Integr. equ. oper. theory 27 (1997) 48 – 70
0378-620X/97/010048-23 $ 1.50+0.20/0

Integral Equations and Operator Theory

WEIGHTING OPERATOR PATTERNS OF PRITCHARD-SALAMON REALIZATIONS

M.A. Kaashoek, C.V.M. van der Mee and A.C.M. Ran

In this paper a complete characterization is given of those operator kernels which appear as weighting patterns of Pritchard-Salamon realizations. The result is based on an extension of the standard shift realization to weighted L_2-spaces of vector-valued functions.

0. INTRODUCTION

This paper concerns a class of infinite dimensional systems which has been introduced in [PS, S1] and is known as the Pritchard-Salamon class (cf., [C, CLTZ, vK]). Systems from this class have been successfully used in the analysis of control and optimization problems involving partial differential equations and/or delay equations (see, e.g., the books [CZ], [vK]).

In this paper we consider for Pritchard-Salamon systems the analogue of the weighting pattern (or, in other words, the impulse-response function), i.e., the inverse Laplace transform of the transfer function. The weighting pattern is a function on $0 \leq t < \infty$ whose values are bounded linear operators acting between (possibly infinite dimensional) input and output spaces. Our main result gives a complete description of the class of all operator-valued functions that can appear as the weighting pattern function of a Pritchard-Salamon system. Furthermore, for such an operator-valued function we show how one may construct a corresponding Pritchard-Salamon realization.

Pritchard-Salamon realizations have two infinite dimensional state spaces, one continuously and densely embedded in the other. The crucial role of this embedding is clarified further by writing the corresponding embedding operator explicitly. The latter helps to simplify the duality theory. As a second by-product we link the stable Pritchard-Salamon systems with the realization triples considered in [BGK 1, 2]. It turns out that after a small modification any system with a stable Pritchard-Salamon realization and finite dimensional input and output spaces is a realization of the type used in [BGK 1, 2]. The converse is not necessarily true.

The paper consists of four sections. In the first section we introduce the weighting pattern, state the main theorem and make the connection with the realization triples from [BGK 1, 2]. In Section 2 we give a new definition of the transfer function of a Pritchard-Salamon system, and show that it leads to the same formulas for the transfer function which usually appear. The third section gives a duality theorem, which we need for our main result. In the fourth section we present the construction of a Pritchard-Salamon realization, starting from the weighting pattern, and complete the proof of the main theorem. At the end of this section we also compare our results with the realization theory developed in [S2].

1. Main theorem and connection with realization triples

Let V and W be complex Hilbert spaces (not necessarily separable), and let $\tau : W \to V$ be a fixed continuous and dense (linear) imbedding. For $A(V \to V)$ a possibly unbounded operator we define the *part* A_W of A in W (with respect to the injection τ) by

$$\mathcal{D}(A_W) = \{x \in W \mid \tau x \in \mathcal{D}(A), A\tau x \in \tau[W]\}, \qquad \tau A_W x = A\tau x, \qquad x \in \mathcal{D}(A_W).$$

Then $A_W(W \to W)$ is a closed operator whenever $A(V \to V)$ is closed, but it may fail to be densely defined, even if $A(V \to V)$ is densely defined.

Let Y and U be complex Hilbert spaces. We call $\theta = (A, B, C; V, W, U, Y)$ a *Pritchard-Salamon realization* (or a *PS-realization* for short) if the following conditions hold:

(1) $-iA(V \to V)$ is densely defined and generates a strongly continuous semigroup $S(\cdot; -iA)$,

(2) $-iA_W(W \to W)$ is densely defined and generates a strongly continuous semigroup $S(\cdot; -iA_W)$ while

$$S(\cdot; -iA)\tau = \tau S(\cdot; -iA_W), \tag{1.1}$$

(3) $B \in \mathcal{L}(U, V)$ and $C \in \mathcal{L}(W, Y)$,

(4) there exist $t > 0$ and $\gamma > 0$ such that

$$\|CS(\cdot; -iA_W)x\|_{L_2([0,\,t],Y)} \leq \gamma \|\tau x\|_V, \qquad x \in W,$$

(5) there exist $t > 0$ and $\beta > 0$ such that

$$\int_0^t S(s; -iA)B\phi(s)\, ds \in \tau[W], \qquad \phi \in L_2([0,\,t], U),$$

and

$$\|\tau^{-1} \int_0^t S(s; -iA)B\phi(s)\, ds\|_W \leq \beta \|\phi\|_{L_2([0,\,t],U)}, \qquad \phi \in L_2([0,\,t], U).$$

The semigroup property guarantees that (4), respectively (5), holds for each $t > 0$, with the choice of the constant $\gamma > 0$, respectively $\beta > 0$, depending on t.

Given a Pritchard-Salamon realization as above the associated control system

$$
\begin{aligned}
x(t) &= S(t;-iA)x_0 + \int_0^t S(t-s;-iA)Bu(s)\,ds, \\
y(t) &= Cx(t),
\end{aligned}
$$

is a so-called *Pritchard-Salamon system* (see [PS]).

Usually, in the analysis of Pritchard-Salamon systems and their realizations the embedding operator τ appears only implicitly because one takes $W \subset V$. However, as we shall see in Section 3, the duality theory for PS-realizations simplifies considerably if one writes τ explicitly. In [PS] and other publications prior to [CLTZ], the requirement $\mathcal{D}(A) \subset W$ is part of the definition of a PS-realization. This circumvents several technical difficulties of the proofs, but restricts its applicability. Both in [CLTZ] and the present paper a more extensive class of PS-realizations is considered where it is not assumed that $\mathcal{D}(A) \subset W$.

Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization. The *input-output operator of θ* is the linear map $T_\theta : L_{2,\mathrm{loc}}(\mathbb{R}^+, U) \to L_{2,\mathrm{loc}}(\mathbb{R}^+, Y)$ defined by

$$(T_\theta \phi)(t) = C\tau^{-1} \int_0^t S(t-s;-iA)B\phi(s)ds.$$

This is well-defined by properties (5), (3) and the remark made directly after (5) (see also [CLTZ]). In fact, from (5) and (3) it follows that $T_\theta\phi$ is a continuous Y-valued function for each $\phi \in L_{2,\mathrm{loc}}(\mathbb{R}^+, U)$. As (4) holds for all t and $\tau[W]$ is dense in V, there is for every $t > 0$ a bounded linear operator $\Lambda_{\theta,t} : V \to L_2([0,t], Y)$ defined by

$$(\Lambda_{\theta,t}\tau w)(s) = CS(s;-iA_W)w, \qquad 0 \le s \le t, \qquad w \in W. \tag{1.2}$$

However, for $0 \le s \le \min\{t_1, t_2\}$ we have $(\Lambda_{\theta,t_1}\tau w)(s) = (\Lambda_{\theta,t_2}\tau w)(s)$. Therefore, there exists a unique linear operator $\Lambda_\theta : V \to L_{2,\mathrm{loc}}(\mathbb{R}^+, Y)$ such that

$$(\Lambda_\theta x)(s) = (\Lambda_{\theta,t}x)(s), \qquad 0 \le s \le t.$$

The operator Λ_θ is called the *observability operator* of θ. By (5) and the remark made directly after (5) there is for each $t > 0$ a bounded linear operator $\Gamma_{\theta,t} : L_2([0,t], U) \to W$ defined by

$$\Gamma_{\theta,t}\phi = \tau^{-1} \int_0^t S(s;-iA)B\phi(s)\,ds. \tag{1.3}$$

Now define the *weighting pattern* of θ to be the operator-valued function

$$k_\theta : \mathbb{R}^+ \to \mathcal{L}(U, Y), \qquad k_\theta(t)u = (\Lambda_\theta Bu)(t),$$

where Λ_θ is the observability operator of θ.

The main problem we consider in this paper is the characterization of those functions $k(\cdot)$ which appear as weighting patterns of a PS-realization. Our main result is the following.

THEOREM 1.1 *Let U and Y be complex Hilbert spaces, and let $k(\cdot) : \mathbb{R}^+ \to \mathcal{L}(U, Y)$. In order that $k(\cdot)$ is the weighting pattern of a PS-realization it is necessary and sufficient that for some $\mu \in \mathbb{R}$ the following hold:*

$$e^{\mu \cdot} k(\cdot) u \in L_2(\mathbb{R}^+, Y) \qquad (u \in U), \qquad e^{\mu \cdot} k(\cdot)^* y \in L_2(\mathbb{R}^+, U) \qquad (y \in Y), \tag{1.4}$$

where the asterisk denotes the adjoint.

As we shall show in Section 4 (see Corollary 4.3) from Theorem 1.1 it follows that the input-output operator and the weighting pattern are related as follows:

$$(T_\theta \phi)(t) = (P) \int_0^t k_\theta(t-s)\phi(s) ds, \qquad t \in \mathbb{R}^+ \qquad \text{a.e.}\ , \tag{1.5}$$

where the symbol (P) refers to the fact that the integral on the right hand side is to be understood as a Pettis integral, i.e.,

$$\langle (T_\theta \phi)(t), y \rangle = \int_0^t \langle k_\theta(t-s)\phi(s), y \rangle ds, \qquad y \in Y.$$

If the input space U and the output space Y are both finite dimensional, $U = \mathbb{C}^m$ and $Y = \mathbb{C}^r$, say, then $k(\cdot)$ may be viewed as an $r \times m$ matrix function, and (1.4) reduces to the requirement that the entries of $k(\cdot)$ belong to $e^{\mu \cdot} L_2(\mathbb{R}^+)$ for some μ. Furthermore, in this case the integral in (1.5) is a usual Lebesgue integral. In general, from condition (1.4) it does not follow that the integrand in (1.5) is Bochner integrable (see [Ka]).

The next lemma is a technical result the proof of which is based on the arguments used to prove Lemma 3.7 in [C]. Among other things, the lemma will be used to prove the necessity of the first part of (1.4).

In the sequel we write $-\omega_\theta$ for the maximum of the growth bounds of the two semigroups associated with a PS-realization θ.

