1(x) \quad (0<x<+\infty);$$

$$(4.8b) \quad (T\tilde{\psi}_0)'(x) = -(I-B_u)\tilde{\psi}_0(x) \quad (0<x<+\infty).$$

(Note that $(I - B_u)Sx = Tx$, $x \in \operatorname{Ker} P_0$). Using the specific form of B_u, Eq. (4.8b) is easily solved in the form

$$(4.8c) \quad \tilde{\psi}_0(x) = e^{-x/u}(I-\rho)\tilde{h}_0 + e^{+x/u}\rho\tilde{h}_0 \quad (0<x<+\infty),$$

where $u > 0$ and $\tilde{h}_0 \in H_0$. Obviously, $\tilde{\psi}_0$ is bounded if and only if $\rho\tilde{h}_0 = 0$. Here $M_+ = \operatorname{Im} \rho$ (resp. $M_- = \operatorname{Ker} \rho$) is a maximal strictly positive (resp. negative) subspace of H_0.

Using the decompositions (4.7a) - (4.7b) and (4.8a) - (4.8b) the boundary value problem ϕ_+ can be reduced to one of the form (4.5) (with a different ϕ_+). The reduction does not depend on the particular choice of the C-admissible space H. Therefore, in H one finds the same result as in $L_2[-1,+1]$. For the result in $L_2[-1,+1]$ we refer to Theorem IV 3.4 of [22]. (The inversion symmetry J referred to in this theorem is the map $(Jh)(\mu) = h(-\mu)$; see Section III.7 of [22]). □

Now Theorem 3.2 has been generalized to the case $a_n \leq +1$ $(n = 0,1,2,\dots)$, and thus the conservative case of the Transport Equation is included in our description.

5. SPECTRAL SUBSPACES, PROJECTIONS AND SEMIGROUPS

Throughout this section H stands for a C-admissible Banach space that is continuously and densily embedded in $L_1[-1,+1]$; it is supposed that T has a dense range on H. If $\hat{g} \in L_r[-1,+1]$ for some $r > 1$ and H is compatible with $\hat{g}$, we employ the (unique) bounded solutions of the boundary value problems (3.1) and (3.2) to construct spectral subspaces, projections and semigroups. In Section 6 these entities will enable us to write down the solutions of these boundary value problems. We put $A = I - B$.

THEOREM 5.1. *Let the C-admissible space* H *be as above, and let* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Assume that* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t)P_n(t)dt \leq 1$ $(n = 0,1,2,\dots)$. *Then there exists a decomposition of the form*

$$(5.1) \qquad H = H_p \oplus H_m \oplus H_0,$$

where H_0 *denotes the singular subspace,* $H_p \oplus H_m = \operatorname{Ker} P_0$ *and the closed subspaces* H_p *and* H_m *have the following properties:*

(i) H_p *and* H_m *are invariant under the associate operator* S *and* $\sigma(S|_{H_p}) \subset [0,+\infty)$ *and* $\sigma(S|_{H_m}) \subset (-\infty,0]$;

(ii) *the operator* $(-S|_{H_p})^{-1}$ *(resp.* $(+S|_{H_m})^{-1}$*) is the infinitesimal generator of a bounded analytic semigroup* $(U_p(x))_{x\geq 0}$ *(resp.* $(U_m(x))_{x\geq 0}$*) on* H_p *(resp.* H_m*);*

(iii) *if* P_p *(resp.* P_m*) denotes the projection of* H *onto* H_p *(resp.* H_m*) along* $H_m \oplus H_0$ *(resp.* $H_p \oplus H_0$*), then for* $0 \leq x < +\infty$ *the operator* $U_p(x) - U_+(x)$ *(resp.* $U_m(x) - U_-(x)$*) is compact. Here for* $h \in H$ *one puts* $U_p(x)h = U_p(x)P_p h$ *and* $U_m(x)h = U_m(x)P_m h$.

Proof. First it is assumed that $\{a_n : n \geq 0\} \subset (-\infty,1)$. Under this hypothesis the singular subspace H_0 is trivial, the operator A is invertible and the associate operator has the form $S = A^{-1}T$.

Let us construct the closed subspaces H_p and H_m. Observe that the symbol of the Wiener-Hopf operator integral equation

$$(5.2) \qquad \psi(x) - \int_{-\infty}^{+\infty} H(x-y)B\psi(y)dy = \omega(x) \qquad (-\infty<x<+\infty)$$

is given by (3.5), belongs to the Wiener algebra $\{cI + {}_{-\infty}\int^{+\infty} e^{x/\lambda}k(x)dx : k \in L_1((-\infty,+\infty);L(H))\}$ and has invertible values only. So for every $1 \le p \le +\infty$ and $\omega \in L_p((-\infty,+\infty);H)$ the above equation has a unique solution ψ_ϕ in $L_p((-\infty,+\infty);H)$. Given $\phi(x) = TH(x)\phi$ (i.e., $\omega(x) = U_+(x)\phi$ for $x > 0$ and $\phi(x) = -U_-(-x)\phi$ for $x < 0$) we define operators P_p and P_m on H by

$$P_p = \lim_{x \downarrow 0} \psi_\phi(x), \qquad P_m = - \lim_{x \uparrow 0} \psi_\phi(x),$$

where ψ_ϕ is the unique solution in $L_1((-\infty,+\infty);H)$ of Eq. (5.2); note that ψ_ϕ is continuous except for a jump at $x = 0$. Observe that $P_p\phi + P_m\phi = \lim_{x\downarrow 0} \psi_\phi(x) - \lim_{x\uparrow 0} \psi_\phi(x) = \lim_{x\downarrow 0} TH(x)\phi - \lim_{x\uparrow 0} TH(x)\phi = P_+\phi + P_-\phi = \phi$, $\phi \in H$. Further, if $P_p\phi = P_m\phi = 0$, then for $\omega(x) = TH(x)\phi$ the solution ψ_ϕ of Eq. (5.2) would be continuous on all of $\mathbb{R}$. From Eq. (5.2) the continuity of $TH(.)\phi$ would follow, and thus $P_+\phi = \lim_{x\downarrow 0} TH(x)\phi = \lim_{x\uparrow 0} TH(x)\phi = -P_-\phi \in H_+ \cap H_- = \{0\}$. Therefore, P_p and P_m are complementary projections. We put $H_p = \operatorname{Im} P_p$ and $H_m = \operatorname{Im} P_m$.

