Integr. equ. oper. theory 30 (1998) 279 - 316
0378-620X/98/030279-38 $ 1.50+0.20/0

Integral Equations and Operator Theory

INVERSE SCATTERING IN ONE-DIMENSIONAL NONCONSERVATIVE MEDIA

Tuncay Aktosun, Martin Klaus, and Cornelis van der Mee

Dedicated to M.G. Kreĭn, one of the founding fathers of inverse scattering theory.

The inverse scattering problem arising in wave propagation in one-dimensional nonconservative media is analyzed. This is done in the frequency domain by considering the Schrödinger equation with the potential $ikP(x) + Q(x)$, where k^2 is the energy and $P(x)$ and $Q(x)$ are real integrable functions. Using a pair of uncoupled Marchenko integral equations, $P(x)$ and $Q(x)$ are recovered from an appropriate set of scattering data including bound-state information. Some illustrative examples are provided.

0. INTRODUCTION

The wave propagation in a one-dimensional medium, where energy absorption or generation may occur, can be described in the frequency domain by the generalized Schrödinger equation

$$\psi^{+\prime\prime}(k,x) + k^2\psi^+(k,x) = [ikP(x) + Q(x)]\,\psi^+(k,x), \qquad x \in \mathbf{R}, \tag{0.1}$$

where $\mathbf{R}$ is the real line, the prime denotes the derivative with respect to the spatial coordinate x, k is the wavenumber, k^2 is the energy, $P(x)$ represents the energy absorption or generation, and $Q(x)$ represents the restoring force density. By changing the sign of $P(x)$ in (0.1) we obtain the associated equation

$$\psi^{-\prime\prime}(k,x) + k^2\psi^-(k,x) = [-ikP(x) + Q(x)]\,\psi^-(k,x), \qquad x \in \mathbf{R}, \tag{0.2}$$

whose scattering data are to be used along with the scattering data from (0.1) in order to recover $P(x)$ and $Q(x)$.

Let $L^p_q(I)$ denote the measurable functions $f(x)$ such that $\int_I dx\,(1+|x|)^q\,|f(x)|^p$ is finite. Note that we have $L^p(I) = L^p_0(I)$. We will assume that $Q(x)$ is real valued and belongs to $L^1_1(\mathbf{R})$ and that $P(x)$ is real valued and satisfies $P \in L^1(\mathbf{R})$. We will use $||f||_p$ to denote the norm on $L^p(\mathbf{R})$ and write $||f||_{1,q}$ for $\int_{-\infty}^{\infty} dx\,(1+|x|)^q\,|f(x)|$. We will later impose further restrictions on $P(x)$ and $Q(x)$.

The scattering solutions of (0.1) and (0.2) comprise those behaving like e^{ikx} or e^{-ikx} as $x \to \pm\infty$, and such solutions occur when $k^2 > 0$. Among the scattering solutions are the

Jost solution from the left $f_l^{\pm}(k,x)$ and the Jost solution from the right $f_r^{\pm}(k,x)$ satisfying the boundary conditions

$$f_l^{\pm}(k,x) = \begin{cases} e^{ikx} + o(1), & x \to +\infty, \\ \dfrac{1}{T^{\pm}(k)} e^{ikx} + \dfrac{L^{\pm}(k)}{T^{\pm}(k)} e^{-ikx} + o(1), & x \to -\infty, \end{cases} \tag{0.3}$$

$$f_r^{\pm}(k,x) = \begin{cases} \dfrac{1}{T^{\pm}(k)} e^{-ikx} + \dfrac{R^{\pm}(k)}{T^{\pm}(k)} e^{ikx} + o(1), & x \to +\infty, \\ e^{-ikx} + o(1), & x \to -\infty, \end{cases} \tag{0.4}$$

where $T^{\pm}(k)$ is the transmission coefficient and $R^{\pm}(k)$ and $L^{\pm}(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrices $\mathbf{S}^{+}(k)$ and $\mathbf{S}^{-}(k)$ associated with (0.1) and (0.2), respectively, are given by

$$\mathbf{S}^{\pm}(k) = \begin{bmatrix} T^{\pm}(k) & R^{\pm}(k) \\ L^{\pm}(k) & T^{\pm}(k) \end{bmatrix}.$$

Let $[F;G] = FG' - F'G$ denote the Wronskian. The scattering coefficients can be expressed in terms of Wronskians of the Jost solutions of (0.1) and (0.2) as

$$[f_l^{\pm}(k,x); f_r^{\pm}(k,x)] = -\frac{2ik}{T^{\pm}(k)}, \qquad k \in \overline{\mathbf{C}^{+}}, \tag{0.5}$$

$$[f_l^{\pm}(k,x); f_r^{\mp}(-k,x)] = \frac{2ik\,L^{\pm}(k)}{T^{\pm}(k)} = -\frac{2ik\,R^{\mp}(-k)}{T^{\mp}(-k)}, \qquad k \in \mathbf{R},$$

$$[f_r^{\pm}(k,x); f_l^{\mp}(-k,x)] = -\frac{2ik\,R^{\pm}(k)}{T^{\pm}(k)} = \frac{2ik\,L^{\mp}(-k)}{T^{\mp}(-k)}, \qquad k \in \mathbf{R}.$$

We have [JJ76a,AKV97]

$$\mathbf{S}^{\pm}(-k) = \overline{\mathbf{S}^{\pm}(k)}, \qquad k \in \mathbf{R}, \tag{0.6}$$

$$\mathbf{S}^{\pm}(k)\,\mathbf{S}^{\mp}(-k)^{t} = \mathbf{I}, \qquad k \in \mathbf{R}, \tag{0.7}$$

where $\mathbf{I}$ is the 2×2 unit matrix, the superscript t denotes the matrix transpose, and the overline denotes complex conjugation. From (0.7) we get

$$L^{\pm}(k)\,T^{\mp}(-k) + T^{\pm}(k)\,R^{\mp}(-k) = 0, \qquad k \in \mathbf{R}, \tag{0.8}$$

$$T^{\pm}(k)\,T^{\mp}(-k) = 1 - R^{\pm}(k)\,R^{\mp}(-k), \qquad k \in \mathbf{R}. \tag{0.9}$$

The bound-state solutions of (0.1) and (0.2) are those nontrivial solutions belonging to $L^2(\mathbf{R})$. Such solutions correspond to the values of $k \in \mathbf{C}^{+}$ at which the Jost solutions

from the left and from the right are linearly dependent. For detailed information on the bound states of (0.1) and (0.2), we refer the reader to [AKV97] and the references therein.