LEMMA 1.2 *Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization, and let $-\omega_\theta$ be the maximum of the growth bounds of the semigroups $S(\cdot; -iA)$ and $S(\cdot; -iA_W)$. Then for every $\mu < \omega_\theta$ there are constants $\gamma(\mu)$ and $\beta(\mu)$, independent of t, such that for each $t > 0$*

$$\| e^{\mu \cdot} \Lambda_{\theta,t} x \|_{L_2([0,t],Y)} \leq \gamma(\mu) \, \|x\|_V\, , \qquad x \in V; \tag{1.6}$$

$$\| \Gamma_{\theta,t}(e^{\mu \cdot} \phi) \|_W \leq \beta(\mu) \, \|\phi\|_{L_2([0,t],U)}\, , \qquad \phi \in L_2([0,t],U). \tag{1.7}$$

Moreover, both statements also hold with L_2 replaced by L_1.

PROOF. Fix $0 < t_1 < \infty$. For $t > t_1$ take N such that $Nt_1 < t < (N+1)t_1$. For $x \in \tau[W]$ we have with $x = \tau y$ using (4)

$$\int_0^t e^{2\mu s} \|(\Lambda_{\theta,t}\tau y)(s)\|_Y^2 \, ds \leq$$

$$\sum_{n=0}^{N} e^{2n\mu t_1} \int_0^{t_1} e^{2\mu s} \, \|CS(s;-iA_W)S(nt_1;-iA_W)y\|_Y^2 \, ds \leq$$

$$\leq \gamma_1 \sum_{n=0}^{N} e^{2n\mu t_1} \, \|S(nt_1;-iA)\tau y\|_V^2 \leq$$

$$\leq \gamma_2 \sum_{n=0}^{N} e^{-2n(\omega-\mu)t_1} \, \|\tau y\|_V^2 = \gamma(\mu) \, \|\tau y\|_V^2 ,$$

which settles (1.6) for $x \in \tau[W]$. Since $\tau[W]$ is dense in V and $\Lambda_{\theta,t} : V \to L_2([0,t],Y)$ is bounded, (1.6) holds for all $x \in V$. For $t \leq t_1$ the same estimate trivially holds.

To prove the second part, let $\phi \in L_2([0,t],U)$. For $t > t_1$ we have, writing $t = Nt_1 + v$,

$$\int_0^t e^{\mu s} S(s;-iA)B\phi(s)\,ds = \sum_{n=0}^{N-1} e^{n\mu t_1} S(nt_1;-iA) \int_0^{t_1} e^{\mu s} S(s;-iA)B\phi(s+nt_1)\,ds +$$

$$+ e^{N\mu t_1} S(Nt_1;-iA) \int_0^{v} e^{\mu s} S(s;-iA)B\phi(s+Nt_1)\,ds.$$

Note that each term of the series belongs to $\tau[W]$. Moreover, by (5), applied for $t = t_1$ and $t = v$,

$$\begin{aligned}
\|\Gamma_{\theta,t}(e^{\mu\cdot}\phi)\|_W &\leq \sum_{n=0}^{N-1} e^{n\mu t_1} \left\| \tau^{-1} S(nt_1;-iA) \int_0^{t_1} e^{\mu s} S(s;-iA)B\phi(s+nt_1)\,ds \right\|_W + \\
&\quad + e^{N\mu t_1} \left\| \tau^{-1} S(Nt_1;-iA) \int_0^{v} e^{\mu s} S(s;-iA)B\phi(s+Nt_1)\,ds \right\|_W = \\
&= \sum_{n=0}^{N-1} e^{n\mu t_1} \left\| S(nt_1;-iA_W)\tau^{-1} \int_0^{t_1} e^{\mu s} S(s;-iA)B\phi(s+nt_1)\,ds \right\|_W + \\
&\quad + e^{N\mu t_1} \left\| S(Nt_1;-iA_W)\tau^{-1} \int_0^{v} e^{\mu s} S(s;-iA)B\phi(s+Nt_1)\,ds \right\|_W \leq \\
&\leq \beta_1 \left(\sum_{n=0}^{N-1} e^{-n(\omega-\mu)t_1} \left\| \tau^{-1} \int_0^{t_1} e^{\mu s} S(s;-iA)B\phi(s+nt_1)\,ds \right\|_W + \right. \\
&\quad \left. + e^{-N(\omega-\mu)t_1} \left\| \tau^{-1} \int_0^{v} e^{\mu s} S(s;-iA)B\phi(s+Nt_1)\,ds \right\|_W \right) \leq \\
&\leq \beta_2 \sum_{n=0}^{N} e^{-n(\omega-\mu)t_1} \, \|\phi\|_{L_2([0,t],U)} = \beta(\mu) \, \|\phi\|_{L_2([0,t],U)} ,
\end{aligned}$$

which completes the proof of (1.7).

By taking μ a little smaller if necessary, we see that the results above also hold with L_1 in place of L_2. Indeed, take $\mu_1 < \mu < \omega$. Then

$$\begin{aligned}\|e^{\mu_1 \cdot}\Lambda_{\theta,t}x\|_{L_1([0,t],Y)} &= \int_0^t e^{(\mu_1-\mu)s}\, \|e^{\mu s}(\Lambda_{\theta,t}x)(s)\|_Y \, ds \\ &\le \left\|e^{(\mu_1-\mu)\cdot}\right\|_{L_2[0,t]} \cdot \|e^{\mu\cdot}\Lambda_{\theta,t}x\|_{L_2([0,t],Y)} \\ &\le \gamma(\mu)\left\|e^{(\mu_1-\mu)\cdot}\right\|_{L_2[0,t]} \|x\|_V, \qquad x \in V.\end{aligned}$$

The analogue of (1.7) is proved in the same way. ■

From formula (1.6) in Lemma 1.2 it follows that for every $\mu < \omega_\theta$ and every $u \in U$ we have $e^{\mu\cdot}k_\theta(\cdot)u = e^{\mu\cdot}(\Lambda_\theta Bu)(\cdot) \in L_2(\mathbb{R}^+, Y)$. This proves the first condition of the necessity part of Theorem 1.1. By taking μ a little smaller, if necessary, and using the same argument as in the last paragraph of the previous proof, we see that $e^{\mu\cdot}k_\theta(\cdot)u \in L_1(\mathbb{R}^+, Y)$.

A PS-realization $\theta = (A, B, C; V, W, U, Y)$ is said to be *stable* if $\omega_\theta > 0$, i.e., if the semigroups $S(\cdot; -iA)$ and $S(\cdot; -iA_W)$ in (1) and (2) are both exponentially decaying semigroups. In this case (see [PS]) θ has the following two additional properties:

(4′) there is a bounded linear operator $\Lambda_\theta : V \to L_2(\mathbb{R}^+, Y)$ such that

$$\Lambda_\theta \tau x = CS(\cdot; -iA_W)x, \qquad x \in W,$$

(5′) there is a bounded linear operator $\Gamma_\theta : L_2(\mathbb{R}^+, U) \to W$ such that

$$\tau\Gamma_\theta\phi = \int_0^\infty S(s; -iA)B\phi(s)\, ds \qquad (\in \tau[W])$$

for $\phi \in L_2(\mathbb{R}^+, U)$.

Note that (4′) and (5′) automatically imply (4) and (5). So $\theta = (A, B, C; V, W, U, Y)$ is a stable PS-realization if and only if (1), (2) and (3) hold, the semigroups in (1) and (2) are exponentially decaying, and (4′) and (5′) are fullfilled. Observe that the operator defined in (4′) is indeed the same as the observability operator defined earlier, i.e., in the particular case of a stable PS-realization the image of Λ_θ is in $L_2(\mathbb{R}^+, Y)$ instead of just in $L_{2,\mathrm{loc}}(\mathbb{R}^+, Y)$.

For the sake of completeness, let us prove (4′) and (5′) for a stable PS-realization θ. Notice that for every $x \in W$ we have $CS(\cdot; -iA_W)x \in L_2(\mathbb{R}^+, Y)$ because $S(\cdot; -iA_W)$ is exponentially decaying. Applying Lemma 1.2, formula (1.6) with $\mu = 0$, the boundedness of Λ_θ viewed as a map from V to $L_2(\mathbb{R}^+, Y)$ follows. To prove (5′) first observe that for every $\phi \in L_2(\mathbb{R}^+, U)$ the integral in (5′) exists and defines a vector, v say, in V. Next, for every positive integer n, consider $w_n = \Gamma_{\theta,n}\phi$, and $v_n = \tau w_n$. Applying (1.7) with $\mu = 0$ we see that for every $m > n$ we have

$$\|\Gamma_{\theta,n}\phi - \Gamma_{\theta,m}\phi\|_W \le \beta(0)\|\phi\|_{L_2([n,m],U)}.$$

As ϕ is in $L_2(\mathbb{R}^+, U)$, we see that w_n is a Cauchy sequence in W. Let w be its limit in W. Then $v_n = \tau w_n \to \tau w$. On the other hand, $v_n \to v$. Thus $v = \tau w \in \tau[W]$, and the operator Γ_θ in (5′) is well-defined. The rest of (5′) is then an easy consequence of Lemma 1.2.

The next lemma shows that for many purposes we may restrict our attention to the stable case.

LEMMA 1.3 *Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization. For any real number μ we have that $\theta(\mu) := (A + i\mu I_V, B, C; V, W, U, Y)$ is a PS-realization with the following properties:*

(1) $(A + i\mu I_V)_W = A_W + i\mu I_W$,
(2) $S(t; -iA + \mu I_V) = e^{\mu t} S(t; -iA)$, *and* $S(t; -iA_W + \mu I_W) = e^{\mu t} S(t; -iA_W)$,
(3) $T_{\theta(\mu)}\phi = e^{\mu t} T_\theta(e^{-\mu \cdot}\phi)$,
(4) $k_{\theta(\mu)}(t)u = e^{\mu t} k_\theta(t)u$ *for all* $u \in U$.

If $\mu < \omega_\theta$, where $-\omega_\theta$ is the maximum of the growth bounds of the semigroups $S(\cdot; -iA)$ and $S(\cdot; -iA_W)$, then $\theta(\mu)$ is a stable PS-realization.