Given $\phi \in H$ let ψ_ϕ be the unique solution of Eq. (5.2) with $\omega(x) = TH(x)\phi$ $(0 \ne x \in \mathbb{R})$. We define operators $U_p(x)$ $(x \ge 0)$ and $U_m(x)$ $(x \ge 0)$ by

$$U_p(x)\phi = \psi_\phi(x) \quad (x > 0), \qquad U_p(0)\phi = \lim_{x\downarrow 0} \psi_\phi(x);$$
$$U_m(x)\phi = -\psi_\phi(-x) \quad (x > 0), \qquad U_m(0)\phi = - \lim_{x\uparrow 0} \psi_\phi(x).$$

So $U_p(0) = P_p$, $U_m(0) = P_m$. Since on $L_\infty((-\infty,+\infty);H)$ the operator $(L\psi)(x) = \psi(x) - {}_{-\infty}\int^{+\infty} H(x-y)B\psi(y)dy$ $(-\infty<x<+\infty)$ has a bounded inverse, it is easily seen that $U_p(x)$ and $U_m(x)$ are bounded and

$$\|U_p(x)\| \le M_H\|L^{-1}\|, \qquad \|U_m(x)\| \le M_H\|L^{-1}\|; \qquad 0 \le x < +\infty.$$

By the continuity of ψ_ϕ on $[0,+\infty)$ and $(-\infty,0]$ the functions U_p, $U_m : [0,+\infty) \to L(H)$ are strongly continuous.

To prove that U_p and U_m are semigroups, we apply the argument of (the first part of) the proof of Theorem 2.1 and obtain

(5.3a) $(T\psi_\phi)'(x) = -(I-B)\psi_\phi(x) \quad (0 \neq x \in \mathbb{R});$

(5.3b) $\lim_{x\downarrow 0} \psi_\phi(x) = P_p\phi, \qquad \lim_{x\uparrow 0} \psi_\phi(x) = -P_m\phi.$

For $0 \le y < +\infty$ the function $\chi_y(x) = \psi_\phi(x+y)$ $(x \ge 0)$, $\chi_y(x) = 0$ $(x < 0)$, is a solution of Eq. (5.2) with $\omega(x) = TH(x)U_p(y)\phi$ $(0 \neq x \in \mathbb{R})$, and thus $\chi_y(x) = U_p(x)U_p(y)\phi$ $(x \ge 0)$ and $\chi_y(x) = -U_m(-x)\cdot 0 = 0$ $(x < 0)$. So we have for $0 \le x < +\infty$:

$$U_p(x)U_p(y) = U_p(x+y) \quad (x \ge 0); \quad U_m(x)U_p(y) = 0 \quad (x \ge 0).$$

Hence, $\operatorname{Im} U_p(x) \subset H_p$ $(x \ge 0)$ and U_p is a semigroup. Analogously, $\operatorname{Im} U_m(x) \subset H_m$ $(x \ge 0)$ and U_m is a semigroup.

Let us show that $A^{-1}T$ leaves invariant H_p and H_m. It is easily checked that for $\phi \in H$ the function $\tilde{\psi}(x) = A^{-1}T\psi_\phi(x)$ satisfies

$$(T\tilde{\psi})'(x) = -TA^{-1}(T\psi_\phi)'(x) = -T\psi_\phi(x) = -A\tilde{\psi}(x) \quad (0 \neq x \in \mathbb{R}).$$

So $A^{-1}TP_p\phi = \lim_{x\downarrow 0} \tilde{\psi}(x) \in H_p$ and $A^{-1}TP_m\phi = -\lim_{x\uparrow 0} \tilde{\psi}(x) \in H_m$.

Let us compute the infinitesimal generators of the bounded strongly continuous semigroups $(U_p(x)|_{H_p})_{x\ge 0}$ and $(U_m(x)|_{H_m})_{x\ge 0}$. For $0<x<+\infty$ we have

$$x^{-1}T\{U_p(x) - U_p(0)\}\phi = x^{-1}\int_0^x (T\psi_\phi)'(y)dy = -x^{-1}\int_0^x A\psi_\phi(y)dy,$$

and by the continuity of ψ_ϕ on $[0,+\infty)$ this tends to $-A\psi_\phi(0) = -AP_p\phi$ as $x \downarrow 0$. Therefore, if $P_p\phi$ belongs to the domain of the generator G_p of (the restriction to H_p of) the semigroup $(U_p(x))_{x\ge 0}$, we have $A^{-1}TG_pP_p\phi = -P_p\phi$. Conversely, if h belongs to the domain of G_p, then $h = P_p\phi = \lim_{x\downarrow 0} \psi_\phi(x)$ for some $\phi \in H$ and $\psi_\phi = U_p(.)\phi$ is differentiable on $(0,+\infty)$, while

$$TU_p(x)G_ph = (T\psi_\phi)'(x) = -A\psi_\phi(x) = -AU_p(x)h, \quad 0<x<+\infty.$$

For $x \downarrow 0$ we get $TG_ph = -Ah$, and thus $h \in A^{-1}T[H_p]$. Hence, $G_p = -(A^{-1}T|_{H_p})^{-1}$. Similarly, $+(A^{-1}T|_{H_m})^{-1}$ is the infinitesimal generator of (the restriction to H_m of) the semigroup $(U_m(x))_{x\ge 0}$.

Since the above semigroups are bounded and $\sigma(A^{-1}T) \subset \mathbb{R}$, we have

(5.4) $\quad \sigma(A^{-1}T|_{H_p}) \subset [0,+\infty), \quad \sigma(A^{-1}T|_{H_m}) \subset (-\infty,0].$

Clearly, for $0<x<+\infty$ we have $\|U_p(x)\phi\| = \|\psi_\phi(x)\| \to 0$ as $x\to+\infty$, where the convergence is uniform in ϕ on bounded subsets of H. Hence,

(5.5) $$\lim_{x\to+\infty} \|U_p(x)\| = 0, \qquad \lim_{x\to+\infty} \|U_m(x)\| = 0.$$

To establish the analyticity of the semigroups take $\phi \in H$ and consider Eq. (5.2) for $\omega(x) = T\mathcal{H}(x)\phi$ $(0 \neq x \in \mathbb{R})$. First we show that the integral $g_\phi(x) = {}_{-\infty}\int^{+\infty}\mathcal{H}(x-y)B\psi_\phi(y)dy$, with ψ_ϕ being the solution, represents an analytic function on $\mathbb{C}\setminus i\mathbb{R}$. Putting $(U(t)h)(\mu) = e^{-it/\mu}h(\mu)$ $(-1\le\mu\le+1)$ and observing that $\|U(t)\| = 1$ $(t\in\mathbb{R})$, we have $\mathcal{H}(z) = U(\text{Im } z)H(\text{Re } z)$ for Re $z \neq 0$. Hence,

$$\|\mathcal{H}(x-y)B\psi_\phi(y)\| \le \|\mathcal{H}(\text{Re } x - y)B\psi_\phi(y)\|; \qquad 0<y<+\infty,\ \text{Re } x > 0,$$

and thus $g_\phi(x)$ is given by an absolutely convergent Bochner integral for every Re $x > 0$. Because the absolute convergence of this integral is uniform in x on strips of the form $\varepsilon_1\le\text{Re } x\le\varepsilon_2$, where $0<\varepsilon_1<\varepsilon_2<+\infty$, it appears that $g_\phi(x)$ depends analytically in x on the open right half-plane. Since $T\mathcal{H}(.)$ is analytic on $\mathbb{C}\setminus i\mathbb{R}$, the functions $U_p(x)\phi$ and $U_m(x)\phi$ depend analytically on x for every $\phi \in H$. Thus U_p and U_m are analytic on the open right half-plane. (Here we employed Theorem 4.4 G of [25]).