When $P(x) = 0$, from (0.1) and (0.2) we get

$$\psi^{[0]\prime\prime}(k,x) + k^2\psi^{[0]}(k,x) = Q(x)\,\psi^{[0]}(k,x), \qquad x \in \mathbf{R}. \tag{0.10}$$

Let $f_l^{[0]}(k,x)$ and $f_r^{[0]}(k,x)$ denote the Jost solutions of (0.10) from the left and from the right, respectively. The zero-energy Jost solutions $f_l^{\pm}(0,x)$ and $f_r^{\pm}(0,x)$ of (0.1) and (0.2) are determined by $Q(x)$ alone, and we have

$$f_l^{\pm}(0,x) = f_l^{[0]}(0,x), \qquad f_r^{\pm}(0,x) = f_r^{[0]}(0,x). \tag{0.11}$$

Let $\mathbf{S}^{[0]}(k)$ denote the scattering matrix associated with (0.10):

$$\mathbf{S}^{[0]}(k) = \begin{bmatrix} T^{[0]}(k) & R^{[0]}(k) \\ L^{[0]}(k) & T^{[0]}(k) \end{bmatrix},$$

where $T^{[0]}(k)$ is the transmission coefficient and $R^{[0]}(k)$ and $L^{[0]}(k)$ are the reflection coefficients from the right and from the left, respectively. Generically $f_l^{[0]}(0,x)$ and $f_r^{[0]}(0,x)$ are linearly independent and $T^{[0]}(0) = 0$. However, in the exceptional case these two functions are linearly dependent and $T^{[0]}(0) \neq 0$; in this case, let us define

$$\gamma = \frac{f_l^{[0]}(0,x)}{f_r^{[0]}(0,x)}. \tag{0.12}$$

Then γ is a nonzero real constant determined by $Q(x)$ alone.

As for (0.10), the generic case for (0.1) and (0.2) occurs if $T^{[0]}(0) = 0$ and the exceptional case occurs if $T^{[0]}(0) \neq 0$. In the generic case we have [JJ76a,AKV97]

$$T^{\pm}(0) = 0, \qquad R^{\pm}(0) = L^{\pm}(0) = -1,$$

and in the exceptional case $T^{+}(0)$ and $T^{-}(0)$ are both nonzero. From Propositions 4.2, 5.1 and 5.3 and Theorem 5.2 of [AKV97], we have the following result.

THEOREM 0.1 *Assume $P, Q \in L^1(\mathbf{R})$ and $1/T^{\pm}(k)$ does not vanish for $k \in \mathbf{R} \setminus \{0\}$; then $\mathbf{S}^{\pm}(k)$ is continuous for $k \in \mathbf{R}\setminus\{0\}$. In the generic case, $\mathbf{S}^{\pm}(k)$ is continuous at $k = 0$ if we further assume $Q \in L_1^1(\mathbf{R})$. In the exceptional case, let us further assume $P, Q \in L_1^1(\mathbf{R})$; then $\mathbf{S}^{\pm}(k)$ is continuous at $k = 0$ if and only if $\int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2 \neq \pm(\gamma^2+1)$, where γ is the constant defined in* (0.12).

The inverse scattering problem for (0.1) considered in this paper consists of the recovery of $P(x)$ and $Q(x)$ from an appropriate set of scattering data. To stay in touch with

the scattering data traditionally adopted [JJ76a,JJ76b,Ja76,SS95], we use as our scattering data the two reflection coefficients $R^+(k)$ and $R^-(k)$ from the right, the bound-state energies or equivalently the poles k_j^+ of $T^+(k)$ in $\mathbf{C}^+$ for $j = 1, \cdots, N^+$ and the poles k_j^- of $T^-(k)$ in $\mathbf{C}^+$ for $j = 1, \cdots, N^-$, the multiplicities n_j^+ and n_j^- for each of these poles, and the bound-state constants $c_{j,s}^+$ for $s = 0, \cdots, n_j^+ - 1$ and $c_{j,s}^-$ for $s = 0, \cdots, n_j^- - 1$ defined in Section 4. In that section, we relate these bound-state constants to the ratio of the Jost solutions in a neighborhood of each bound-state wavenumber by generalizing the relationship between the Jost solutions of (0.10) and the so-called bound-state norming constants. In order to have a unique solution of the inverse problem, the total number of bound-state constants must agree with the total number of bound states including multiplicities. Let $N(P,Q)$ and $N(-P,Q)$ denote the number of bound states of (0.1) and (0.2), respectively, including multiplicities. We then have $N(\pm P,Q) = \sum_{j=1}^{N^\pm} n_j^\pm$. Thus, the total number of bound-state constants in our scattering data is $N(P,Q) + N(-P,Q)$.

We recover $P(x)$ and $Q(x)$ as follows. In terms of the scattering data, we first evaluate the two real-valued functions $\hat{S}_l^+(z)$ and $\hat{S}_l^-(z)$ defined in (5.4). These functions are used to obtain the two kernel functions $K_l^+(x;y,z)$ and $K_l^-(x;y,z)$ defined in (5.12). Then, the pair of uncoupled Marchenko equations (5.14) with kernels $K_l^+(x;y,z)$ and $K_l^-(x;y,z)$, respectively, is solved, and from their solutions $a_l^+(x,y)$ and $a_l^-(x,y)$, the functions $b_l^+(x,y)$ and $b_l^-(x,y)$ are constructed by simple integration as in (5.19). The four functions $a_l^+(x,y)$, $a_l^-(x,y)$, $b_l^+(x,y)$, and $b_l^-(x,y)$ are used to recover $P(x)$ and $Q(x)$ as indicated in Theorem 5.5. Note that in order to obtain our uncoupled pair of Marchenko equations, we first convert the Riemann-Hilbert problem given in (5.2) into the pair of two coupled Marchenko integral equations (5.9) and (5.10). We then further decouple the two equations (5.9) into the pair of uncoupled Marchenko equations (5.14).

In the inverse scattering problem for the Schrödinger equation (0.10), the scattering data usually consist of a reflection coefficient, $N(0,Q)$ bound-state energies, and $N(0,Q)$ bound-state norming constants [Fa64,DT79,CS89]. In this comparatively easy case, the poles of the transmission coefficient $T^{[0]}(k)$ in $\mathbf{C}^+$ are all simple and located on the imaginary axis. The scattering matrix $\mathbf{S}^{[0]}(k)$ can be uniquely constructed [Fa64,DT79,CS89] from a reflection coefficient and the poles of $T^{[0]}(k)$ in $\mathbf{C}^+$. The poles of $T^+(k)$ in $\mathbf{C}^+$, however, are not necessarily restricted to the imaginary axis, and the multiplicity of each such pole may be larger than one. The scattering matrix $\mathbf{S}^+(k)$ is not unitary and cannot be constructed from a reflection coefficient and the poles of $T^+(k)$ in $\mathbf{C}^+$. On the other hand, from (0.7) we see that

$$\frac{1}{T^-(k)} = \frac{T^+(-k)^2 - L^+(-k)\,R^+(-k)}{T^+(-k)}, \qquad k \in \mathbf{R}, \tag{0.13}$$

$$R^-(k) = \frac{-L^+(-k)}{T^+(-k)^2 - L^+(-k)\,R^+(-k)}, \qquad k \in \mathbf{R}, \tag{0.14}$$

$$L^-(k) = \frac{-R^+(-k)}{T^+(-k)^2 - L^+(-k)\,R^+(-k)}, \qquad k \in \mathbf{R}.$$