PROOF. Items (1) and (2) are straightforward. From item (2) it also follows that $\theta(\mu)$ is a PS-realization, as is readily checked. For item (3), compute using (2),

$$(T_{\theta(\mu)}\phi)(t) = C\tau^{-1} \int_0^t e^{\mu(t-s)} S(t-s; -iA) B\phi(s)\, ds = e^{\mu t} T_\theta(e^{-\mu \cdot}\phi).$$

Next, (4) is a consequence of the fact that for any $t > 0$, and any $w \in W$ we have $(\Lambda_{\theta(\mu),t}\tau w)(s) = e^{\mu s} C S(s; -iA_W) w$, and hence $(\Lambda_{\theta(\mu)} v)(s) = e^{\mu s}(\Lambda_\theta v)(s)$ for all $v \in V$. Finally, the stability of $\theta(\mu)$ is clear in case $\mu < \omega_\theta$. ■

One may view $\mathcal{D}(A)$ as a Hilbert space by endowing it with the graph norm $\|x\|_{\mathcal{D}(A)} = [\|x\|_V^2 + \|Ax\|_V^2]^{1/2}$ where $x \in \mathcal{D}(A)$.

PROPOSITION 1.4 *Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization. Then there exists a unique $\tilde{C} : \mathcal{D}(A) \to Y$ such that $Cx = \tilde{C}\tau x$, $x \in \mathcal{D}(A_W)$, and $\tilde{C}$ is A-bounded, i.e., $\tilde{C} \in \mathcal{L}(\mathcal{D}(A), Y)$ where $\mathcal{D}(A)$ is endowed with the graph norm $\|\cdot\|_{\mathcal{D}(A)}$.*

The operator $\tilde{C}$ defined in the above proposition will be called the *extended output operator* associated with θ.

PROOF. First, we prove that we may assume without loss of generality that θ is stable. To see this, let $\mu < \omega_\theta$. Let $\theta(\mu)$ be the stable PS-realization as constructed in

Lemma 1.3. Since the graph norms associated with $A + i\mu I_V$ and A are equivalent norms on $\mathcal{D}(A + i\mu I_V) = \mathcal{D}(A)$, we see that it suffices to prove the proposition for $\theta(\mu)$ in place of θ.

Let us now assume that θ is stable. Then, by property (4′), with $L_2(\mathbb{R}^+, Y)$ replaced by $L_1(\mathbb{R}^+, Y)$, there exists a constant $\gamma \geq 0$ such that

$$\int_0^\infty \|CS(t; -iA_W)z\|_Y \, dt \leq \gamma \|\tau z\|_V, \qquad z \in W. \tag{1.8}$$

By taking the Fourier transform, we see that for $\operatorname{Im} \lambda > 0$

$$\|C(\lambda - A_W)^{-1} z\|_Y \leq \gamma \|\tau z\|_V, \qquad z \in W. \tag{1.9}$$

Thus, $C(\lambda - A_W)^{-1}\tau^{-1}$ extends to a bounded operator on V, which we denote by $\tilde{C}(\lambda)$. Now, fix λ, and define $\tilde{C} : \mathcal{D}(A) \to Y$ by $\tilde{C} = \tilde{C}(\lambda)(\lambda - A)$. Then, for $x \in \mathcal{D}(A)$, we have by (1.9)

$$\begin{aligned} \|\tilde{C}x\|_Y &= \|\tilde{C}(\lambda)(\lambda - A)x\|_Y \leq \|\tilde{C}(\lambda)\|(|\lambda|\|x\|_V + \|Ax\|_V) \leq \\ &\leq \gamma_1 \|\tilde{C}(\lambda)\|(|\lambda| + 1)(\|x\|_V + \|Ax\|_V) \leq \gamma_2 \|x\|_{\mathcal{D}(A)}. \end{aligned}$$

Thus $\tilde{C}$ is A-bounded. Next, for $z \in \mathcal{D}(A_W)$:

$$\begin{aligned} \tilde{C}\tau z &= \tilde{C}(\lambda)(\lambda - A)\tau z = \tilde{C}(\lambda)\tau(\lambda - A_W)z = \\ &= C(\lambda - A_W)^{-1}\tau^{-1}\tau(\lambda - A_W)z = Cz, \end{aligned}$$

where we use $\tau(\lambda - A_W)x = (\lambda - A)\tau z$ for $z \in \mathcal{D}(A_W)$, which holds because of (1.1).

It remains to prove the uniqueness of $\tilde{C}$. Let $\check{C} : \mathcal{D}(A) \to Y$ be A-bounded and satisfy $\check{C}\tau z = Cz$ for $z \in \mathcal{D}(A_W)$. Then

$$C(\lambda - A_W)^{-1} = \check{C}\tau(\lambda - A_W)^{-1} = \check{C}(\lambda - A)^{-1}\tau,$$

so for $z \in \tau[W]$ we have $C(\lambda - A_W)^{-1}\tau^{-1}z = \check{C}(\lambda - A)^{-1}z$. Therefore, $\check{C}(\lambda - A)^{-1} = \tilde{C}(\lambda)$. Hence $\check{C} = \tilde{C}(\lambda)(\lambda - A) = \tilde{C}$. ■

The extended output operator $\tilde{C}$ defined in Proposition 1.4 is very useful, since it allows one to work with realizations having one state space. (See also the remarks below.)

We conclude this section with a remark which relates stable PS-realizations to the realization triples appearing in [BGK1,2]. Let $\theta = (A, B, C; V, W, U, Y)$ be a stable PS-realization. Consider the triple of operators $\tilde{\theta} = (A, B, \tilde{C}; V, U, Y)$, where $\tilde{C}$ is as in Proposition 1.4. This triple has the following properties:

(i) $-iA(V \to V)$ generates a strongly continuous exponentially decaying semigroup $S(\cdot; -iA)$ with growth bound $\leq -\omega$,

(ii) $\mathcal{D}(\tilde{C}) \supset \mathcal{D}(A)$ and $\tilde{C}$ is A-bounded, $B \in \mathcal{L}(U, V)$,

(iii) there exists a linear operator $\Lambda_{\tilde\theta} : V \to L_1(\mathbb{R}^+, Y)$ such that the following two properties hold:

$$\sup_{\|x\|\le 1} \int_0^\infty e^{\mu t} \|(\Lambda_{\tilde\theta} x)(t)\|\, dt \;<\; \infty, \qquad \mu < \omega,$$

and $\Lambda_{\tilde\theta}$ maps $\mathcal{D}(A)$ into $D_1(\mathbb{R}^+, Y) := \{f \in L_1(\mathbb{R}^+, Y) \mid f' \in L_1(\mathbb{R}^+, Y) \text{ a.e.}\}$ and

$$\Lambda_{\tilde\theta} x = \tilde{C} S(\cdot; -iA)x, \qquad x \in \mathcal{D}(A).$$

In property (iii) the derivative is taken in the strong sense. Properties (i) and (ii) are immediate, it remains to prove (iii). Take $\Lambda_{\tilde\theta} = \Lambda_\theta$. Applying property (4′) and the identity $e^{\mu t} = e^{-(\nu-\mu)t} e^{\nu t}$ with $\mu < \nu < \omega$, we find using Cauchy-Schwarz's inequality

$$\|e^{\mu \cdot} \Lambda_\theta x\|_{L_1(\mathbb{R}^+, Y)} \le \text{const.}\ \|x\|_V, \qquad x \in V,$$

where $\mu < \omega$ is arbitrary. Therefore, the first property in (iii) holds. Furthermore, for $x \in \mathcal{D}(A)$, we have $S(\cdot; -iA)x \in \mathcal{D}(A)$, so $\tilde{C} S(\cdot; -iA)x$ is well-defined, and moreover, $S(\cdot; -iA)x$ is strongly differentiable with derivative $AS(\cdot; -iA)x$. This implies that $\tilde{C} S(\cdot; -iA)x \in D_1(\mathbb{R}^+, Y)$. Now for $x \in \mathcal{D}(A_W)$ we have, as $\tau\mathcal{D}(A_W) \subset \mathcal{D}(A) \cap \tau[W]$, using Proposition 1.4 and the definition of a PS-realization:

$$\Lambda_{\tilde\theta} \tau x = \Lambda_\theta \tau x = CS(\cdot; -iA_W)x = CS(\cdot; -iA_W)\tau^{-1}\tau x = \tilde{C} S(\cdot; -iA)\tau x.$$

As $\mathcal{D}(A_W)$ is dense in W and τ is continuous and injective with dense range, $\tau\mathcal{D}(A_W)$ is dense in V. Thus $\Lambda_{\tilde\theta}$ and $\tilde{C} S(\cdot; -iA)$ coincide, whenever they are both defined. So the second property also holds.

It follows that $\tilde\theta$ has the properties of a realization triple in the sense of [BGK 1], Section I.2 if U and Y are both finite dimensional.

In the other direction, let U and Y be finite dimensional, and let $\tilde\theta = (A, B, \tilde{C}; V, U, Y)$ be a realization triple in the sense of [BGK 1], Section I.2, i.e., assume properties (i), (ii) and (iii) above hold. Put $W = \mathcal{D}(A)$ endowed with the graph norm, and define $\tau : W \to V$ by $\tau x = x$. Also, define C by $C = \tilde{C}|_{\mathcal{D}(A)}$. Consider $\theta = (A, B, C; V, W, U, Y)$. Then for θ the first four properties of a PS-realization hold. On the other hand, from [BGK 1] we know that a matrix function $k(\cdot)$ is the weighting pattern of a realization triple in the sense of [BGK 1] if and only if there is a positive number μ such that $e^{\mu \cdot} k(\cdot) \in L_1(\mathbb{R}^+; \mathcal{L}(U, Y))$. It follows (use Theorem 1.1) that the class of weighting patterns that allow a PS-realization with finite dimensional input space U and finite dimensional output space Y is strictly smaller than the class of weighting patterns that allow a realization in the sense of [BGK 1, 2].

2. The transfer function and the input-output operator

Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization, and let k_θ be its weighting pattern. We define the *transfer function* of θ to be the operator function $W_\theta(\lambda)$ given by

$$W_\theta(\lambda)u = -i \int_0^\infty e^{i\lambda t} k_\theta(t) u \, dt, \qquad \operatorname{Im} \lambda > -\omega_\theta.$$

Here u is an arbitrary vector in U and $-\omega_\theta$ is the maximum of the growth bounds of the two semigroups associated with θ. We will show that this definition coincides with the one given in, e.g., [CZ], see also [CLTZ].