It remains to prove that

(5.6) $$\lim_{x\to 0,\,|\arg x|\le\delta} \|U_p(x)\phi - P_p\phi\| = 0, \qquad \phi \in H$$

(and similarly for U_m). Remark that

$$\int_{-\infty}^{+\infty}\Big[\mathcal{H}(x-y) - \mathcal{H}(-y)\Big]B\psi_\phi(y)dy = P_+\int_{-\infty}^{+\infty}\mathcal{H}(x-y)B\psi_\phi(y)dy +$$
$$+ [P_- - U_-(x)]\int_{-\infty}^{+\infty}\mathcal{H}(x-y)B\psi_\phi(y)dy - \int_0^{\text{Re } x}\mathcal{H}(-y)B\psi_\phi(y)dy.$$

Since $(U_-(x))_{x\ge0}$ is an analytic semigroup, we have

$$\lim_{x\to0,\,|\arg x|\le\delta}\left\|\int_{-\infty}^{+\infty}\Big[\mathcal{H}(x-y) - \mathcal{H}(-y)\Big]B\psi_\phi(y)dy\right\| = 0, \qquad 0<\delta<\tfrac{1}{2}\pi.$$

From this relationship formula (5.6) is clear. The semigroups $(U_p(x)|_{H_p})_{x\ge0}$ and $(U_m(x)|_{H_m})_{x\ge0}$ are analytic, indeed.

To prove the compactness of $U_p(x) - U_+(x)$ and $U_m(x) - U_-(x)$, we consider the convolution equation

$$(5.7) \qquad \psi_\tau(x) - \int_{-\tau}^{+\tau} H(x-y)B\psi_\tau(y)dy = \omega(x) \qquad (-\tau<x<+\tau).$$

By the "projection method" (Theorem 3 of [7]) for τ large enough, for every $1\le p\le+\infty$ and $\omega \in L_p((-\tau,+\tau);H)$ Eq. (5.7) has a unique solution ψ_τ in $L_p((-\tau,+\tau);H)$. Furthermore, if we set $\psi_\tau(x) = 0$ for $|x| \ge \tau$, and if ψ denotes the unique solution of Eq. (5.2) (with $\omega \in L_p((-\infty,+\infty);H)$), we have

$$(5.8) \qquad \lim_{\tau\to+\infty} \|\psi_\tau - \psi\|_{L_p((-\infty,+\infty);H)} = 0.$$

On $L_p((-\tau,+\tau);H)$ $(0<\tau<+\infty,\ 1\le p\le+\infty)$ the operator $(K\psi_\tau)(x) = {}_{-\tau}\int^{+\tau} H(x-y)B\psi_\tau(y)dy$ $(-\tau<x<+\tau)$ is compact. (The proof is analogous to the proof of Lemma 1.1 of [8], because B is a limit in the norm of operators of finite rank). Further, if $\omega(x) = TH(x)\phi$ $(0 \ne x \in \mathbb{R})$, the solution ψ_τ of Eq. (5.7) is continuous on $[-\tau,+\tau]$ except for a possible jump at $x = 0$. Using the compactness of K on $L_\infty((-\tau,+\tau);H)$ it follows that the linear operators on H that map ϕ into $\lim_{x\downarrow 0} (K\omega)(x)$, are compact. But from (5.8) it is clear that these operators converge in the norm of $L(H)$ to $U_p(x) - U_+(x)$ $(x \ge 0)$ and $U_m(-x) - U_-(-x)$ $(x \le 0)$, respectively. Hence, the operators $U_p(x) - U_+(x)$ $(x \ge 0)$ and $U_m(x) - U_-(x)$ $(x \ge 0)$ are compact.

Finally, we extend our results to the case $\{a_n : n \ge 0\} \subset (-\infty,1]$, where H_0 might fail to be trivial and A might fail to be invertible. If $S : \operatorname{Ker} P_0 \to \operatorname{Ker} P_0$ denotes the associate operator (see Section 4), take $u > \max(\|T\|,\|S\|)$ and let M_+ (M_-) be a maximal strictly positive (negative) subspace of H_0 (endowed with the indefinite inner product (4.3)). Then $M_+ \oplus M_- = H_0$. Let ρ be the projection of H_0 onto M_+ along M_- and put

$$B_u = B(I - P_0) + (I + u^{-1}T)(I - \rho)P_0 + (I - u^{-1}T)\rho P_0;$$
$$A_u = I - B_u = AP_0 - u^{-1}T(I - \rho)P_0 + u^{-1}T\rho P_0$$

(see Proposition 4.2). Then H is compatible with B_u and

$$A_u^{-1}T = S \oplus (-u)I_{M_-} \oplus uI_{M_+}, \qquad \sigma(A_u^{-1}T) = \sigma(S) \cup \{u,-u\},$$

where $\sigma(S) \cap \{u,-u\} = \emptyset$. Using the first part of the proof one discovers that Ker P_0 (i.e., the subspace on which S is defined) can be decomposed as the direct sum $H_p \oplus H_m$ = Ker P_0, where H_p and H_m are closed S-invariant subspaces, $S[H_p]$ is dense in H_p, $S[H_m]$ is dense in H_m and the operators $(-S|_{H_p})^{-1}$ and $(+S|_{H_m})^{-1}$ generate bounded analytic semigroups. In fact, for the operator B_u the analogues of H_p and H_m are the $A_u^{-1}T$-invariant subspaces $H_p \oplus M_+$ and $H_m \oplus M_-$, respectively. To see this, note that the spectrum of the restriction of $A_u^{-1}T$ to $H_p \oplus M_+$ (resp. $H_m \oplus M_-$) is contained in $[0,+\infty)$ (resp. $(-\infty,0]$). Now the theorem has been extended to the case $\{a_n : n \geq 0\} \subset (-\infty,1]$. □

COROLLARY 5.2. *Let the* C*-admissible space* H *be as above, and let* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Assume* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t)P_n(t)dt \leq 1$ $(n = 0,1,2,\dots)$. *Put* $H_p^+ = A[H_p]$, $H_m^+ = A[H_m]$ *and* $H_0^+ = T[H_0]$. *Then we have the identities*

(5.9a) $\overline{T[H_p]} = H_p^+, \quad \overline{T[H_m]} = H_m^+, \quad A[H_0] \subset H_0^+,$

the decompositions

(5.9b) $H_p \oplus H_m \oplus H_0 = H, \quad H_p^+ \oplus H_m^+ \oplus H_0^+ = H,$

and the intertwining properties

(5.9c) $TP_p = P_p^+T, \quad TP_m = P_m^+T, \quad TP_0 = P_0^+T;$

(5.9d) $AP_p = P_p^+A, \quad AP_m = P_m^+A, \quad AP_0 = P_0^+A,$

where P_p^+ (P_m^+) *is the projection of* H *onto* H_p^+ (H_m^+) *along* $H_m^+ \oplus H_0^+$ $(H_p^+ \oplus H_0^+)$.