If $\mathbf{S}^+(k)$ is continuous and invertible for $k \in \mathbf{R}$, using (0.13) we can uniquely construct $1/T^-(k)$ in $\mathbf{C}^+$ by an analytic continuation from $\mathbf{R}$ to $\mathbf{C}^+$. Thus, with the help of (0.13) and (0.14) we can construct the scattering data $\{R^+(k), R^-(k), k_j^+, k_m^-, n_j^+, n_m^-, c_{j,s}^+, c_{m,u}^-\}$ by using $\mathbf{S}^+(k)$ for $k \in \mathbf{R}$ and a set of $N(P,Q) + N(-P,Q)$ constants, where $N(-P,Q)$ is equal to the number of zeros in $\mathbf{C}^+$ of the analytic continuation of the right-hand side of (0.13). Thus, it is possible to formulate and solve the inverse scattering problem for (0.1) without using (0.2) but by using the scattering data consisting of $\mathbf{S}^+(k)$ for $k \in \mathbf{R}$ and a set of $N(P,Q) + N(-P,Q)$ constants.

Let us now discuss the history of the inverse scattering problem for (0.1). In the radial case, when there are no bound states, Jaulent and Jean presented an inversion method [JJ72] when $P(x)$ is complex and $Q(x)$ is real. In [JJ76a,JJ76b] they applied their method to solve the one-dimensional inverse problem with real $Q(x)$ and imaginary $P(x)$; Jaulent [Ja76] also applied this method when $P(x)$ is real, although many details were not given. As indicated in Section IV of [Ja76], in this method, in our own terminology, using the scattering data $\{R^+(k), R^-(k)\}$, a pair of coupled Marchenko integral equations similar to our (5.9) and (5.10) was obtained. From the solutions of one of these pairs, by solving a differential equation, $P(x)$ and $Q(x)$ were recovered. This extra differential equation was needed in the solution of the inverse problem; in our own notation this is because the coupled Marchenko equations given in (5.9), in addition to containing the two unknown functions $B_l^+(x,y)$ and $B_l^-(x,y)$, also contain the unknown function $\zeta(x)$ defined in (1.2). In [Ja76] no details and no proofs were given in the one-dimensional case with real $P(x)$, and it was only mentioned that the results could be obtained analogously to the radial case.

When $P(x)$ is purely imaginary and $\int_{-\infty}^{\infty} dz\, P(z) = 0$, Sattinger and Szmigielski [SS95] showed that one can simplify the method of Jaulent and Jean and recover $P(x)$ by solving an algebraic equation rather than a differential equation. The pair of two coupled Marchenko equations (3.5) and (3.6) of [SS95] corresponds to our (5.20), and the algebraic equation of [SS95] corresponds to our (5.25). If $P(x)$ is purely imaginary, then the scattering matrix $\mathbf{S}^{\pm}(k)$ is unitary, the reflection coefficient $R^{\pm}(k)$ cannot exceed one in absolute value, and $1/T^{\pm}(k)$ cannot vanish on the real axis. If $P(x)$ is real and nontrivial, then $\mathbf{S}^{\pm}(k)$ is no longer unitary, $R^{\pm}(k)$ is not necessarily bounded by one in absolute value, and $1/T^{\pm}(k)$ may vanish on the real axis. Thus, the analysis of the inverse scattering problem with real $P(x)$ is more complicated than with purely imaginary $P(x)$.

We should also mention the study by Kaup [Ka75] on the direct and inverse scattering problem for

$$\phi'' + \left[k^2 + \frac{1}{4\beta^2}\right]\phi = [ikP(x) + Q(x)]\phi, \tag{0.15}$$

where β is a positive constant and $P, Q \in L^1_1(\mathbf{R})$. Under additional assumptions on $P(x)$, Tsutsumi [Ts81] analyzed the direct problem for (0.15) with $\beta = 1/2$ by using a 2×2 matrix analog of (0.1) with k replaced by $\sqrt{k^2+1}$. When $\beta = 1/2$, $\int_{-\infty}^{\infty} dx\, P(x) = 0$, and $P(x)$ and $Q(x)$ are in the Schwartz space, Sattinger and Szmigielski [SS96] studied the inverse scattering problem for (0.15) by analyzing an associated Riemann-Hilbert problem. Equation (0.15) is important for the solution of the initial-value problem to a coupled system of two nonlinear evolution equations by the inverse scattering transform [Ka75,SS96].

This paper is organized as follows. In Section 1 we introduce the auxiliary functions $\eta_l^{\pm}(k,x)$ and $\eta_r^{\pm}(k,x)$ in terms of the Jost solutions of (0.1) and (0.2), establish their analyticity in $\mathbf{C}^+$, and obtain their large-k asymptotics. In Section 2 we study various properties of the Fourier transforms of $\eta_l^{\pm}(k,x) - 1$ and $\eta_r^{\pm}(k,x) - 1$. In Section 3 we analyze certain properties of the scattering coefficients and their Fourier transforms. In Section 4 we analyze the bound states of (0.1) and (0.2) and define the bound-state constants. In Section 5, using the results of the prior sections, a pair of uncoupled Marchenko integral equations is obtained, the compactness of the corresponding integral operators is analyzed, and the recovery of $P(x)$ and $Q(x)$ from the solutions of the uncoupled Marchenko equations is described. In Section 6 we present some conditions for the unique solvability of the Marchenko equations. Finally, in Section 7 we present some examples illustrating the recovery of $P(x)$ and $Q(x)$.

1. Properties of solutions

In terms of the Jost solutions of (0.1) and (0.2), let

$$\eta_l^{\pm}(k,x) = e^{-ikx\pm\zeta} f_l^{\pm}(k,x), \qquad \eta_r^{\pm}(k,x) = e^{ikx\pm p\mp\zeta} f_r^{\pm}(k,x), \tag{1.1}$$

where we have defined

$$\zeta = \zeta(x) = \frac{1}{2}\int_x^{\infty} dz\, P(z), \qquad p = \frac{1}{2}\int_{-\infty}^{\infty} dz\, P(z). \tag{1.2}$$

Since $P(x)$ and $Q(x)$ are real, from (0.1)-(0.4) and (1.1) we get

$$\eta_l^{\pm}(-k,x) = \overline{\eta_l^{\pm}(k,x)}, \qquad \eta_r^{\pm}(-k,x) = \overline{\eta_r^{\pm}(k,x)}, \qquad k \in \mathbf{R}. \tag{1.3}$$

The functions $\eta_l^{\pm}(k,x)$ and $\eta_r^{\pm}(k,x)$ will be used to formulate the coupled Marchenko equations (5.9) and (5.10). In this section we analyze certain properties of $\eta_l^{\pm}(k,x)$ and $\eta_r^{\pm}(k,x)$.