Observe that the integral above is well-defined by the part of Theorem 1.1 which was already proved in Section 1. Furthermore, if θ is a stable PS-realization, then the function $W_\theta(\cdot)u$ is analytic and uniformly bounded in the open right half plane. To see this, take $0 < \mu < \omega_\theta$ and notice that $k_\theta(\cdot)u \in e^{-\mu \cdot} L_2(\mathbb{R}^+, Y)$, by the remark made in the first paragraph after the proof of Lemma 1.2.

PROPOSITION 2.1 *Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization, and let $\tilde{C} : \mathcal{D}(A) \rightarrow Y$ be the extended output operator associated with θ. Then*

$$W_\theta(\lambda) = \tilde{C}(\lambda - A)^{-1} B, \qquad \operatorname{Im} \lambda > -\omega_\theta. \tag{2.1}$$

PROOF. Without loss of generality we may assume that θ is stable. Thus assume $-\omega_\theta < 0$, and fix $\operatorname{Im} \lambda > 0$. We claim that

$$-i \int_0^\infty e^{i\lambda t} (\Lambda_\theta x)(t) \, dt = \tilde{C}(\lambda - A)^{-1} x, \qquad x \in V. \tag{2.2}$$

To prove this, take $x = \tau y$ with $y \in W$. Then, by (4′),

$$\begin{aligned} -i \int_0^\infty e^{i\lambda t} (\Lambda_\theta x)(t) \, dt &= -i \int_0^\infty e^{i\lambda t} C S(t; -iA_W) y \, dt = \\ &= C(\lambda - A_W)^{-1} y = \tilde{C}(\lambda - A)^{-1} x, \end{aligned}$$

where we use the definition of $\tilde{C}$. Now use that the map $\Lambda_\theta : V \rightarrow L_2(\mathbb{R}^+, Y)$ is a bounded linear operator, and that the map $x \rightarrow \tilde{C}(\lambda - A)^{-1} x$ is a bounded linear operator from V into Y. Since (2.1) holds for each $x \in \tau[W]$ and $\tau[W]$ is dense in V, a continuity argument yields (2.2). From (2.2) and $k_\theta(t)u = (\Lambda_\theta B u)(t)$ it is clear that (2.1) holds. ■

PROPOSITION 2.2 *Let $\theta = (A, B, C; V, W, U, Y)$ be a stable PS-realization, and let $\tilde{C} : \mathcal{D}(A) \rightarrow Y$ be the extended output operator associated with θ. Define bounded linear operators*

$$\hat{C} = \tilde{C} A^{-1} : V \rightarrow Y, \quad \hat{B} = \tau^{-1} A^{-1} B : U \rightarrow W.$$

Then

$$W_\theta(\lambda) = \hat{C}A(\lambda - A)^{-1}B = CA_W(\lambda - A_W)^{-1}\hat{B}. \tag{2.3}$$

PROOF. The first equality in (2.3) follows immediately from (2.2). To prove the second equality, first note that $A(\lambda - A)^{-1}$ and $A_W(\lambda - A_W)^{-1}$ are well-defined bounded linear operators and (cf., [GGK], page 410) for $x \in V$ and $y \in W$ we have

$$\lim_{\lambda \in \mathbf{R},\, \lambda \to \infty} A(\lambda - A)^{-1}x = 0, \quad \lim_{\lambda \in \mathbf{R},\, \lambda \to \infty} A_W(\lambda - A_W)^{-1}y = 0.$$

It follows that

$$\lim_{\lambda \in \mathbf{R},\, \lambda \to \infty} \hat{C}A(\lambda - A)^{-1}Bu = \lim_{\lambda \in \mathbf{R},\, \lambda \to \infty} CA_W(\lambda - A_W)^{-1}\hat{B}u = 0. \tag{2.4}$$

Next, we compute

$$\begin{aligned}
&\frac{d}{d\lambda}\hat{C}A(\lambda - A)^{-1}Bu = -\hat{C}A(\lambda - A)^{-2}Bu = \\
&= -\hat{C}A(\lambda - A)^{-2}A\tau\hat{B}u = -\hat{C}A^2(\lambda - A)^{-2}\tau\hat{B}u = \\
&= -\hat{C}A^2\tau(\lambda - A_W)^{-2}\hat{B}u = -\hat{C}\tau A_W^2(\lambda - A_W)^{-2}\hat{B}u = \\
&= -CA_W^{-1}\tau^{-1}\tau A_W^2(\lambda - A_W)^{-2}\hat{B}u = -CA_W(\lambda - A_W)^{-2}\hat{B}u.
\end{aligned}$$

Here we used that on $\tau[W]$ the operator $\hat{C}$ coincides with $CA_W^{-1}\tau^{-1}$. From the above calculation we see that

$$\frac{d}{d\lambda}\hat{C}A(\lambda - A)^{-1}Bu = \frac{d}{d\lambda}CA_W(\lambda - A_W)^{-1}\hat{B}u.$$

This identity, together with (2.4), yields the second equality in (2.3). ∎

One easily sees that $\hat{\theta} = (A_W, \hat{B}, CA_W; W, U, Y)$ is a realization triple in the sense of [BGK1] in case $\theta = (A, B, C; V, W, U, Y)$ is a stable PS-realization and U and Y are finite dimensional.

COROLLARY 2.3 *Let $\theta = (A, B, C; V, W, U, Y)$ be a stable PS-realization. Then T_θ maps $L_2(\mathbb{R}^+, U)$ into $L_2(\mathbb{R}^+, Y)$, and*

$$\widehat{T_\theta\phi}(\lambda) = W_\theta(\lambda)\hat{\phi}(\lambda), \qquad \lambda \in \mathbb{R}, \tag{2.5}$$

where $\hat{\psi}$ denotes the Fourier transform of the L_2-function ψ.

PROOF. Put

$$(R\phi)(t) = \int_0^t S(t - s; -iA)B\phi(s)\,ds, \qquad \phi \in L_2(\mathbb{R}^+, U).$$

Since $S(\cdot;-iA)$ is exponentially decaying, we know that R is a bounded linear operator from $L_2(\mathbb{R}^+,U)$ into $L_2(\mathbb{R}^+,V)$. Notice that for $\operatorname{Im}\lambda>0$ we have

$$\begin{aligned}-i\int_0^\infty e^{i\lambda t}(R\phi)(t)\,dt &= (\lambda-A)^{-1}B\hat{\phi}(\lambda)=\\ &=(\lambda-A)^{-1}A\tau\hat{B}\hat{\phi}(\lambda)=A(\lambda-A)^{-1}\tau\hat{B}\hat{\phi}(\lambda)=\\ &=A\tau(\lambda-A_W)^{-1}\hat{B}\hat{\phi}(\lambda)=\tau A_W(\lambda-A_W)^{-1}\hat{B}\hat{\phi}(\lambda).\end{aligned}$$

Writing $(R\phi)(t)=\tau\Gamma_{\theta,t}\phi_t$, where ϕ_t is defined by $\phi_t(s)=\phi(t-s)$ for $0\le s\le t$, and $\phi_t(s)=0$ elsewhere on $\mathbb{R}^+$, we see that $\tau^{-1}(R\phi)\in L_\infty(\mathbb{R}^+,W)$. Thus for $\operatorname{Im}\lambda>0$ both sides of the equality

$$\tau^{-1}\int_0^\infty e^{i\lambda t}(R\phi)(t)\,dt=\int_0^\infty e^{i\lambda t}\tau^{-1}(R\phi)(t)\,dt$$

are well-defined. Moreover, because of the boundedness of τ, they coincide.

Next observe that $(T_\theta\phi)(t)=C\tau^{-1}(R\phi)(t)$. Now using the boundedness of C, we have for $\operatorname{Im}\lambda>0$

$$\begin{aligned}-i\int_0^\infty e^{i\lambda t}(T_\theta\phi)(t)\,dt &= -iC\tau^{-1}\int_0^\infty e^{i\lambda t}(R\phi)(t)\,dt=\\ &=C\tau^{-1}\tau A_W(\lambda-A_W)^{-1}\hat{B}\hat{\phi}(\lambda)=CA_W(\lambda-A_W)^{-1}\hat{B}\hat{\phi}(\lambda)=\\ &=W_\theta(\lambda)\hat{\phi}(\lambda).\end{aligned}$$

Since θ is stable, $W_\theta(\cdot)$ is analytic and bounded on $\operatorname{Im}\lambda>0$. Thus, by the Paley-Wiener theorem, $W_\theta(\cdot)\hat{\phi}(\cdot)$ is the Fourier transform of a function in $L_2(\mathbb{R}^+,Y)$. It follows that T_θ maps $L_2(\mathbb{R}^+,U)$ into $L_2(\mathbb{R}^+,Y)$, and moreover, (2.5) holds. ∎

The result of Corollary 2.3 is known, see [vK], Section 2.3.

3. Duality

Let $\theta=(A,B,C;V,W,U,Y)$ be a PS-realization. Since $\tau:W\to V$ is a continuous injection with dense range, also $\tau^*:V\to W$, defined by $\langle\tau^*v,\,w\rangle_W=\langle v,\,\tau w\rangle_V$, is a continuous injection with dense range. Also observe that $B^*:V\to U$ and $C^*:Y\to W$, defined by $\langle B^*v,\,y\rangle_U=\langle v,\,By\rangle_V$, and $\langle C^*y,\,w\rangle_W=\langle y,\,Cw\rangle_Y$ are well-defined bounded linear operators. We also denote the adjoints of operators acting in V and W by superscripted $*$.

PROPOSITION 3.1 *Let $\theta=(A,B,C;V,W,U,Y)$ be a PS-realization, and let A_W be the part of A in W. Then $\theta^*=(-A_W^*,C^*,-B^*;W,V,Y,U)$ is also a PS-realization. Moreover, the transfer function of θ^* is given by $W_{\theta^*}(-\bar{\lambda})=W_\theta(\lambda)^*$, and its weighting pattern is given by $k_{\theta^*}(t)=k_\theta(t)^*$.*

Observe the reversal of the roles of V and W, as well as the reversal of the roles of U and Y.