Proof. The corollary is a direct consequence of the following facts:

(1) A acts as an invertible operator from Ker $P_0 = H_p \oplus H_m$ onto Ker $P_0^+ = H_p^+ \oplus H_m^+$;
(2) T maps Ker P_0 into Ker P_0^+ and H_0 into H_0^+;
(3) T has a dense range and $T[H_0] = H_0^+$;
(4) the subspaces H_p and H_m are invariant under S, and
(5) ASh = Th for every $h \in$ Ker P_0. □

With the operator S^+ on the subspace Ker P_0^+ = A[Ker P_0] given by $AS = S^+A$, one can associate bounded analytic semigroups on H_p^+

and H_m^+ that satisfy the relationships

(5.10a) $AU_p(x) = U_p^+(x)A, \quad AU_m(x) = U_m^+(x)A;$

(5.10b) $TU_p(x) = U_p^+(x)T, \quad TU_m(x) = U_m^+(x)T.$

The infinitesimal generators are $(-S^+|_{H_p^+})^{-1}$ and $(+S^+|_{H_m^+})^{-1}$. Using the compactness of $B = I - A$ one easily shows that for $x \geq 0$ the operators $U_p^+(x) - U_+(x)$ and $U_m^+(x) - U_-(x)$ (and therefore $P_p^+ - P_+$ and $P_m^+ - P_-$) are compact.

If $H = L_2[-1,+1]$, the operator S is self-adjoint with respect to the inner product

$$\langle f,g\rangle_A = \langle Af,g\rangle \quad (f,g \in \operatorname{Ker} P_0),$$

which is equivalent to the usual one. If F denotes the resolution of the identity of the opretor S, then $P_p h = F((0,+\infty))(I-P_0)h$ and $P_m h = F((-\infty,0))(I-P_0)h$ for every $h \in L_2[-1,+1]$ (see Section III.3 of [22]; for an anisotropic case see [15]).

THEOREM 5.3. *Let the C-admissible space* H *be as above, and let* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Assume* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t)P_n(t)dt \leq 1$ $(n = 0,1,2,\ldots)$. *Then there exists a subspace* N *of* Ker $(I - B)$ *such that*

(5.11a) $H_p \oplus N \oplus H_- = H, \quad H_m \oplus N \oplus H_- = H.$

Furthermore, if $\{a_n, a_{n+1}\} \neq \{1\}$ $(n = 0,1,2,\ldots)$, *then*

(5.11b) $H_p \oplus \operatorname{Ker} A \oplus H_- = H, \quad H_m \oplus \operatorname{Ker} A \oplus H_+ = H.$

Proof. For later use (in multigroup Transport Theory) we shall postpone the application of Theorems 3.1 and 4.3 until the second half of the proof. Fist we assume that $a_n < +1$ $(n = 0,1,2,\ldots)$. Consider the operator integral equations

(5.12a) $$\psi_+(x) - \int_0^{+\infty} H(x-y)B\psi_+(y)dy = \omega(x) \quad (0<x<+\infty);$$

(5.12b) $$\psi_-(x) - \int_{-\infty}^{0} H(x-y)B\psi_-(y)dy = \omega(x) \quad (-\infty<x<0).$$

Let the subspaces $\tilde{H}_p$ and $\tilde{H}_m$ be defined by

(5.13a) $\tilde{H}_p = \{-\lim_{x\downarrow 0} \psi_+(x) \,|\, \exists\phi \in H: \psi_+ \in L_\infty((0,+\infty);H)$ satisfies (5.12a) for $\omega(x) = TH(x)\phi\};$

(5.13b) $\tilde{H}_m = \{-\lim_{x\uparrow 0}\psi_-(x) \,|\, \exists\phi\in H: \psi_-\in L_\infty((-\infty,0);H)$ satisfies (5.12b) for $\omega(x) = TH(x)\phi\}$.

(Note that ψ_+ and ψ_- have continuous extensions to $[0,+\infty)$ and $(-\infty,0]$, respectively). For $\omega(x) = TH(x)\phi$ $(0\neq x\in\mathbb{R})$ the functions ψ_+ and ψ_- satisfy the operator differential equation $(T\psi)'(x) = -(I-B)\psi(x)$ on $(0,+\infty)$ and $(-\infty,0)$, respectively. Hence, $\tilde{H}_p\subset\tilde{H}_p$ and $\tilde{H}_m\subset\tilde{H}_m$.

Let us prove that for $\omega(x) = TH(x)\phi$ with $\phi\in H$ every solution of Eq. (5.12a) and Eq. (5.12b) is uniquely determined by its value at $x=0$. Let ψ_+ be a solution of Eq. (5.12a) with $\omega(x) = TH(x)\phi$ $(0<x<+\infty)$ and $\lim_{x\downarrow 0}\psi_+(x) = 0$. As in the proof of Theorem 2.1 one sees that $(T\psi)'(x) = -(I-B)\psi(x)$ $(0<x<+\infty)$, and thus the right derivative of $T\psi$ at $x=0$ exists and vanishes. Putting $\psi(x)=0$ for $x<0$, the function $\psi:(-\infty,+\infty)\to H$ is bounded and satisfies the equation $(T\psi)'(x) = -(I-B)\psi(x)$ $(x\in\mathbb{R})$ and the identity $\psi(0)=0$. So ψ is a solution of Eq. (5.2) in $L_\infty((-\infty,+\infty);H)$ with right-hand side $\omega(x)\equiv 0$. Thus $\psi=0$. Hence, every solution is uniquely determined by its value at $x=0$.

Since $TH(x)\phi = TH(x)P_+\phi$ $(x>0)$ and $TH(x)\phi = TH(x)P_-\phi$ $(x<0)$, we have

(5.14a) $\dim\{\psi_+ \,|\, \psi_+(x) - \int_0^{+\infty} H(x-y)B\psi_+(y)dy = 0 \text{ for } 0<x<+\infty\} = \dim[\tilde{H}_p\cap H_-]<+\infty;$

(5.14b) $\dim\{\psi_- \,|\, \psi_-(x) - \int_{-\infty}^{0} H(x-y)B\psi_-(y)dy = 0 \text{ for } +\infty<x<0\} = \dim[\tilde{H}_m\cap H_+]<+\infty.$

Further, $\tilde{H}_p + H_-$ (resp. $\tilde{H}_m + H_+$) is the set of those vectors $\phi\in H$ for which Eq. (5.12a) (resp. (5.12b)) with right-hand side $\omega(x) = TH(x)$ has a solution $\psi_+\in L_\infty((0,+\infty);H)$ (resp. $\psi_-\in L_\infty((-\infty,0);H)$). Hence,

(5.15a) $\operatorname{codim}[\tilde{H}_p + H_-]\leq \operatorname{codim}\{\omega\in L_\infty((0,+\infty);H) \,|\, \text{Eq. (5.12a) has a solution}\}<+\infty;$

(5.15b) $\operatorname{codim}[\tilde{H}_m + H_+]\leq \operatorname{codim}\{\omega\in L_\infty((-\infty,0);H) \,|\, \text{Eq. (5.12b) has a solution}\}<+\infty.$

But $\tilde{H}_p\subset H_p$ and $\tilde{H}_m\subset H_m$, and therefore

(5.16a) $\dim[\tilde{H}_p\cap H_-]\leq\dim[H_p\cap H_-]$, $\quad\dim[\tilde{H}_m\cap H_+]\leq\dim[H_m\cap H_+]$;

(5.16b) $\operatorname{codim}[\tilde{H}_p+H_-]\geq\operatorname{codim}[H_p+H_-]$, $\operatorname{codim}[\tilde{H}_m+H_+]\geq\operatorname{codim}[H_m+H_+]$.