Using (0.1)-(0.3), (1.1), and (1.2) we obtain

$$\eta_l^{\pm\prime\prime}(k,x) + [2ik \pm P(x)]\eta_l^{\pm\prime}(k,x) = W^{\pm}(x)\,\eta_l^{\pm}(k,x), \qquad x \in \mathbf{R}, \tag{1.4}$$

$$\eta_l^{\pm}(k,+\infty) = 1, \qquad \eta_l^{\pm\prime}(k,+\infty) = 0, \tag{1.5}$$

where we have defined

$$W^{\pm}(x) = Q(x) \mp \frac{1}{2}P'(x) - \frac{1}{4}P(x)^2. \tag{1.6}$$

Note that, if $P(x)$ is piecewise continuous, then the discontinuities of $P(x)$ lead to Dirac delta contributions in (1.6). We will elaborate on this at the end of Section 3.

Let $\mu_l^{\pm}(k,x) = e^{2ikx \mp 2\zeta}$. Multiplying (1.4) by $\mu_l^{\pm}(k,x)$ we obtain

$$[\mu_l^{\pm}(k,x)\,\eta_l^{\pm\prime}(k,x)]' = \mu_l^{\pm}(k,x)\,W^{\pm}(x)\,\eta_l^{\pm}(k,x), \qquad x \in \mathbf{R}. \tag{1.7}$$

Integrating (1.7) and using (1.5) we get

$$\eta_l^{\pm\prime}(k,x) = -\int_x^{\infty} dy\,\frac{\mu_l^{\pm}(k,y)}{\mu_l^{\pm}(k,x)}\,W^{\pm}(y)\,\eta_l^{\pm}(k,y). \tag{1.8}$$

Integrating (1.8) and using (1.5) once again, we find

$$\eta_l^{\pm}(k,x) = 1 + \int_x^{\infty} dt \int_t^{\infty} dy\,\frac{\mu_l^{\pm}(k,y)}{\mu_l^{\pm}(k,t)} W^{\pm}(y)\,\eta_l^{\pm}(k,y). \tag{1.9}$$

Changing the order of integration in (1.9) we have

$$\eta_l^{\pm}(k,x) = 1 + \int_x^{\infty} dy\,G_l^{\pm}(k;x,y)\,W^{\pm}(y)\,\eta_l^{\pm}(k,y), \tag{1.10}$$

where we have defined

$$\begin{aligned} G_l^{\pm}(k;x,y) &= \int_x^y dt \frac{\mu_l^{\pm}(k,y)}{\mu_l^{\pm}(k,t)} = \int_x^y dt\, e^{2ik(y-t)\pm\int_t^y P} \\ &= \frac{1}{2ik}\left[e^{2ik(y-x)\pm\int_x^y P} - 1\right] \mp \frac{1}{2ik}\int_x^y dt\,P(t)\,e^{2ik(y-t)\pm\int_t^y P}. \end{aligned} \tag{1.11}$$

Similarly, using (0.1), (0.2), (0.4), (1.1), and (1.2) we obtain

$$\eta_r^{\pm\prime\prime}(k,x) - [2ik \pm P(x)]\,\eta_r^{\pm\prime}(k,x) = W^{\mp}(x)\,\eta_r^{\pm}(k,x), \tag{1.12}$$

$$\eta_r^{\pm}(k,-\infty) = 1, \qquad \eta_r^{\pm\prime}(k,-\infty) = 0. \tag{1.13}$$

Integrating (1.12) twice and using (1.13) we get

$$\eta_r^{\pm\prime}(k,x) = \int_{-\infty}^{x} dy\, W^{\mp}(y)\, \eta_r^{\pm}(k,y)\, e^{2ik(x-y)\pm\int_y^x P}, \tag{1.14}$$

$$\eta_r^{\pm}(k,x) = 1 + \int_{-\infty}^{x} dy\, G_r^{\pm}(k;x,y)\, W^{\mp}(y)\, \eta_r^{\pm}(k,y), \tag{1.15}$$

where we have defined

$$\begin{aligned} G_r^{\pm}(k;x,y) &= \int_y^x dt\, e^{2ik(t-y)\pm\int_y^t P} \\ &= \frac{1}{2ik}\left[e^{2ik(x-y)\pm\int_y^x P} - 1\right] \pm \frac{1}{2ik}\int_y^x dt\, P(t)\, e^{2ik(t-y)\pm\int_y^t P}. \end{aligned}$$

THEOREM 1.1 *Assume $P \in L^1(\mathbf{R})$ and $Q \in L_1^1(\mathbf{R})$. Then, for each fixed $x \in \mathbf{R}$, the functions $\eta_l^{\pm}(k,x)$ and $\eta_r^{\pm}(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, and*

$$\eta_l^{\pm}(k,x) = 1 + o(1), \quad \eta_r^{\pm}(k,x) = 1 + o(1), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$$

If we further assume that $W^+, W^- \in L^1(\mathbf{R})$, then

$$\eta_l^{\pm}(k,x) = 1 + O(1/k), \quad \eta_r^{\pm}(k,x) = 1 + O(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}. \tag{1.16}$$

PROOF. We only prove (1.16) because the rest of the proof is given in Theorem 3.1 of [AKV97]. Note that for $y \geq x$, from (1.11) we get

$$|G_l^{\pm}(k;x,y)| \leq \frac{C}{|k|}, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{1.17}$$

where $C = \frac{1}{2}\left(1 + (1 + ||P||_1)e^{||P||_1}\right)$. Thus, iterating (1.10) and using (1.17) we obtain

$$|\eta_l^{\pm}(k,x) - 1| \leq \frac{C}{|k|}\left[\int_x^{\infty} dt\, |W^{\pm}(t)|\right] \exp\left(\int_x^{\infty} dz\, |W^{\pm}(z)|\right), \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\},$$

from which we have (1.16) for $\eta_l^{\pm}(k,x)$ whenever $W^{\pm} \in L^1(\mathbf{R})$. The proof of (1.16) for $\eta_r^{\pm}(k,x)$ is obtained in a similar manner. ∎

From Theorem 1.1 we obtain the following result which will be used in Section 2.