PROOF. First we show that $A^*(V \to V)$ is the part of $A_W^*(W \to W)$ in V (with respect to the injection τ^*). For the time being let us denote the part of $A_W^*(W \to W)$ in V by T. Let $v \in \mathcal{D}(A^*)$. We have to show that $\tau^* v \in \mathcal{D}(A_W^*)$ and $A_W^* \tau^* v \in \tau^*[V]$. For $x \in \mathcal{D}(A_W)$ we have:

$$\langle \tau^* v,\, A_W x\rangle_W = \langle v,\, \tau A_W x\rangle_V = \langle v,\, A\tau x\rangle_V = \langle A^* v,\, \tau x\rangle_V = \langle \tau^* A^* v,\, x\rangle_W.$$

So by the definition of A_W^* we have indeed that $\tau^* v \in \mathcal{D}(A_W^*)$ and $A_W^* \tau^* v = \tau^* A^* v \in \tau^*[V]$. In particular,

$$\mathcal{D}(A^*) \subset \{v \in V \mid \tau^* v \in \mathcal{D}(A_W^*) \text{ and } A_W^* \tau^* v \in \tau^*[V]\}.$$

This proves that $A^* \subset T$. To prove the converse, let again $-\omega_\theta$ be the maximum of the exponential growth bounds of the semigroups $S(\cdot; -iA)$ and $S(\cdot; -iA_W)$. We claim that for $\operatorname{Im}\lambda < \omega_\theta$, the operator $\lambda - iT$ is invertible. Indeed, for all $x \in V$ we have $(\lambda - iA^*)^{-1}x \in \mathcal{D}(A^*) \subset \mathcal{D}(T)$, and hence

$$(\lambda - iT)(\lambda - iA^*)^{-1}x = (\lambda - iA^*)(\lambda - iA^*)^{-1}x = x. \tag{3.1}$$

Thus, $\lambda - iT$ is onto. Now suppose $x \in \mathcal{D}(T)$ satisfies $(\lambda - iT)x = 0$. Then $\tau^* x \in \mathcal{D}(A_W^*)$ and $A_W^* \tau^* x = \tau^* T x$. Thus $(\lambda - iA_W^*)\tau^* x = \tau^*(\lambda - iT)x = 0$. As τ^* is injective, and also $\lambda - iA_W^*$ is injective it follows that $x = 0$. So, $\lambda - iT$ is invertible, and moreover, $(\lambda - iT)^{-1} = (\lambda - iA^*)^{-1}$, by (3.1). But then A^* and T must coincide. Thus A^* is the part of A_W^* in V.

Using [P], Corollary I.10.6, one sees that iA_W^* and $iA^* (= (iA_W^*|_V))$ are generators of C_0-semigroups, and that

$$S(\cdot; iA_W^*) = S(\cdot; -iA_W)^*, \qquad S(\cdot; iA^*) = S(\cdot; -iA)^*.$$

Hence

$$S(t; iA_W^*)\tau^* = (\tau S(t; -iA_W))^* = (S(t; -iA)\tau)^* = \tau^* S(t; iA^*).$$

In order to show that θ^* is a PS-realization it remains to define bounded linear operators $\Gamma_{\theta^*,t} : L_2([0,t], Y) \to V$ and $\Lambda_{\theta^*,t} : W \to L_2([0,t], U)$ satisfying

$$\tau^* \Gamma_{\theta^*,t}\phi = \int_0^t S(s; iA_W^*) C^* \phi(s)\, ds\ , \qquad \phi \in L_2([0,t], Y),$$

and $\Lambda_{\theta^*,t}\tau^* x = -B^* S(\cdot; iA^*)x$, for $x \in V$. Taking $\Gamma_{\theta^*,t} = \Lambda_{\theta,t}^*$ and $\Lambda_{\theta^*,t} = -\Gamma_{\theta,t}^*$, one easily verifies that the conditions just mentioned hold.

To show the statement concerning the transfer functions, we use Proposition 2.2. For simplicity we assume that θ is stable, which may be done because of Lemma 1.3. Also, put $\alpha = -A_W^*$, $\beta = C^*$, $\gamma = -B^*$, then $\alpha_V = -A^*$, and $\theta^* = (\alpha, \beta, \gamma; V, W, Y, U)$. Recall from Proposition 2.2 the definitions of $\hat{C}$ and $\hat{B}$. Define $\hat{\gamma}$ and $\hat{\beta}$ in a similar way for θ^*. By formula (2.3) we have

$$W_\theta(\lambda) = \hat{C}A(\lambda - A)^{-1}B, \qquad W_{\theta^*}(\lambda) = \gamma\alpha_V(\lambda - \alpha_V)^{-1}\hat{\beta}.$$

We first show that

$$(\hat{C})^* = -\hat{\beta}, \qquad (\hat{B})^* = -\hat{\gamma}. \tag{3.2}$$

To prove the first equality, recall that $\hat{C}|_{\tau[W]} = CA_W^{-1}\tau^{-1}|_{\tau[W]}$. Fix $w \in W$ and $y \in Y$. We know that $\alpha^{-1}\beta y \in \tau^*[V]$, and thus $\alpha^{-1}\beta y = \tau^*\tau^{*-1}\alpha^{-1}\beta y$. Hence

$$\begin{aligned}&\langle \hat{C}\tau w,\ y\rangle = \langle CA_W^{-1}w,\ y\rangle = \langle w,\ A_W^{*-1}C^*y\rangle = \\ &=\langle w,\ -\alpha^{-1}\beta y\rangle = \langle w,\ -\tau^*\tau^{*-1}\alpha^{-1}\beta y\rangle = \langle \tau w,\ -\tau^{*-1}\alpha^{-1}\beta y\rangle.\end{aligned}$$

Now use that $\tau[W]$ is dense in V. We see that $(\hat{C})^* = -\tau^{*-1}\alpha^{-1}\beta = -\hat{\beta}$.

To prove the second equality in (3.2), recall that $\tau\hat{B} = A^{-1}B$. Thus

$$(\hat{B})^*\tau^* = B^*A^{*-1} = B^*A^{*-1}\tau^{*-1}\tau^* = \gamma\alpha_V^{-1}\tau^{*-1}\tau^*.$$

Therefore, $(\hat{B})^*|_{\tau^*[V]} = \hat{\gamma}|_{\tau^*[V]}$. Both $(\hat{B})^*$ and $\hat{\gamma}$ are bounded, and $\tau^*[V]$ is dense in W, so $(\hat{B})^* = -\hat{\gamma}$.

Now using (3.2) we compute

$$W_\theta(\lambda)^* = \{\hat{C}A(\lambda - A)^{-1}B\}^* = \gamma\alpha_V(-\bar{\lambda} - \alpha_V)^{-1}\hat{\beta} = W_{\theta^*}(-\bar{\lambda}).$$

From this the statement concerning the weighting operator functions is obtained by taking inverse Fourier transforms. ∎

Note that if $\mathcal{D}(A) \subset \tau[W]$, then $\mathcal{D}(A_W^*) \subset \tau^*[V]$. To see this assume that θ is a stable PS-realization. Then A^{-1} is bounded, and the inclusion $\mathcal{D}(A) \subset \tau[W]$ means that $\tau^{-1}A^{-1} : V \to W$ is well-defined. As it is clearly closed, it is bounded. Also, from $\tau A_W x = A\tau x$ for $x \in \mathcal{D}(A_W)$, we see that $A_W^{-1}\tau^{-1} \subset \tau^{-1}A^{-1}$. Hence $(\tau^{-1}A^{-1})^* \subset (A_W^{-1}\tau^{-1})^*$. As A_W^{-1} is bounded (for the same reason as A^{-1} is bounded), we see from [R], Theorem 13.2: $(\tau^{-1}A^{-1})^* \subset (A_W^{-1}\tau^{-1})^* = \tau^{*-1}A_W^{*-1}$. Thus, $(\tau^{-1}A^{-1})^* \subset \tau^{*-1}A_W^{*-1}$. But $(\tau^{-1}A^{-1})^*$ is bounded, as it is the adjoint of a bounded operator. Since $\tau^{*-1}A_W^{*-1}$ is closed, it must also be bounded, which means $\mathcal{D}(A_W^*) \subset \tau^*[V]$. See [vK], Theorem 2.17 (iii), for an analogous result.

4. Proof of Theorem 1.1 and Realization Theorem

In this section we prove Theorem 1.1. Using the results of the previous sections the proof of the necessity part is now easy. Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization, and let $k_\theta(t)u = -i(\Lambda_\theta Bu)(t)$ be its weighting pattern. As before, let $-\omega_\theta$ be the maximum of the exponential growth bounds of $S(\cdot; -iA)$ and $S(\cdot; -iA_W)$. We already know (see the first paragraph after the proof of Lemma 1.2) that for each $\mu < \omega_\theta$ the function $e^{\mu\cdot} k_\theta(\cdot)u \in L_2(\mathbb{R}^+, Y)$ for all $u \in U$. Since $k_\theta(\cdot)^* = k_{\theta^*}(\cdot)$, by Proposition 3.1, we also know that $e^{\mu\cdot} k_\theta(\cdot)^* y \in L_2(\mathbb{R}^+, U)$ for all $y \in Y$ and every $\mu < \omega_\theta = \omega_{\theta^*}$. Thus (1.5) holds for some $\mu \in \mathbb{R}$.

The reverse implication will first be proved for the stable case. To do this, let $k(\cdot) : \mathbb{R}^+ \to \mathcal{L}(U, Y)$ and $\mu > 0$ be such that $e^{\mu\cdot} k(\cdot)u \in L_2(\mathbb{R}^+, Y)$ for all $u \in U$ and $e^{\mu\cdot} k(\cdot)^* y \in L_2(\mathbb{R}^+, U)$ for all $y \in Y$. We shall produce a stable PS-realization θ such that $k = k_\theta$. Observe that by taking μ a little smaller if necessary, we may assume from the start that in addition $e^{\mu\cdot} k(\cdot)u \in L_1(\mathbb{R}^+, Y)$ for all $u \in U$ and $e^{\mu\cdot} k(\cdot)^* y \in L_1(\mathbb{R}^+, U)$ for all $y \in Y$.