Put $V = P_+P_p + P_-P_m$. Then $I - V = P_+P_m + P_-P_p = (P_- - P_+)(P_p - P_+)$ is a compact operator (see Theorem 5.1, third part). Since $\operatorname{Ker} V = [H_p \cap H_-] \oplus [H_m \cap H_+]$ and $\operatorname{Im} V = [H_p + H_-] \cap [H_m + H_+]$, one gets

$$(5.17)\quad \dim[H_p \cap H_-] + \dim[H_m \cap H_+] = \operatorname{codim}[H_p + H_-] + \operatorname{codim}[H_m + H_+].$$

From Theorem 1 of [7] (whose proof is based on methods from [9]) it follows that the right (left) indices $\kappa_1,\dots,\kappa_n$ $(\rho_1,\dots,\rho_m)$ for the Wiener-Hopf factorization of the symbol (3.4) of Eq. (5.12a) with respect to the imaginary axis $(-i\infty,+i\infty)$ satisfy

$$(5.18a)\quad \dim\{\psi_+ \mid \psi_+(x) - \int_0^{+\infty} H(x-y)B\psi_+(y)dy = 0 \text{ for } 0<x<+\infty\} = -\sum_{\kappa_i<0}\kappa_i;$$

$$(5.18b)\quad \operatorname{codim}\{\omega \in L_\infty((0,+\infty);H) \mid \text{Eq. (5.12a) has a solution}\} = +\sum_{\kappa_i>0}\kappa_i;$$

$$(5.18c)\quad \dim\{\psi_- \mid \psi_-(x) - \int_{-\infty}^{0} H(x-y)B\psi_-(y)dy = 0 \text{ for } -\infty<x<0\} = +\sum_{\rho_j>0}\rho_j;$$

$$(5.18d)\quad \operatorname{codim}\{\omega \in L_\infty((-\infty,0);H) \mid \text{Eq. (5.12b) has a solution}\} = -\sum_{\rho_j<0}\rho_j.$$

However, a slight change of B in the norm of $L(H)$ does not change the right (left) sum index $\sum_{i=1}^{n}\kappa_i$ $(\sum_{j=1}^{m}\rho_j)$ (cf. [11], Lemma 7.2), and for an operator B of finite rank the right and left sum indices are the same (cf. [11], Theorem 8.2). Because B is a limit in the norm of $L(H)$ of operators of finite rank, we have

$$(5.19)\quad \sum_{i=1}^{n}\kappa_i = \sum_{j=1}^{m}\rho_j.$$

Next we combine (5.14) - (5.19) and the inclusions $\tilde{H}_p \subset H_p$ and $\tilde{H}_m \subset H_m$ and conclude that the equality sign holds in (5.15) - (5.16) and that

$$\tilde{H}_p \cap H_- = H_p \cap H_-, \qquad \tilde{H}_m \cap H_+ = H_m \cap H_+;$$
$$\tilde{H}_p + H_- = H_p + H_-, \qquad \tilde{H}_m + H_+ = H_m + H_+.$$

So if $h_p \in H_p$, then $h_p = \tilde{h}_p + h_-$ for some $\tilde{h}_p \in \tilde{H}_p$ and $h_- \in H_-$, and therefore $h_p - \tilde{h}_p = h_- \in H_p \cap H_-$. So $h_p - \tilde{h}_p \in \tilde{H}_p \cap H_- \subset \tilde{H}_p$, and thus $h_p \in \tilde{H}_p$. We may conclude that $\tilde{H}_p = H_p$ and $\tilde{H}_m = H_m$. Hence, the identities (5.13) provide a description of H_p and H_m different from the one given in the proof of Theorem 5.1. In particular, for some (and hence every) $1 \le p \le +\infty$ and every $\omega \in L_p((0,+\infty);H)$ (resp. $\omega \in L_p((-\infty,0);H)$) the Wiener-Hopf operator integral equation

(5.12a) (resp. (5.12b)) has a unique solution $\psi_+ \in L_p((0,+\infty);H)$ (resp. $\psi_- \in L_p((-\infty,0);H)$) if and only if the decomposition $H_p \oplus H_- = H$ (resp. $H_m \oplus H_+ = H$) holds true.

At this moment we incorporate Theorem 3.1 and conclude that

$$(5.20) \quad H_p \oplus H_- = H, \quad H_m \oplus H_+ = H.$$

Finally, we drop the assumption that $a_n < +1$ $(n = 0,1,2,\dots)$. Let M_+ (M_-) be a maximal strictly positive (negative) subspace of the singular subspace H_0 (endowed with the indefinite inner product (4.3)). Using the operators B_u and A_u defined at the end of the proof of Theorem 5.1, it follows from (5.20) that

$$H_p \oplus M_+ \oplus H_- = H, \quad H_m \oplus M_- \oplus H_+ = H.$$

So $H_p \cap H_- = H_m \cap H_+ = \{0\}$ and $[H_p \oplus H_0] + H_- = [H_m \oplus H_0] + H_+ = H$. But for a subspace N of H_0 we have $H_p \oplus N \oplus H_- = H$ (resp. $H_m \oplus N \oplus H_+ = H$) if and only if $N \oplus \{[H_p \oplus H_-] \cap H_0\} = H_0$ (resp. $N \oplus \{[H_m \oplus H_+] \cap H_0\} = H_0$). By Proposition III 5.5 of [22] the subspace $[H_p \oplus H_-] \cap H_0$ is strictly negative and the subspace $[H_m \oplus H_+] \cap H_0$ is strictly positive. (Actually this is proved in $L_2[-1,+1]$, but H_0 does not depend on the specific form of H). By formula (III 7.5) of [22] there exists a subspace N of Ker A that is both maximal positive and maximal negative (and hence neutral). For this subspace N formula (5.11a) is clear. One may take N = Ker A if and only if $\dim \operatorname{Ker} A = \frac{1}{2} \dim H_0$, and the latter is true if and only if $\{a_n, a_{n+1}\} \neq \{1\}$ $(n \geq 0)$ (cf. the paragraph following the statement of Theorem 4.3), and therefore in this case formula (5.11b) is clear. □

COROLLARY 5.4. *Let the C-admissible space* H *be as above, and let* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Assume* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t) P_n(t)\,dt \leq 1$ $(n = 0,1,2,\dots)$. *If for some subspace* N *of* H_0 *the decomposition* $H_p \oplus N \oplus H_- = H$ *holds, then also*

$$(5.21) \quad H_p^+ \oplus T[N] \oplus H_- = H,$$

where $T[N] \subset H_0^+$. *Further, if* P *is the projection of* H *onto* $H_p \oplus N$ *along* H_- *and* P^+ *is the projection of* H *onto* $H_p^+ \oplus T[N]$ *along* H_-, *then we have the intertwining property*

(5.22) $TP = P^+T$.