COROLLARY 1.2 *Assume $Q \in L_1^1(\mathbf{R})$ and $P, W^+, W^- \in L^1(\mathbf{R})$, where $W^+(x)$ and $W^-(x)$ are the functions defined in* (1.6). *Then, for each fixed $x \in \mathbf{R}$, the functions $\eta_l^{\pm}(\cdot,x) - 1$ and $\eta_r^{\pm}(\cdot,x) - 1$ belong to the Hardy space $\mathbf{H}_+^2(\mathbf{R})$, and thus their Fourier transforms defined in* (2.1) *are L^2-functions having their support on the positive half-line.*

Using (1.10) and (1.15), it is possible to improve (1.16) and prove that, for each fixed $x \in \mathbf{R}$, as $k \to \infty$ in $\overline{\mathbf{C}^+}$ we have

$$\eta_l^{\pm}(k,x) = 1 - \frac{1}{2ik}\int_x^{\infty} dy\, W^{\pm}(y) + o(1/k), \tag{1.18}$$

$$\eta_r^{\pm}(k,x) = 1 - \frac{1}{2ik}\int_{-\infty}^{x} dy\, W^{\mp}(y) + o(1/k). \tag{1.19}$$

Similarly, from (1.8) and (1.14), as $k \to \infty$ in $\overline{\mathbf{C}^+}$ we get

$$\eta_l^{\pm\prime}(k,x) = -\int_x^{\infty} dy\, W^{\pm}(y)\, e^{2ik(y-x)\pm\int_x^y P} + o(1/k), \tag{1.20}$$

$$\eta_r^{\pm\prime}(k,x) = \int_{-\infty}^{x} dy\, W^{\mp}(y)\, e^{2ik(x-y)\pm\int_y^x P} + o(1/k). \tag{1.21}$$

2. Fourier transforms of solutions

In this section we analyze the properties of the Fourier transforms of $\eta_l^{\pm}(k,x)-1$ and $\eta_r^{\pm}(k,x)-1$.

Assume $Q \in L_1^1(\mathbf{R})$ and $P, W^+, W^- \in L^1(\mathbf{R})$. Define

$$B_l^{\pm}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-iky}[\eta_l^{\pm}(k,x)-1], \quad B_r^{\pm}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-iky}[\eta_r^{\pm}(k,x)-1]. \tag{2.1}$$

From Corollary 1.2 it follows that, for each fixed $x \in \mathbf{R}$, the functions $B_l^{\pm}(x,\cdot)$ and $B_r^{\pm}(x,\cdot)$ belong to $L^2(\mathbf{R})$, and moreover we have $B_l^{\pm}(x,y) = B_r^{\pm}(x,y) = 0$ for $y < 0$. Thus,

$$\eta_l^{\pm}(k,x) = 1 + \int_0^{\infty} dy\, e^{iky} B_l^{\pm}(x,y), \qquad \eta_r^{\pm}(k,x) = 1 + \int_0^{\infty} dy\, e^{iky} B_r^{\pm}(x,y). \tag{2.2}$$

Using (2.2) in (1.10) and (1.15), we obtain the integral relations

$$\begin{aligned} B_l^{\pm}(x,y) = {} & \frac{1}{2}\int_{x+y/2}^{\infty} dt\, W^{\pm}(t)\, e^{\pm\int_{t-y/2}^{t} P} \\ & + \frac{1}{2}\int_0^y dz \int_{x+z/2}^{\infty} dt\, W^{\pm}(t)\, B_l^{\pm}(t, y-z)\, e^{\pm\int_{t-z/2}^{t} P}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} B_r^{\pm}(x,y) = {} & \frac{1}{2}\int_{-\infty}^{x-y/2} dt\, W^{\mp}(t)\, e^{\pm\int_{t}^{t+y/2} P} \\ & + \frac{1}{2}\int_0^y dz \int_{-\infty}^{x-z/2} dt\, W^{\mp}(t)\, B_r^{\pm}(t, y-z)\, e^{\pm\int_{t}^{t+z/2} P}. \end{aligned} \tag{2.4}$$

THEOREM 2.1 *Assume* $P \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, *and* $W^+, W^- \in L^1_{1+\alpha}(\mathbf{R})$ *for some* $\alpha \geq 0$. *Then, for each fixed* $x \in \mathbf{R}$, *the functions* $B^\pm_l(x,\cdot)$ *and* $B^\pm_r(x,\cdot)$ *are continuous in* $y \in [0,+\infty)$ *and are* $o((1+y)^{-1-\alpha})$ *as* $y \to +\infty$. *Moreover, for each fixed* $x \in \mathbf{R}$, *we have*

$$|B^\pm_l(x,y)| \leq \frac{C}{2}\,\sigma^\pm_l(x+y/2)\,e^{C\gamma^\pm_l(x)}, \qquad |B^\pm_r(x,y)| \leq \frac{C}{2}\,\sigma^\pm_r(x+y/2)\,e^{C\gamma^\pm_r(x)}, \tag{2.5}$$

where $C = e^{\|P\|_1}$ *and*

$$\sigma^\pm_l(x) = \int_x^\infty dt\,|W^\pm(t)|, \qquad \sigma^\pm_r(x) = \int_{-\infty}^x dt\,|W^\mp(t)|, \tag{2.6}$$

$$\gamma^\pm_l(x) = \int_x^\infty dt\,(t-x)\,|W^\pm(t)|, \qquad \gamma^\pm_r(x) = \int_{-\infty}^x dt\,(x-t)\,|W^\mp(t)|.$$

PROOF. Let us iterate (2.3) by writing

$$B^\pm_l(x,y) = \sum_{n=0}^\infty B^\pm_{l,n}(x,y), \tag{2.7}$$

where we have

$$B^\pm_{l,n+1}(x,y) = \frac{1}{2}\int_0^y dz \int_{x+z/2}^\infty dt\,W^\pm(t)\,B^\pm_{l,n}(t,y-z)\,e^{\pm\int_{t-z/2}^t P}, \qquad n = 0,1,\cdots, \tag{2.8}$$

and $B^\pm_{l,0}(x,y)$ is given by the first term on the right-hand side of (2.3). Note that

$$|B^\pm_{l,0}(x,y)| \leq \frac{C}{2}\int_{x+y/2}^\infty dt\;|W^\pm(t)| = \frac{C}{2}\,\sigma^\pm_l(x+y/2).$$

Starting from the induction hypothesis

$$|B^\pm_{l,n}(x,y)| \leq \frac{C^{n+1}}{2\,n!}\,\sigma^\pm_l(x+y/2)\,\gamma^\pm_l(x)^n,$$

and using (2.8) and the fact that $d\gamma^\pm_l(x)/dx = -\sigma^\pm_l(x)$, we obtain (2.5) for $B^\pm_l(x,y)$. The proof of (2.5) for $B^\pm_r(x,y)$ is similar. For each fixed $x \in \mathbf{R}$, using (2.8) and $W^\pm \in L^1_1(\mathbf{R})$, we see that $B^\pm_{l,n}(x,y)$ are continuous in $y \in [0,+\infty)$. Since $\sigma^\pm_l(x)$ and $\gamma^\pm_l(x)$ are decreasing functions of x, it follows that for each fixed $x \in \mathbf{R}$, the terms $B^\pm_{l,n}(x,y)$ are uniformly bounded in $y \in [0,+\infty)$. Thus, for each fixed $x \in \mathbf{R}$, the series in (2.7) is uniformly convergent and hence $B^\pm_l(x,y)$ are continuous in $y \in [0,+\infty)$. From (2.5), (2.6), and $W^\pm \in L^1_{1+\alpha}(\mathbf{R})$, we get $B^\pm_l(x,y) = o((1+y)^{-1-\alpha})$ as $y \to +\infty$. The proof for $B^\pm_r(x,y)$ is similar. ∎