Before we state and prove the realization result in detail we need some preparations. For every $\mu > 0$ let $L_2^{\mp\mu}(\mathbb{R}^+; Y)$ be the complex Hilbert space of all strongly measurable functions $f : \mathbb{R}^+ \to Y$ which are bounded with respect to the norm $\|f\| = \left[\int_0^\infty e^{\mp 2\mu t} \|f(t)\|^2 \, dt\right]^{1/2}$. We have in the sense of continuous and dense imbeddings

$$L_2^{-\mu}(\mathbb{R}^+; Y) \subset L_2(\mathbb{R}^+; Y) \subset L_2^{\mu}(\mathbb{R}^+; Y),$$

and similarly if Y is replaced by U.

Proposition 4.1 *Let U and Y be complex Hilbert spaces, and let $k(\cdot) : \mathbb{R}^+ \to \mathcal{L}(U, Y)$ be such that for some $\mu > 0$ we have $e^{\mu\cdot} k(\cdot)u \in L_1(\mathbb{R}^+, Y) \cap L_2(\mathbb{R}^+, Y)$ for all $u \in U$, and $e^{\mu\cdot} k(\cdot)^* y \in L_1(\mathbb{R}^+, U) \cap L_2(\mathbb{R}^+, U)$ for all $y \in Y$. Then the operator H defined by*

$$(Hf)(t) = (P) \int_0^\infty k(t + \alpha) f(\alpha) \, d\alpha \qquad t > 0 \tag{4.1a}$$

is bounded from $L_2^{\mu}(\mathbb{R}^+; U)$ into $L_2^{-\mu}(\mathbb{R}^+; Y)$, and is also bounded from $L_2(\mathbb{R}^+; U)$ into $L_2(\mathbb{R}^+; Y)$, where the integral is to be interpreted as a Pettis integral in the following sense

$$\langle (Hf)(t), y \rangle = \int_0^\infty \langle k(t + s) f(s), y \rangle_Y \, ds, \qquad t > 0. \tag{4.1b}$$

In the proof we shall show that the formula defining $(Hf)(t)$ may be interpreted as a Bochner integral in case f is a measurable step function of compact support.

Proof. First we prove some auxiliary statements. The map $J : U \to L_1(\mathbb{R}^+; Y)$ defined by $Ju = k(\cdot)u$ is easily seen to be a closed operator. As it is everywhere defined it is a bounded operator, and we have $\|k(\cdot)u\|_{L_1(\mathbf{R}^+;Y)} \leq \gamma\|u\|_U$, for all $u \in U$, and for some positive γ. The same argument shows that also $\|k(\cdot)^*y\|_{L_1(\mathbf{R}^+;U)} \leq \gamma\|y\|_Y$ for $y \in Y$.

We first show that the integral defining $(Hf)(t)$ is well-defined. Let $y \in Y$, and $f \in L_2(\mathbb{R}^+, U)$. Then

$$\langle k(t+\cdot)f(\cdot), y\rangle_Y = \langle f(\cdot), k(t+\cdot)^*y\rangle_Y$$

is a function in $L_1(\mathbb{R}^+)$. Thus $\int_0^\infty \langle k(t+\alpha)f(\alpha), y\rangle_Y \, d\alpha$ is well-defined. Moreover, by the result of the previous paragraph, this expression is continuous in y. Hence, by the Riesz representation theorem, there is a unique vector $(Hf)(t)$ such that (4.1b) holds. Thus (4.1a) is well-defined as a Pettis integral. Moreover, it is clearly linear in f.

We define $\hat{k}(\lambda)$ by

$$\hat{k}(\lambda)u = \widehat{k(\cdot)u}(\lambda), \qquad u \in U,$$

where $\widehat{k(\cdot)u}$ is the Fourier transform of $k(\cdot)u$. For each $u \in U$ we have

$$\sup_{\lambda\in\mathbf{R}} \|\hat{k}(\lambda)u\| = \sup_{\lambda\in\mathbf{R}} \|\int_0^\infty e^{i\lambda t}k(t)u\,dt\| \leq \int_0^\infty \|k(t)u\|\,dt = \|k(\cdot)u\|_{L_1(\mathbf{R}^+;Y)} \leq \gamma\|u\|_U.$$

Hence $\sup_{\lambda\in\mathbf{R}} \|\hat{k}(\lambda)\| = \sup_{\lambda\in\mathbf{R},\|u\|\leq 1} \|\hat{k}(\lambda)u\| \leq \gamma$ is finite.

Now let $f \in L_2(\mathbb{R}^+; U)$ be a step function, i.e.,

$$f(t) = \begin{cases} f_j, & t \in E_j, j = 1, \cdots, r \\ 0, & \text{otherwise,} \end{cases}$$

where $E_1, \cdots, E_r$ are mutually disjoint subsets of $\mathbb{R}^+$ of finite Lebesgue measure and $f_1, \cdots, f_r$ are vectors in U. Observe that by our assumption on $k(\cdot)$ we have

$$(Hf)(t) = \sum_{j=1}^r \int_{E_j} k(t+\alpha)f_j\,d\alpha \in L_2(\mathbb{R}^+, Y).$$

Then for $t > 0$

$$(Hf)(t) = \int_{-\infty}^0 k(t-s)f(-s)\,ds = \sum_{j=1}^r \int_{-E_j} k(t-s)f_j\,ds.$$

Taking Fourier transforms it follows that $(\widehat{-Hf})(\lambda) = \hat{k}(\lambda)\hat{f}(-\lambda)$ for $\lambda \in \mathbb{R}$. So

$$\|\widehat{Hf}\|_{L_2(\mathbf{R};Y)} \leq \left(\sup_{\lambda\in\mathbf{R}} \|\hat{k}(\lambda)\|\right) \cdot \|\hat{f}\|_{L_2(\mathbf{R};U)},$$

whence also

$$\|Hf\|_{L_2(\mathbf{R}^+;Y)} \leq \left(\sup_{\lambda\in\mathbf{R}} \|\hat{k}(\lambda)\|\right) \cdot \|f\|_{L_2(\mathbf{R}^+;U)} \leq \gamma\|f\|_{L_2(\mathbf{R}^+;U)}.$$

Now let f be an arbitrary element of $L_2(\mathbb{R}^+;U)$, and take a sequence of step functions f_n converging to f in $L_2(\mathbb{R}^+;U)$, again using [DU], Section II.2 to see that such a sequence exists. The estimate in the previous paragraph shows that Hf_n is a Cauchy sequence in $L_2(\mathbb{R}^+;Y)$. Thus $Hf_n \to g$ for some $g \in L_2(\mathbb{R}^+;Y)$. Now fix $t > 0$ and compute for $y \in Y$

$$\begin{aligned}\langle (Hf_n)(t), y\rangle &= \int_0^\infty \langle k(t+\alpha)f_n(\alpha), y\rangle_Y \, d\alpha = \\ &= \int_0^\infty \langle f_n(\alpha), k(t+\alpha)^* y\rangle_Y \, d\alpha = \\ &= \langle f_n(\cdot), k(t+\cdot)^* y\rangle_{L_2(\mathbb{R}^+;U)} \to \langle f(\cdot), k(t+\cdot)^* y\rangle_{L_2(\mathbb{R}^+;U)} = \\ &= \langle (Hf)(t), y\rangle.\end{aligned}$$

Thus Hf_n converges pointwise to Hf, and converges in $L_2(\mathbb{R}^+;Y)$ to g. But then $Hf = g$. This shows that $Hf \in L_2(\mathbb{R}^+;Y)$ for all $f \in L_2(\mathbb{R}^+;U)$.

Now we show that H is a bounded linear map from $L_2(\mathbb{R}^+;U)$ to $L_2(\mathbb{R}^+;Y)$. Again, let f_n be a sequence of step functions converging to f in $L_2(\mathbb{R}^+;U)$. From the previous paragraph we now that Hf_n converges to Hf in $L_2(\mathbb{R}^+;Y)$. We also have that $\|Hf_n\| \leq \gamma \cdot \|f_n\|$. Take $\epsilon > 0$, and let n be such that $\|Hf - Hf_n\| \leq \epsilon$ and $\|f - f_n\| \leq \epsilon$. Then we have

$$\begin{aligned}\|Hf\| &\leq \|Hf - Hf_n\| + \|Hf_n\| \leq \epsilon + \gamma\|f_n\| \\ &\leq \epsilon + \gamma(\|f\| + \|f - f_n\|) \leq (1+\gamma)\epsilon + \gamma\|f\|.\end{aligned}$$

Letting $\epsilon \to 0$, we obtain that $\|Hf\| \leq \gamma\|f\|$.

Replacing $k(\cdot)$ by $e^{\mu \cdot}k(\cdot)$ it is easily seen that the same arguments show that H is bounded as a linear map from $L_2^{\mu}(\mathbb{R}^+;U)$ into $L_2^{-\mu}(\mathbb{R}^+;Y)$. ∎

For $t > 0$ we define

$$[S(t)f](\alpha) = f(t+\alpha), \qquad t, \alpha > 0,$$

$$[S^{\#}(t)f](\alpha) = \begin{cases} f(\alpha - t), & \alpha > t > 0 \\ 0, & \text{otherwise.}\end{cases}$$