Proof. Note that $P_p^+ - P_+$ is a compact operator (see the paragraph following the proof of Corollary 5.2). Put $V^+ = P_+P_p^+ + P_-P_m^+$. Then $I - V^+ = P_+P_m^+ + P_-P_p^+ = (P_- - P_+)(P_p^+ - P_+)$ is a compact operator and $TV = V^+T$. But

$$\operatorname{Im} V = [H_p \oplus H_-] \cap [H_m \oplus H_+], \qquad \operatorname{Im} V^+ = [H_p^+ \oplus H_-] \cap [H_m^+ \oplus H_+];$$

$$\operatorname{Ker} V = [H_p \cap H_-] \oplus [H_m \cap H_+] \oplus H_0, \; \operatorname{Ker} V^+ = [H_p^+ \cap H_-] \oplus [H_m^+ \cap H_+] \oplus H_0^+.$$

From these identities we get $\operatorname{codim}[H_p^+ \oplus H_-] = \operatorname{codim}[H_p \oplus H_+]$ and $[H_p^+ \oplus H_-] \cap \operatorname{Im} T = T[H_p \oplus H_-]$, and thus $[H_p^+ \oplus H_-] \cap T[N] = \{0\}$. Therefore, $\operatorname{codim}[H_p^+ \oplus H_-] = \operatorname{codim}[H_p \oplus H_-] = \dim N = \dim T[N]$, and formula (5.21) is clear. The intertwining property (5.22) is obvious. □

We conclude this section with some historical remarks. For $H = L_2[-1,+1]$ the decompositions (5.11b) are due to Hangelbroek in the non-conservative degenerate case ($0<a_0<1$, $-a_0 \le a_n \le a_0$ for $1 \le n \le N$, $a_n = 0$ for $n \ge N+1$) and to Lekkerkerker [20] in the conservative isotropic case ($a_0 = 1$, $a_n = 0$ for $n \ge 1$). For non-conservative cases formula (5.22) originates from Hangelbroek. For $H = L_2[-1,+1]$ the results of this section can be found in Sections III.3 - III.5 of [22].

6. FORMAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS

In Sections 3 and 4 the finite-slab problem (3.1) and the half-space problems (3.2) and (4.6) were shown to be well-posed under certain conditions on the phase function and the underlying function space. In this section the semigroup and intertwining properties derived in Section 5 will be exploited to obtain formal expressions for the solution of somewhat more general boundary value problems. Throughout this section H stands for a C-admissible Banach space that is continuously and densily embedded in $L_1[-1,+1]$; it is supposed that T has a dense range.

THEOREM 6.1. *Let the C-admissible space* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r > 1$. *Assume that* $a_n = (n+\tfrac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t) P_n(t)\,dt \le +1$ $(n = 0,1,2,\ldots)$. *Then for every* $\chi \in H$ *there exists a unique function* $\psi : (0,\tau) \to H$ *such that* $T\psi$ *is strongly differentiable on*

$(0,\tau)$ *and satisfies the equations*

$$(6.1a)\quad (T\psi)'(x) = -(I-B)\psi(x) \quad (0<x<\tau<+\infty);$$

$$(6.1b)\quad \lim_{x\downarrow 0} \|TP_+\psi(x) - P_+\chi\|_H = 0, \quad \lim_{x\uparrow\tau} \|TP_-\psi(x) - P_-\chi\|_H = 0.$$

The solution ψ *has the form*

$$(6.2a)\quad \psi(x) = [T^{-1}U_p^+(x) + T^{-1}U_m^+(\tau-x) + T^{-1}(I - xAT^{-1})P_0^+](V_\tau^+)^{-1}\chi,$$

where V_τ^+ *is the invertible operator given by*

$$(6.2b)\quad V_\tau^+ = P_+[P_p^+ + U_m^+(\tau)] + P_-[P_m^+ + U_p^+(\tau)] + P_0^+ - \tau P_-AT^{-1}P_0^+.$$

Note that $\psi(x)$ is well-defined for $0<x<\tau$, because $(U_p^+(x))_{x\geq 0}$ and $(U_m^+(x))_{x\geq 0}$ are analytic semigroups and $T[H_0] = H_0^+$.

THEOREM 6.2. *Let the C-admissible space* H *be compatible with* $\hat{g} \in L_r[-1,+1]$ *for some* $r>1$. *Assume that* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}}\int_{-1}^{+1}\hat{g}(t)P_n(t)dt \leq 1$ $(n=0,1,2,\ldots)$ *and that* $\{a_n,a_{n+1}\} \neq \{1\}$ *for every* $n\geq 0$. *Then there exists a unique function* $\psi : (0,+\infty)\to H$ *such that* $T\psi$ *is strongly differentiable on* $(0,+\infty)$ *and satisfies the equations*

$$(6.3a)\quad (T\psi)'(x) = -(I-B)\psi(x) \quad (0<x<+\infty);$$

$$(6.3b)\quad \lim_{x\downarrow 0} \|TP_+\psi(x) - \chi_+\|_H = 0, \quad \|T\psi(x)\|_H = O(1) \quad (x\to+\infty),$$

for every $\chi_+ \in H_+$. *The solution* ψ *has the form*

$$(6.4)\quad \psi(x) = T^{-1}U_p^+(x)P^+\chi_+, \quad 0<x<+\infty,$$

where P^+ *is the projection of* H *along* H_- *onto* $H_p^+ \oplus \mathrm{span}\{TP_n : a_n=+1\}$.

Note that $\psi(x)$ is well-defined for $0<x<+\infty$. The existence of the projection P^+ will be derived in the proof of Theorem 6.2.