From (2.3) we obtain

$$2\frac{\partial B_l^{\pm}(x,y)}{\partial x} + W^{\pm}(x+y/2)\, e^{\pm\int_x^{x+y/2} P} = -\int_0^y dz\, W^{\pm}(x+z/2)\, B_l^{\pm}(x+z/2, y-z)\, e^{\pm\int_x^{x+z/2} P}. \tag{2.9}$$

Using (2.5) in (2.9), we get

$$\left|2\frac{\partial B_l^{\pm}(x,y)}{\partial x} + W^{\pm}(x+y/2)\, e^{\pm\int_x^{x+y/2} P}\right| \le C^2 e^{C\gamma_l^{\pm}(x)}\, \sigma_l^{\pm}(x)\, \sigma_l^{\pm}(x+y/2). \tag{2.10}$$

In a similar way, using (2.4) we obtain

$$2\frac{\partial B_r^{\pm}(x,y)}{\partial x} - W^{\mp}(x-y/2)\, e^{\pm\int_{x-y/2}^{x} P} = \int_0^y dz\, W^{\mp}(x-z/2)\, B_r^{\pm}(x-z/2, y-z)\, e^{\pm\int_{x-z/2}^{x} P}, \tag{2.11}$$

and using (2.5) in (2.11) we get

$$\left|2\frac{\partial B_r^{\pm}(x,y)}{\partial x} - W^{\mp}(x-y/2)\, e^{\pm\int_{x-y/2}^{x} P}\right| \le C^2 e^{C\gamma_r^{\pm}(x)}\, \sigma_r^{\pm}(x)\, \sigma_r^{\pm}(x-y/2). \tag{2.12}$$

As in the proof of Theorem 2.1, one can show that if $P \in L^1(\mathbf{R})$ and $Q, W^+, W^- \in L_1^1(\mathbf{R})$, then the left-hand sides of (2.9) and (2.11) are continuous in $y \ge 0$ for every $x \in \mathbf{R}$.

PROPOSITION 2.2 *Assume $P \in L^1(\mathbf{R})$, $Q \in L_1^1(\mathbf{R})$, and $W^+, W^- \in L_{1+\alpha}^1(\mathbf{R})$ for some $\alpha \ge 0$. Then, for each fixed $x \in \mathbf{R}$, the functions $B_l^{\pm}(x,\cdot)$, $B_r^{\pm}(x,\cdot)$, $\partial B_l^{\pm}(x,\cdot)/\partial x$, and $\partial B_r^{\pm}(x,\cdot)/\partial x$ belong to $L_\alpha^1(\mathbf{R}^+)$.*

PROOF. Note that

$$\int_0^\infty dy\, (1+y)^\alpha \sigma_l^{\pm}(x+y/2) = \int_x^\infty dz \int_0^{2(z-x)} dy\, (1+y)^\alpha |W^{\pm}(z)| \le \frac{2^{1+\alpha}}{1+\alpha}\, [1+\max\{0,-x\}]^{1+\alpha}\, \|W^{\pm}\|_{1,1+\alpha}. \tag{2.13}$$

Using (2.13) in (2.5) we see that $B_l^{\pm}(x,\cdot)$ and $B_r^{\pm}(x,\cdot)$ are in $L_\alpha^1(\mathbf{R}^+)$. Similarly, using (2.13) in (2.10) and (2.12) we see that $\partial B_l^{\pm}(x,\cdot)/\partial x$ and $\partial B_r^{\pm}(x,\cdot)/\partial x$ belong to $L_\alpha^1(\mathbf{R}^+)$. ∎

From (2.3) and (2.4) we have

$$B_l^{\pm}(x,0+) = \frac{1}{2}\int_x^\infty dt\, W^{\pm}(t), \qquad B_r^{\pm}(x,0+) = \frac{1}{2}\int_{-\infty}^x dt\, W^{\mp}(t), \tag{2.14}$$

and from (2.2) and (2.14) it follows that

$$ik[1-\eta_l^{\pm}(k,x)] = \frac{1}{2}\int_x^{\infty} dt\, W^{\pm}(t) + \int_0^{\infty} dy\, e^{iky}\frac{\partial B_l^{\pm}(x,y)}{\partial y},$$

$$ik[1-\eta_r^{\pm}(k,x)] = \frac{1}{2}\int_{-\infty}^{x} dt\, W^{\mp}(t) + \int_0^{\infty} dy\, e^{iky}\frac{\partial B_r^{\pm}(x,y)}{\partial y}.$$

3. Scattering coefficients

In this section we analyze certain properties of the scattering coefficients and their Fourier transforms. At the end of the section we discuss the large-k behavior of the reflection coefficients in case $W^+(x)$ or $W^-(x)$ contains some delta-function terms.

For $\alpha \geq 0$ let $\mathcal{W}_\alpha$ denote the set of all functions $\phi(k)$ of the form $\phi(k) = c + \int_{-\infty}^{\infty} dt\, e^{ikt}h(t)$ where c is a complex constant and $h \in L^1_\alpha(\mathbf{R})$. Then $\mathcal{W}_\alpha$ endowed with the norm

$$\|\phi\|_{\mathcal{W}_\alpha} = |c| + \int_{-\infty}^{\infty} dt\,(1+|t|)^\alpha |h(t)|$$

is a commutative Banach algebra with unit element. Its multiplicative linear functionals are the maps $\phi \mapsto c = \phi(\pm\infty)$ and, for every $k \in \mathbf{R}$, $\phi \mapsto \phi(k)$. We have the following result ([GRS64]; Example (c) in Section XXIX.2 and Example (vii) in Section XXX.1 of [GGK93]).