Let us note that $S(\cdot)$ induces strongly continuous semigroups on $L_2^{-\mu}(\mathbb{R}^+;Y)$ and $L_2^{\mu}(\mathbb{R}^+;Y)$. We denote these semigroups by $S_-(\cdot)$ and $S_+(\cdot)$, respectively. Similarly, $S_-^{\#}(\cdot)$ and $S_+^{\#}(\cdot)$ are the C_0-semigroups induced by $S^{\#}(\cdot)$ on $L_2^{-\mu}(\mathbb{R}^+;U)$ and $L_2^{\mu}(\mathbb{R}^+;U)$, respectively. The semigroups $S_-(\cdot)$ and $S_+^{\#}(\cdot)$ are both exponentially decaying. In fact, $S_-(t)$ and $S_+^{\#}(t)$ both have norm $e^{-\mu t}$. In the sequel, we shall use that

$$S_-(t)^* = e^{-2\mu t}S_-^{\#}(t), \; S_+(t)^* = e^{2\mu t}S_+^{\#}(t).$$

Now, let H be as in Proposition 4.1. Then

$$S_-(t)H = HS_+^{\#}(t), \qquad t \geq 0,$$

and

$$S_+(t)H^* = H^* S_-^{\#}(t), \qquad t \geq 0,$$

Thus $\operatorname{Im} H$ is invariant under $S_-(t)$. We define V to be the closure of $\operatorname{Im} H$ in $L_2^{-\mu}(\mathbb{R}^+;Y)$. Then $S_-(t)|_V$ is an exponentially decaying C_0-semigroup on V; its generator will be denoted by $-iA$. Thus

$$S(t;-iA) = S_-(t)|_V : V \to V, \qquad t \geq 0.$$

Next, let Q be the orthogonal projection of $L_2^{\mu}(\mathbb{R}^+;U)$ along $\operatorname{Ker} H$. Since $\operatorname{Ker} H$ is invariant under $S_+^{\#}(t)$, we have $QS_+^{\#}(t) = QS_+^{\#}(t)Q$ for each $t \in \mathbb{R}^+$. If $Hg = f$ for some $g \in L_2^{\mu}(\mathbb{R}^+;U)$, then the vector Qg is uniquely determined by f. We define W to be the complex Hilbert space which one obtains if $\operatorname{Im} H \subset L_2^{-\mu}(\mathbb{R}^+;Y)$ is endowed with the norm

$$\|f\|_W = \left[\|f\|^2_{L_2^{-\mu}(\mathbb{R}^+;Y)} + \|Qg\|^2_{L_2^{\mu}(\mathbb{R}^+;U)}\right]^{1/2},$$

where g is some vector such that $Hg = f$. For the case when $\operatorname{Ker} H = \{0\}$, the space W is just the space $\operatorname{Im} H$ endowed with the graph norm corresponding to H^{-1}. If $Hg = f$, then $S_-(t)f = HS_+^{\#}(t)g$. Since $QS_+^{\#}(t)g = QS_+^{\#}(t)Qg$, we see that

$$\begin{aligned}\|S_-(t)f\|_W^2 &= \|S_-(t)f\|^2_{L_2^{-\mu}(\mathbb{R}^+;Y)} + \|QS_+^{\#}(t)Qg\|^2_{L_2^{\mu}(\mathbb{R}^+;U)} \\ &\leq e^{-\mu t}\|f\|^2_{L_2^{-\mu}(\mathbb{R}^+;Y)} + e^{-\mu t}\|Qg\|^2_{L_2^{\mu}(\mathbb{R}^+;U)} = e^{-\mu t}\|f\|_W^2.\end{aligned}$$

Thus $S_-(t)$ induces an exponentially decaying C_0-semigroup in W. Let $\tau : W \to V$ be the canonical embedding of W into V. Then $\tau[W]$ is dense in V. As before, we write A_W for the part of A in W. Then

$$S(t;-iA_W) = S_-(t)|_W : W \to W.$$

THEOREM 4.2 *Let U and Y be complex Hilbert spaces, and let $k(\cdot)$ be such that for some $\mu > 0$ we have that $e^{\mu\cdot}k(\cdot)u$ belongs to $L_1(\mathbb{R}^+,Y) \cap L_2(\mathbb{R}^+,Y)$ for all $u \in U$ and $e^{\mu\cdot}k(\cdot)^*y$ belongs to $L_1(\mathbb{R}^+,U) \cap L_2(\mathbb{R}^+,U)$ for all $y \in Y$. Suppose V, W, A and A_W are as above. Define*

$$(Bu)(t) = k(t)u, \qquad t \in \mathbb{R}^+,\ u \in U,$$

$$Cf = (P)\int_0^\infty k(t)(Qg)(t)\,dt, \qquad f = Hg,$$

where Q is defined as above and the integral is to be interpreted as a Pettis integral. Then $B \in \mathcal{L}(U,V)$, $C \in \mathcal{L}(W,Y)$ and $\theta = (A,B,C;V,W,U,Y)$ is a stable PS-realization, whose weighting pattern is precisely $k(\cdot)$.

PROOF. Let us define

$$(B_n u)(t) = n \int_0^{1/n} k(t+\alpha)u\, d\alpha, \qquad t > 0.$$

Then $B_n u \in \operatorname{Im} H$,

$$\|B_n u - Bu\|^2_{L_2^{-\mu}(\mathbb{R}^+;Y)} = \int_0^\infty e^{2\mu t} \left\| n \int_0^{1/n} \{k(t+\alpha) - k(t)\}u\, d\alpha \right\|^2 dt.$$

Now

$$\left\| n \int_0^{1/n} \{k(t+\alpha) - k(t)\}u\, d\alpha \right\|^2 \leq \left\{ n \int_0^{1/n} \|\{k(t+\alpha) - k(t)\}u\|\, d\alpha \right\}^2 \leq$$

$$\leq n^2 \left(\int_0^{1/n} \|\{k(t+\alpha) - k(t)\}u\|^2\, d\alpha\right)\left(\int_0^{1/n} 1\, d\alpha\right) = n \int_0^{1/n} \|\{k(t+\alpha) - k(t)\}u\|^2\, d\alpha.$$

Thus we see that

$$\int_0^\infty e^{2\mu t} \left\| n \int_0^{1/n} \{k(t+\alpha) - k(t)\}u\, d\alpha \right\|^2 dt \leq$$

$$\leq \int_0^\infty \{n \int_0^{1/n} e^{2\mu t} \|\{k(t+\alpha) - k(t)\}u\|^2\, d\alpha\}\, dt =$$

$$= n \int_0^{1/n} \left(\int_0^\infty e^{2\mu t} \|\{k(t+\alpha) - k(t)\}u\|^2\, dt\right) d\alpha.$$

As $\int_0^\infty e^{2\mu t} \|\{k(t+\alpha) - k(t)\}u\|^2\, dt$ is continuous in α we have that $B_n u$ tends to Bu in the norm of $L_2^{-\mu}(\mathbb{R}^+;Y)$. Thus B is a bounded linear operator from U to V, i.e., $B \in \mathcal{L}(U,V)$.

Take $\phi \in L_2(\mathbb{R}^+;U)$. As $S(\cdot;-iA)$ is an exponentially decaying semigroup it follows that $S(t;-iA)B\phi(t) \in L_1(\mathbb{R}^+;V)$. Moreover, $B\phi(t) \in L_2^{-\mu}(\mathbb{R}^+;Y)$. So, $S(t;-iA)B\phi(t) = S_-(t)k(\cdot)\phi(t)$. This gives

$$\int_0^\infty S(t;-iA)B\phi(t)\, dt = \int_0^\infty S_-(t)k(\cdot)\phi(t)\, dt = (P)\int_0^\infty k(\cdot+t)\phi(t)\, dt = H\phi \in W.$$

Therefore, for $\phi \in L_2(\mathbb{R}^+;U)$ we have

$$\|\int_0^\infty S(t;-iA)B\phi(t)\, dt\|^2_W = \|H\phi\|^2_W = \|H\phi\|^2_{L_2^{-\mu}(\mathbb{R}^+;Y)} + \|Q\phi\|^2_{L_2^{\mu}(\mathbb{R}^+;U)} \leq$$

$$\leq (\gamma^2+1)\|\phi\|^2_{L_2^{\mu}(\mathbb{R}^+;U)} \leq \tilde{\gamma}\|\phi\|^2_{L_2(\mathbb{R}^+;U)},$$

for some constants γ and $\tilde{\gamma}$. To get the one but last inequality we used that H is bounded (by Proposition 4.1) and that Q is also bounded.

On the other hand, if $f = Hg$ for some $g \in L_2^\mu(\mathbb{R}^+;U)$, we have

$$\begin{aligned}\|Cf\|_Y &= \sup_{\|y\|=1, y\in Y} |< Cf, y >| \le \sup_{\|y\|=1,y\in Y} \int_0^\infty |< (Qg)(t), k(t)^* y >|\, dt \\ &\le \sup_{\|y\|=1,y\in Y} \left[\int_0^\infty e^{2\mu t}\|k(t)^* y\|^2\, dt\right]^{1/2} \cdot \left[\int_0^\infty e^{-2\mu t}\|(Qg)(t)\|^2\, dt\right]^{1/2} \\ &\le \text{const.}\, \|Qg\|_{L_2^\mu(\mathbb{R}^+;U)} \le \text{const.}\, \|f\|_W, \qquad f \in \operatorname{Im} H,\end{aligned}$$

so that $C \in \mathcal{L}(W,Y)$.

Moreover, $CHg = (P)\int_0^\infty k(s)g(s)\, ds$. Indeed, for $y \in Y$ we have

$$\langle (Hg)(t), y\rangle = \int_0^\infty \langle k(t+\alpha)g(\alpha), y\rangle\, d\alpha =$$
$$\int_0^\infty \langle g(\alpha), k(t+\alpha)^* y\rangle\, d\alpha = \int_0^\infty \langle g(\alpha), (S_-(t)k(\cdot)^* y)(\alpha)\rangle\, d\alpha = \langle g, S_-(t)k(\cdot)^* y\rangle,$$

which, as $t \downarrow 0$, tends to $\langle g, k(\cdot)^* y\rangle = \langle (P)\int_0^\infty k(\alpha)g(\alpha)\, d\alpha, y\rangle$. Thus

$$\langle CHg, y\rangle = \langle (P)\int_0^\infty k(\alpha)(Qg)(\alpha)\, d\alpha, y\rangle = \lim_{t\downarrow 0}\langle (HQg)(t), y\rangle = \lim_{t\downarrow 0}\langle (Hg)(t), y\rangle,$$

because $H(I-Q) = 0$, and so $\langle CHg, y\rangle = \langle (P)\int_0^\infty k(\alpha)g(\alpha)\, d\alpha, y\rangle$. Hence, $CHg = (P)\int_0^\infty k(s)g(s)\, ds$.

Using this we have for $f \in W$

$$CS(t; -iA_W)f = CS_-(t)Hg = CHS^\#(t)g = (P)\int_0^\infty k(\alpha)(S^\#(t)g)(\alpha)\, d\alpha =$$
$$= (P)\int_t^\infty k(\alpha)g(\alpha - t)\, d\alpha = (Hg)(t) = f(t),$$

whence $CS(\cdot, -iA_W)f \in L_2(\mathbb{R}^+;Y)$ and

$$\|CS(\cdot, -iA_W)f\|_{L_2(\mathbb{R}^+;Y)} = \|f\|_{L_2(\mathbb{R}^+;Y)} \le \|f\|_{L_2^{-\mu}(\mathbb{R}^+;Y)}, \qquad f \in L_2^{-\mu}(\mathbb{R}^+;Y).$$

Thus θ is a PS-realization.