Proof of Theorem 6.1. Let $\psi : (0,\tau)\to H$ be a solution of Eq. (6.1). By definition, $T\psi$ is strongly differentiable on $(0,\tau)$ with derivative $-(I-B)\psi$. Using (5.9c) - (5.9d) it is clear that $TP_p\psi(.)$ and $TP_m\psi(.)$ are differentiable with derivatives $-AP_p\psi(.)$ and $-AP_m\psi(.)$, respectively. So both functions satisfy the equation $\dot{\phi} = -AT^{-1}\phi$. From the contents of Section IX 1.3 of [17] it follows that on H_p^+ and H_m^+ the initial value problems

$$\dot{\phi}(x) = -AT^{-1}\phi(x), \quad x\in[\tau_1,\tau_2]\subset(0,\tau),$$

with initial values $TP_p\psi(\tau_1) \in H_p^+$ and $TP_m\psi(\tau_2) \in H_m^+$ have unique solutions given by the respective expressions

$$\phi_1(x) = U_p^+(x-\tau_1)TP_p\psi(\tau_1) = TU_p(x-\tau_1)P_p\psi(\tau_1), \quad x \geq \tau_1;$$

$$\phi_2(x) = U_m^+(\tau_2-x)TP_m\psi(\tau_2) = TU_m(\tau_2-x)P_m\psi(\tau_2), \quad x \leq \tau_2.$$

Here we have employed (5.10b). But $TP_p\psi(.)$ and $TP_m\psi(.)$ also satisfy these initial value problems. By the uniqueness of the solution of these problems and the triviality of $\operatorname{Ker} T$ we obtain for $\tau_1 \leq x \leq \tau_2$:

(6.5) $\quad P_p\psi(x) = U_p(x-\tau_1)P_p\psi(\tau_1), \quad P_m\psi(x) = U_m(\tau_2-x)P_m\psi(\tau_2).$

By (5.9c) - (5.9d) the function $TP_0\psi(.)$ satisfies the equation $\dot{\phi} = -AT^{-1}\phi$ on $(0,\tau)$. But T acts as an invertible operator from H_0 onto H_0^+, H_0 has a finite dimension and $(T^{-1}A)^2h = 0$ for $h \in H_0$. Therefore,

(6.6) $\quad P_0\psi(x) = e^{-xT^{-1}A}\phi_0 = (I - xT^{-1}A)\phi_0, \quad 0<x<\tau,$

where $\phi_0 \in H_0$. From (6.5) and (6.6) we get for $\tau_1 \leq x \leq \tau_2$:

(6.7) $\quad \psi(x) = U_p(x-\tau_1)P_p\psi(\tau_1) + U_m(\tau_2-x)P_m\psi(\tau_2) + (I - xT^{-1}A)\phi_0.$

Suppose that ψ satisfies (6.1b) for some $x \in H$. Substituting $x = \tau_1$ $(x = \tau_2)$ into (6.7), premultiplying by TP_+ (TP_-) and taking the limit as $\tau_1 \downarrow 0$ $(\tau_2 \uparrow \tau)$ we see that $\lim_{x\downarrow 0} TP_+P_p\psi(x)$ and $\lim_{x\uparrow\tau} TP_-P_m\psi(x)$ exist. Suppose that for some $N \subset H_0$ the decompositions

$$H_p \oplus N \oplus H_- = H_m \oplus N \oplus H_- = H$$

hold true (cf. Theorem 5.3, where the existence of such $N \subset \operatorname{Ker} A$ is proved). Then by Corollary 5.4 we have

$$H_p^+ \oplus T[N] \oplus H_- = H_m^+ \oplus T[N] \oplus H_- = H.$$

Denoting by P^+ (Q^+) the projection of H onto $H_p^+ \oplus T[N]$ $(H_m^+ \oplus T[N])$ along H_- (H_+) and by P (Q) the projection of H onto $H_p \oplus N$ $(H_m \oplus N)$ along H_- (H_+), one has

$$TP = P^+T, \quad TQ = Q^+T$$

(cf. (5.22)). But $P^+(TP_+P_p) = TPP_+P_p = TP_p$ and $Q^+(TP_-P_m) = TQP_-P_m$

$= TP_m$. So $\hat{\phi}_p = \lim_{x\downarrow 0} TP_p\psi(x)$ and $\hat{\phi}_m = \lim_{x\uparrow\tau} TP_m\psi(x)$ exist. Taking $\tau_1 \downarrow 0$ and $\tau_2 \uparrow \tau$ in (6.7) and employing (5.10b), we see that

$$\psi(x) = T^{-1}U_p^+(x)\hat{\phi}_p + T^{-1}U_m^+(\tau-x)\hat{\phi}_m + (I - xT^{-1}A)\phi_0, \qquad 0<x<\tau,$$

where $\hat{\phi}_p \in H_p^+$, $\hat{\phi}_m \in H_m^+$ but $\phi_0 \in H_0$. Substituting (6.1b) one obtains

$$V_\tau^+(\hat{\phi}_p + \hat{\phi}_m + T\phi_0) = \chi,$$

where V_τ^+ has the form (6.2b).

It remains to prove that V_τ^+ is invertible. Because the boundary value problem (6.1) certainly has a solution for $\chi = T\phi \in \operatorname{Im} T$ (cf. (3.1)), the operator V_τ^+ has a dense range. From the paragraph following the statement of Corollary 5.2 it is clear that $U_p^+(x) - U_+(x)$ and $U_m^+(x) - U_-(x)$ are compact operators. But then $V_\tau^+ - I$ is easily shown to be compact and the invertibility of V_τ^+ is established. □

Proof of Theorem 6.2. Let $\psi : (0,+\infty) \to H$ be a solution of the boundary value problem (6.3). According to Theorem 6.1 for every $0<\tau<+\infty$ there exists a (unique) $\chi_\tau \in H$ with $P_+\chi_\tau = \chi_+$ such that ψ has the form (6.2a) on $(0,\tau)$. More precisely, for $0<x<\tau$ we have

$$T\psi(x) = U_p^+(x)(V_\tau^+)^{-1}\chi_\tau + U_m^+(\tau-x)(V_\tau^+)^{-1}\chi_\tau + (I - xAT^{-1})P_0^+(V_\tau^+)^{-1}\chi_\tau.$$

So $\lim_{x\downarrow 0} T\psi(x)$ exists and this limit

(6.8) $$\lim_{x\downarrow 0} T\psi(x) = P_p^+(V_\tau^+)^{-1}\chi_\tau + U_m^+(\tau)(V_\tau^+)^{-1}\chi_\tau + P_0^+(V_\tau^+)^{-1}\chi_\tau$$

does not depend on τ, and hence none of the three terms at the right of (6.8) depends on τ. Further,

$$T\psi(\tau) = U_p^+(\tau)P_p^+(V_\tau^+)^{-1}\chi_\tau + P_m^+(V_\tau^+)^{-1}\chi_+ + (I - \tau AT^{-1})P_0^+(V_\tau^+)^{-1}\chi_\tau.$$