Proposition 3.1 *If $\phi \in \mathcal{W}_\alpha$ and $\phi(k) \neq 0$ for every $k \in \mathbf{R}$ and $\phi(\pm\infty) \neq 0$, then the function $1/\phi$ belongs to $\mathcal{W}_\alpha$.*

From (6.3) and (6.4) of [AKV97] we have

$$-\frac{2ik\,R^{\pm}(k)}{T^{\pm}(k)} = e^{-2ikx\mp p\pm 2\zeta}[\eta_r^{\pm}(k,x);\eta_l^{\mp}(-k,x)], \qquad k \in \mathbf{R}, \tag{3.1}$$

$$-\frac{2ik\,L^{\pm}(k)}{T^{\pm}(k)} = e^{2ikx\pm p\mp 2\zeta}[\eta_l^{\pm}(k,x);\eta_r^{\mp}(-k,x)], \qquad k \in \mathbf{R}, \tag{3.2}$$

$$\frac{2ik}{T^{\pm}(k)}e^{\pm p} = [2ik \pm P(x)]\,\eta_l^{\pm}(k,x)\,\eta_r^{\pm}(k,x) + [\eta_r^{\pm}(k,x);\eta_l^{\pm}(k,x)], \qquad k \in \overline{\mathbf{C}^+}. \tag{3.3}$$

Proposition 3.2 *Assume $P, Q, W^+, W^- \in L^1(\mathbf{R})$. Then*

$$R^{\pm}(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} dy\, W^{\mp}(y)\, e^{-2iky\pm 2\zeta(y)} + o(1/k), \qquad |k| \to +\infty \text{ in } \mathbf{R}, \tag{3.4}$$

$$L^{\pm}(k) = -\frac{1}{2ik}\int_{-\infty}^{\infty} dy\, W^{\pm}(y)\, e^{2iky\mp 2\zeta(y)} + o(1/k), \qquad |k| \to +\infty \text{ in } \mathbf{R}, \tag{3.5}$$

and as $k \to \infty$ *in* $\overline{\mathbf{C}^+}$ *we have*

$$e^{\mp p}\, T^{\pm}(k) - 1 = \frac{1}{2ik} \int_{-\infty}^{\infty} dy\, W^{\pm}(y) + o(1/k) = \frac{1}{2ik} \int_{-\infty}^{\infty} dy\, W^{\mp}(y) + o(1/k). \tag{3.6}$$

PROOF. When $P, Q, W^+, W^- \in L^1(\mathbf{R})$, for each fixed $x \in \mathbf{R}$, one can show that (1.18)-(1.21) hold as $k \to \infty$ in $\overline{\mathbf{C}^+}$ when $|k| \geq a$ for any positive constant a. Using (1.18)-(1.21) in (3.1) and (3.2), we get

$$\frac{2ik\, R^{\pm}(k)}{T^{\pm}(k)} = e^{\mp p} \int_{-\infty}^{\infty} dy\, W^{\mp}(y)\, e^{-2iky \pm 2\zeta(y)} + o(1), \qquad |k| \to +\infty \text{ in } \mathbf{R}, \tag{3.7}$$

$$\frac{2ik\, L^{\pm}(k)}{T^{\pm}(k)} = -e^{\pm p} \int_{-\infty}^{\infty} dy\, W^{\pm}(y)\, e^{2iky \mp 2\zeta(y)} + o(1), \qquad |k| \to +\infty \text{ in } \mathbf{R}, \tag{3.8}$$

and as $k \to \infty$ in $\overline{\mathbf{C}^+}$, using (1.18)-(1.21) in (3.3) we obtain

$$\frac{e^{\pm p}}{T^{\pm}(k)} = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dy\, W^{\pm}(y) + o(1/k) = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} dy\, W^{\mp}(y) + o(1/k). \tag{3.9}$$

Thus, from (3.7)-(3.9) we obtain (3.4)-(3.6). ∎

PROPOSITION 3.3 *Assume* $P, W^+, W^- \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, *and* $1/T^{\pm}(k)$ *does not vanish for* $k \in \mathbf{R} \setminus \{0\}$; *then in the generic case, the three functions* $e^{\mp p} T^{\pm}(k) - 1$, $R^{\pm}(k)$, *and* $L^{\pm}(k)$ *belong to* $L^2(\mathbf{R})$. *In the exceptional case, these three functions belong to* $L^2(\mathbf{R})$ *if we further assume that* $P \in L^1_1(\mathbf{R})$ *and* $\int_{-\infty}^{\infty} dx\, P(x)\, f_l^{[0]}(0,x)^2 \neq \pm(\gamma^2 + 1)$, *where* γ *is the constant defined in* (0.12).

PROOF. From Proposition 4.2 of [AKV97], it follows that $T^+(k)$, $R^+(k)$, and $L^+(k)$ are continuous for $k \in \mathbf{R} \setminus \{0\}$ if we assume that $P, Q \in L^1(\mathbf{R})$ and $1/T^+(k)$ does not vanish for $k \in \mathbf{R} \setminus \{0\}$. By Proposition 3.2 we see that $e^{-p} T^+(k) - 1$, $R^+(k)$, and $L^+(k)$ are $O(1/k)$ as $k \to +\infty$ in $\mathbf{R}$ if we further assume $W^+, W^- \in L^1(\mathbf{R})$. From Proposition 5.1 of [AKV97] we see that $T^+(k)$, $R^+(k)$, and $L^+(k)$ are continuous also at $k = 0$ in the generic case when $P \in L^1(\mathbf{R})$ and $Q \in L^1_1(\mathbf{R})$. In the exceptional case, when $P, Q \in L^1_1(\mathbf{R})$, these three functions are continuous at $k = 0$ if and only if $\int_{-\infty}^{\infty} dx\, P(x)\, f_l^+(0,x)^2 \neq \gamma^2 + 1$, as shown in Theorem 5.2 of [AKV97]. The proof for the three functions related to (0.2) is obtained in a similar manner. ∎

THEOREM 3.4 *Assume* $P \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, *and* $W^+, W^- \in L^1_{1+\alpha}(\mathbf{R})$ *for some* $\alpha \geq 0$. *Then, the quantities* $2ik[1 - e^{\pm p}/T^{\pm}(k)]$, $2ikR^{\pm}(k)/T^{\pm}(k)$, *and* $2ikL^{\pm}(k)/T^{\pm}(k)$ *are Fourier transforms of real functions in* $L^1_{\alpha}(\mathbf{R})$.

PROOF. From Proposition 2.2 it follows that the right-hand sides in (3.1) and (3.2) belong to $\mathcal{W}_{\alpha}$ and vanish as $|k| \to +\infty$ in $\mathbf{R}$. On the other hand, the right-hand side of (3.3) equals $2ik$ plus a function in $\mathcal{W}_{\alpha}$ that vanishes as $k \to \infty$ in $\overline{\mathbf{C}^+}$. ∎

THEOREM 3.5 *Assume* $P \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, *and* $W^+, W^- \in L^1_{1+\alpha}(\mathbf{R})$ *for some* $\alpha \geq 0$, *and suppose* $1/T^{\pm}(k)$ *does not vanish for* $k \in \mathbf{R} \setminus \{0\}$. *Then, in the generic case* $T^{\pm}(k) - e^{\pm p}$, $R^{\pm}(k)$, *and* $L^{\pm}(k)$ *are all Fourier transforms of functions in* $L^1_\alpha(\mathbf{R})$.