Finally, since the weighting pattern $k_\theta(\cdot)$ of θ is given by $k_\theta(\cdot)u = \Lambda_\theta Bu$, it is straightforward to see that $k_\theta(\cdot) = k(\cdot)$ as desired. ∎

Observe that the realization above is just the standard shift realization on weighted L_2 spaces.

PROOF OF THEOREM 1.1. The proof of Theorem 1.1 for the general case is reduced to the stable case by using Lemma 1.3. Let $k(\cdot) : \mathbb{R}^+ \to \mathcal{L}(U,Y)$ and $\mu \in \mathbb{R}$ be such that

$e^{\mu \cdot}k(\cdot)u \in L_2(\mathbb{R}^+, Y)$ for all $u \in U$ and $e^{\mu \cdot}k(\cdot)^*y \in L_2(\mathbb{R}^+, U)$ for all $y \in Y$. Taking μ a little smaller if necessary we may assume without loss of generality that $e^{\mu \cdot}k(\cdot)u \in L_2(\mathbb{R}^+, Y) \cap L_1(\mathbb{R}^+, Y)$ for all $u \in U$ and $e^{\mu \cdot}k(\cdot)^*y \in L_2(\mathbb{R}^+, U) \cap L_1(\mathbb{R}^+, U)$ for all $y \in Y$. We may also assume that $\mu \leq 0$, as the case $\mu > 0$ has already been dealt with. Take $\rho > 0$ fixed, and put $\tilde{k}(t) = e^{(\mu-\rho)t}k(t)$. For $\tilde{k}(\cdot)$ we have $e^{\rho \cdot}\tilde{k}(\cdot)u \in L_2(\mathbb{R}^+, Y) \cap L_1(\mathbb{R}^+, Y)$ for all $u \in U$ and $e^{\rho \cdot}\tilde{k}(\cdot)^*y \in L_2(\mathbb{R}^+, U) \cap L_1(\mathbb{R}^+, U)$ for all $y \in Y$. Thus, by Theorem 4.2, there is a stable PS-realization $\tilde{\theta} = (\tilde{A}, B, C; V, W, U, Y)$ such that $\tilde{k} = k_{\tilde{\theta}}$. Define A by $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$ and $-iA = -i\tilde{A} - (\mu - \rho)I_V$. Then $\theta = (A, B, C; V, W, U, Y)$ is a PS-realization and $k_\theta(t)u = e^{-(\mu-\rho)t}k_{\tilde{\theta}}(t)u = e^{-(\mu-\rho)t}\tilde{k}(t)u = k(t)u$ for all $u \in U$ by Lemma 1.3. ∎

COROLLARY 4.3 *Let $\theta = (A, B, C; V, W, U, Y)$ be a PS-realization. Then*

$$(T_\theta \phi)(t) = (P)\int_0^t k_\theta(t-s)\phi(s)\,ds, \quad t \in \mathbb{R}, \text{ a.e.}, \quad \phi \in L_{2,\text{loc}}(\mathbb{R}^+, U). \tag{4.2}$$

Here the integral on the right hand side of (4.2) is to be understood as a Pettis integral.

The right hand side of (4.2) is well-defined as a Bochner integral in case ϕ is a measurable step function of compact support.

PROOF. Without loss of generality we may assume that $\theta = (A, B, C; V, W, U, Y)$ is stable.

First we show that the right hand side of (4.2) is well-defined as a Pettis integral. Take $\phi \in L_2(\mathbb{R}^+, U)$ and put $\psi(t) = (P)\int_0^t k_\theta(t-s)\phi(s)\,ds$. Let $y \in Y$ then

$$\langle k(t-\cdot)\phi(\cdot), y\rangle_Y = \langle \phi(\cdot), k(t-\cdot)^*y\rangle_Y$$

is a function in $L_1([0,\ t])$. Thus $\int_0^t \langle k(t-\alpha)\phi(\alpha), y\rangle_Y\,d\alpha$ is well-defined. Moreover, by the result of the previous paragraph, this expression is continuous in y. Hence, by the Riesz representation theorem, there is a unique vector $g(t)$ such that

$$\langle g(t), y\rangle = \int_0^t \langle k(t-s)\phi(s), y\rangle_Y\,ds, \qquad t > 0.$$

Thus the right hand side of (4.2) is well-defined as a Pettis integral. Moreover, it is clearly linear in ϕ.

For each $y \in Y$ we have $\langle \psi(t), y\rangle$ in $L_2(\mathbb{R}^+)$. Moreover,

$$\begin{aligned}\langle \widehat{\psi(\cdot)}, y\rangle &= \int_0^\infty e^{i\lambda t}\langle \psi(t), y\rangle\,dt = \int_0^\infty e^{i\lambda t}\Big(\int_0^t \langle k_\theta(t-s)\phi(s), y\rangle\,ds\Big)\,dt = \\ &= \int_0^\infty \int_s^\infty e^{i\lambda t}(\langle k_\theta(t-s)\phi(s), y\rangle\,dt)\,ds = \int_0^\infty \Big(\int_0^\infty e^{i\lambda(\alpha+s)}\langle k_\theta(\alpha)\phi(s), y\rangle\,d\alpha\Big)\,ds \\ &= \int_0^\infty e^{i\lambda s}\langle W_\theta(\lambda)\phi(s), y\rangle\,ds = \langle W_\theta(\lambda)\hat{\phi}(\lambda), y\rangle\end{aligned}$$

Now using Corollary 2.3 we have $\langle W_\theta(\lambda)\hat{\phi}(\lambda), y\rangle = \langle \widehat{T_\theta\phi}(\lambda), y\rangle$. This proves the corollary. ■

The Pritchard-Salamon realizations studied in this paper are different from the systems considered by Salamon in [S2]. For instance, the systems in [S2] have three state spaces W, H and V such that $W \subset H \subset V$, and for $\phi \in L_2([0,t];U)$ the vector $\int_0^t S(s;-iA)B\phi(s)\,ds$ belongs to H rather than to W as is required for PS-realizations. On the other hand, the main result of the present paper can be used to rederive Theorem 5.2(i) in [S2]. To see this, assume that $U = \mathbb{C}^n$ and $Y = \mathbb{C}^p$, and let $k \in e^{-\mu\cdot}L_2(\mathbb{R}^+;\mathbb{C}^{n\times p})$. Then, by Theorem 1.1, there exists a PS-realization $\theta_0 = (A_0, B_0, C_0; V_0, W_0, \mathbb{C}^n, \mathbb{C}^p)$ with weighting pattern k. Now, let W be $\mathcal{D}(A_0)$ endowed with the graph norm, and put $H = V_0$ and $A = A_0$. Next choose V such that $V^* = \mathcal{D}(A^*) \subset H^*$, set $B = B_0$ and $C = \tilde{C}_0$ where $\tilde{C}_0$ is the extended output operator associated with θ_0, and let $G(\lambda) = \tilde{C}(\lambda - A)^{-1}B$. Then $(A, B, C, G(\lambda))$ is a well-posed system in the sense of [S2]. Its weighting pattern is k and $B : U \to H$ and $C : W \to Y$ are bounded.

ACKNOWLEDGEMENT. The authors are grateful to Ruth F. Curtain for comments on an earlier version of this paper.

REFERENCES

[BGK 1] Bart, H., Gohberg, I., and Kaashoek, M.A.: *Wiener-Hopf equations with symbols analytic in a strip;* in: *Constructive Methods of Wiener-Hopf Factorization.* Birkhäuser OT **21**, Basel and Boston, 1986, pp. 39-74.

[BGK 2] Bart, H., Gohberg, I., and Kaashoek, M.A.: *Wiener-Hopf factorization, inverse Fourier transforms and exponentially dichotomous operators.* J. Funct. Anal. **68**, 1-42 (1986).

[C] Curtain, R.F.: *Equivalence of input-output stability and exponential stability for infinite-dimensional systems.* Math. Systems Theory **21**, 19-48 (1988).

[CLTZ] Curtain, R.F., Logemann, H., Townley, S., and Zwart, H.: *Well-posedness, stabilizability and admissibility for Pritchard-Salamon systems*, J. Math. Systems. Estim. Control, **4**, 493-496 (1994).

[CZ] Curtain, R.F. and Zwart, H.: *An Introduction to Infinite-Dimensional Linear System Theory*, Springer Verlag TAM **21**, New York etc., 1995.

[DU] Diestel, J., and Uhl, J.J., Jr.: *Vector Measures.* Math. Surveys, Vol. **15**, Amer. Math. Soc., Providence, R.I., 1977.

[GGK] Gohberg, I., Goldberg, S. and Kaashoek, M.A.: *Classes of Linear Operators*, Vol. I, Birkhäuser OT **49**, Basel and Boston, 1990.

[Ka] Kaballo, W.: *An example of a weakly square integrable operator function*, Private communication.

[vK] van Keulen, B.A.M.: *$\mathcal{H}_\infty$-control for Infinite-Dimensional Systems: a state-space approach*, Birkhäuser, Boston, 1993.

[P] Pazy, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences, Vol. **44**, Springer-Verlag, New

York, etc., 1983.

[PS] Pritchard, A.J., and Salamon, D.: *The linear quadratic optimal control problem for infinite dimensional systems with unbounded input and output operators.* SIAM J. Control and Optimiz. **25**, 121-144 (1987).

[R] Rudin, W.: *Functional Analysis.* McGraw-Hill, New York, 1973.

[S1] Salamon, D.: *Control and Observation of Neutral Systems.* Research Notes in Mathematics, Vol. **91**, Pitman, Boston, 1984.

[S2] Salamon, D.: *Realization theory in Hilbert space.* Math. Systems Theory **21**, 147-164 (1989).

M.A. Kaashoek and A.C.M. Ran
Faculteit Wiskunde en Informatica
Vrije Universiteit
de Boelelaan 1081a
1081 HV Amsterdam
The Netherlands

C.V.M. van der Mee
Dipartimento di Matematica
Università di Cagliari
Via Ospedale 72
09124 Cagliari
Italy

MSC Primary 93B28, 93B15, Secondary 47B35

Submitted: May 19, 1996