Since $P_p^+(V_\tau^+)^{-1}\chi_\tau$ does not depend on τ, one has $\|U_p^+(\tau)P_p^+(V_\tau^+)^{-1}\chi_\tau\| \to 0$ as $\tau \to +\infty$ (cf. (5.5)). Because $P_0^+(V_\tau^+)^{-1}\chi_\tau$ does not depend on τ, it follows from the estimate $\|T\psi(x)\| = O(1)$ $(x \to +\infty)$ that $P_0^+(V_\tau^+)^{-1}\chi_\tau \in T[\operatorname{Ker} A]$. From this estimate it is also clear that $\|P_m^+(V_\tau^+)^{-1}\chi_\tau\| = O(1)$ $(\tau \to +\infty)$. Since $U_m^+(\tau)P_m^+(V_\tau^+)^{-1}\chi_\tau$ does not depend on τ, it follows that $P_m^+(V_\tau^+)^{-1}\chi_\tau = 0$. Thus

$$T\psi(x) = U_p^+(x)\hat{\phi}_p + \hat{\phi}_0 \qquad (0<x<+\infty),$$

where $\Phi_p \in H_p^+$ and $\Phi_0 \in T[\mathrm{Ker}\ A]$. Substituting (6.1b) we get

(6.9) $\quad P_+(\Phi_p + \Phi_0) = \chi_+.$

First assume that $a_n < +1$ $(n = 0,1,2,\ldots)$. From Theorem 5.3 and Corollary 5.4 we have $H_p \oplus H_- = H$, $H_p^+ \oplus H_- = H$ and $TP = P^+T$, where P^+ denotes the projection of H onto H_p^+ along H_- and P denotes the projection of H onto H_p along H_-. Therefore, $\lim_{x\downarrow 0} T\psi(x) = P^+\chi_+$, and formula (6.4) is clear.

Next assume that $a_n \le 1$ $(n = 0,1,2,\ldots)$. Then there exists a subspace N of Ker A such that $H_p \oplus N \oplus H_- = H$ (see Theorem 5.3), and therefore $H_p^+ \oplus T[N] \oplus H_- = H$ (see Corollary 5.4). So given $\chi_+ \in H_+$ there exist $\Phi_p \in \hat{H}_p$ and $\Phi_0 \in T[N] \subset T[\mathrm{Ker}\ A]$ such that (6.9) holds true. Hence, for every $\chi_+ \in H_+$ the boundary value problem (6.3) has at least one solution. By Theorem 5.3 and Corollary 5.4 we have $H_p^+ \oplus T[\mathrm{Ker}\ A] \oplus H_- = H$ if and only if $\{a_n, a_{n+1}\} \ne \{1\}$ for every $n \ge 0$. But $\mathrm{Ker}\ A = \mathrm{span}\ \{P_n : a_n = 1\}$. Hence, the boundary value problem (6.3) has a unique solution for every $\chi_+ \in H_+$ if and only if $\{a_n, a_{n+1}\} \ne \{1\}$ for every $n \ge 0$. □

The unique solution of the boundary value problem (3.1) is obtained from (6.2a) by inserting $\chi = T\phi$. Intruducing the operator

$$V_\tau = P_+[P_p + U_m(\tau)] + P_-[P_m + U_p(\tau)] + P_0 - \tau P_- T^{-1} A P_0,$$

it follows from Corollary 5.2 that $V_\tau - I$ is a compact operator, and from (5.9c), (5.10b) and (6.2b) that $TV_\tau = V_\tau^+ T$. Since V_τ^+ is invertible and T has a dense range, it is clear that V_τ is invertible. Using $TV_\tau = V_\tau^+ T$ and (5.10b) one sees that the unique solution ψ of the boundary value problem (3.1) is given by

(6.10) $\quad \psi(x) = [U_p(x) + U_m(\tau - x) + (I - xT^{-1}A)P_0](V_\tau)^{-1}\phi, \quad 0<x<\tau.$

Similarly one solves in a formal way the boundary value problem (4.6) if $a_n \le +1$ and $\{a_n, a_{n+1}\} \ne \{1\}$ $(n = 0,1,2,\ldots)$.

A statement of the finite-slab problem in $L_2[-1,+1]$ by Hangelbroek [15] stimulated the author to investigate this problem. Independently of and parallel to the research leading to [22] Hangelbroek proved the invertibility of V_τ in $L_2[-1,+1]$ for a non-conservative case. In [22] in $L_2[-1,+1]$ this result has been extended to the conservative case. In [15] Hangelbroek announced an expression of the form (6.10) on $L_2[-1,+1]$ in the non-conservative case, using the boundary conditions (3.1b) and assuming the solution to be continuous on $[0,\tau]$.

Except Eq. (0.1) astrophysicists also study the more general equation

$$(6.11)\quad (\cos\theta)\frac{d\psi}{dx}(x,\omega) + \psi(x,\omega) = \frac{1}{2\pi}\int_\Omega \hat{g}(\omega\cdot\omega')\psi(x,\omega')d\omega' + f(x,\omega) \quad (\omega\in\Omega,\ 0<x<\tau),$$

where Ω is the unit sphere in $\mathbb{R}^3$ and $\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\in\Omega$. From (6.11) one obtains Eq. (0.1) by averaging over azimuth (i.e., by setting $\tilde{\psi}(x,\cos\theta) = (2\pi)^{-1}\int_0^{2\pi}\psi(x,\omega)d\phi$). Defining $\psi(x)$, $f(x)$, T and B by

$$\psi(x)(\omega) = \psi(x,\omega), \qquad f(x)(\omega) = f(x,\omega);$$
$$(Th)(\omega) = (\cos\theta)\psi(\omega),\ (Bh)(\omega) = (2\pi)^{-1}\int_\Omega \hat{g}(\omega\cdot\omega')h(\omega')d\omega', \quad (\omega\in\Omega)$$

one gets an operator differential equation of the form (0.4). Putting

$$(P_+h)(\omega) = \begin{cases} h(\omega), & \cos\theta\ge 0;\\ 0, & \cos\theta<0;\end{cases} \qquad (P_-h)(\omega) = \begin{cases} 0 & \cos\theta\ge 0;\\ h(\omega), & \cos\theta<0,\end{cases}$$

one may impose boundary conditions. The "natural" spaces to study Eq. (6.11) in are so-called C_Ω-admissible Banach spaces H of functions $h:\Omega\to\mathbb{C}$ with the following properties:

(1) functions in H that only differ at $\omega\in\Omega$ with $\cos\theta = 0$ are identified;

(2) for every $\phi\in C$ the operator T_ϕ defined by
$$(T_\phi h)(\omega) = \phi(\cos\theta)h(\omega)\ (\omega=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\in\Omega)$$
is bounded and $\|T_\phi\|\le M_H\sup\{|\phi(\mu)| : -1\le\mu\le 1\}$ for some finite constant M_H only depending on H.

Examples of such spaces are $L_p(\Omega)$ $(1\le p\le+\infty)$ and $C(\Omega_-)\oplus C(\Omega_+)$. For Eq. (6.11) and C_Ω-admissible Banach spaces the results of Sections 1 to 6 can be reproduced. The space $L_2(\Omega)$ appears in [21]; in [6] the spaces $L_p(\Omega)$ $(1\le p<+\infty)$ are considered.

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C.V.M. van der Mee
Wiskundig Seminarium
(= Dept. of Mathematics)
Vrije Universiteit
Amsterdam, The Netherlands

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