PROOF. Note that

$$\frac{k}{k+i}\frac{e^p}{T^+(k)} - 1 = -\frac{i}{k+i} - \frac{1}{2i(k+i)}2ik\left[1 - \frac{e^p}{T^+(k)}\right]. \tag{3.10}$$

Using Theorem 3.4 we see that the left-hand side of (3.10) is the Fourier transform of a function in $L^1_\alpha(\mathbf{R})$. Using Proposition 3.1 and the absence of zeros of $k/[(k+i)T^+(k)]$ for $k \in \mathbf{R}$, we see that $[(k+i)/k]T^+(k)$ is e^p plus the Fourier transform of a function in $L^1_\alpha(\mathbf{R})$. Hence $T^+(k) - e^p$ is the Fourier transform of a function in $L^1_\alpha(\mathbf{R})$. Similarly, since

$$R^+(k) = \frac{1}{2i(k+i)} \cdot \frac{k+i}{k}T^+(k) \cdot \frac{2ikR^+(k)}{T^+(k)}, \quad L^+(k) = \frac{1}{2i(k+i)} \cdot \frac{k+i}{k}T^+(k) \cdot \frac{2ikL^+(k)}{T^+(k)},$$

using Theorem 3.4 and the fact that $\mathcal{W}_\alpha$ is an algebra, we conclude that the inverse Fourier transforms of $R^+(k)$ and $L^+(k)$ belong to $L^1_\alpha(\mathbf{R})$. The proof for the quantities related to (0.2) is obtained in a similar manner. ∎

THEOREM 3.6 *Assume* $P, Q \in L^1_1(\mathbf{R})$ *and* $W^+, W^- \in L^1_{2+\alpha}(\mathbf{R})$ *for some* $\alpha \geq 0$, *and suppose* $1/T^{\pm}(k)$ *does not have any real zeros. Then, in the exceptional case* $T^{\pm}(k) - e^{\pm p}$, $R^{\pm}(k)$, *and* $L^{\pm}(k)$ *are all Fourier transforms of functions in* $L^1_\alpha(\mathbf{R})$.

PROOF. Let a caret denote the Fourier transform, and let $\hat{f}(k)$ denote the left-hand side of (3.10). From the proof of Theorem 3.5, we know that $f \in L^1_{1+\alpha}(\mathbf{R})$, where f is the inverse Fourier transform of $\hat{f}$. Define

$$\hat{g}(k) = -1 + \frac{k+i}{k}[1 + \hat{f}(k)]. \tag{3.11}$$

Using $\int_{-\infty}^{\infty} dt\, f(t) = \hat{f}(0) = -1$ in (3.11), we obtain

$$\begin{aligned}\hat{g}(k) &= \int_{-\infty}^{\infty} dt\, e^{ikt} f(t) - \int_{-\infty}^{\infty} dt\, \frac{e^{ikt}-1}{ik} f(t) \\ &= \int_{-\infty}^{\infty} dt\, e^{ikt} f(t) - \int_0^{\infty} dz\, e^{ikz} \int_z^{\infty} dt\, f(t) + \int_{-\infty}^{0} dz\, e^{ikz} \int_{-\infty}^{z} dt\, f(t).\end{aligned} \tag{3.12}$$

Since $f \in L^1_{1+\alpha}(\mathbf{R})$, from (3.12) it follows that $\hat{g}(k)$ is the Fourier transform of a function in $L^1_\alpha(\mathbf{R})$. Because $1/T^+(k)$ is assumed not to have any real zeros, it follows that $T^+(k) - e^p$ is the Fourier transform of a function in $L^1_\alpha(\mathbf{R})$. Using Theorem 3.4 and an argument similar to that used for $T^+(k)$, one sees that $R^+(k)/T^+(k)$ and $L^+(k)/T^+(k)$ are Fourier transforms of functions in $L^1_\alpha(\mathbf{R})$. In the exceptional case, $1/T^+(k)$ is continuous on $\mathbf{R}$,

and since $1/T^+(k)$ is assumed not to have any real zeros, it follows that $R^+(k)$ and $L^+(k)$ are also Fourier transforms of functions in $L^1_\alpha(\mathbf{R})$. The proof for the quantities related to (0.2) is obtained similarly. ∎

Using (3.5) and (3.6), we see that if $W^+(x)$ or $W^-(x)$ contains any delta-function terms, the coefficients in those terms can be obtained from the large-k asymptotics of the reflection coefficients. For example, assume that $Q(x)$ contains the delta-function term $q_0\,\delta(x)$ and that $P(x)$ is discontinuous at $x = 0$ resulting in the delta-function term for $P'(x)$ given by $p_0'\,\delta(x)$. In other words, $p_0' = P(0+) - P(0-)$. Then, from (1.6) we see that $W^\pm(x)$ contains $[q_0 \mp p_0'/2]\,\delta(x)$. Using (3.5) and (3.6) we obtain

$$\lim_{|k|\to+\infty} 2ik\,R^\pm(k) = \left[q_0 \pm \frac{p_0'}{2}\right] \exp\left(\pm\int_0^\infty dz\,P(z)\right), \tag{3.13}$$

$$\lim_{|k|\to+\infty} 2ik\,L^\pm(k) = \left[q_0 \mp \frac{p_0'}{2}\right] \exp\left(\mp\int_0^\infty dz\,P(z)\right).$$

We will use (3.13) in Section 7.

4. Bound states

In this section we analyze the bound states of (0.1) and (0.2) and introduce the bound-state constants which will be used in the scattering data to solve the inverse scattering problem. Recall that the bound states of (0.1) correspond to the zeros of $1/T^+(k)$ in $\mathbf{C}^+$ and that such zeros are either situated on the positive imaginary axis or are symmetrically located with respect to the positive imaginary axis; moreover, the multiplicity of each such zero may be larger than one [AKV97]. If $P \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, and there are no zeros of $1/T^+(k)$ for $k \in \mathbf{R}$, then the number of zeros (including multiplicities) of $1/T^+(k)$ in $\mathbf{C}^+$ is finite [AKV97].

Let $k_j^\pm$ for $j = 1, \cdots, N^\pm$ correspond to the poles of $T^\pm(k)$ in $\mathbf{C}^+$ and let $n_j^\pm$ denote the multiplicity of the pole of $T^\pm(k)$ at $k_j^\pm$. Let us also define

$$c^\pm(k,x) = \frac{1}{d^\pm(k,x)} = e^{\pm px}\frac{f_r^\pm(k,x)}{f_l^\pm(k,x)}. \tag{4.1}$$

Note that, in case $f_l^\pm(k_j^\pm,x)$ vanishes at some x, then $f_r^\pm(k_j^\pm,x)$ also vanishes at the same x because $f_l^\pm(k_j^\pm,x)$ and $f_r^\pm(k_j^\pm,x)$ are linearly dependent.

PROPOSITION 4.1 *Assume $P \in L^1(\mathbf{R})$, $Q \in L^1_1(\mathbf{R})$, and suppose $1/T^\pm(k)$ has a zero at $k_j^\pm \in \mathbf{C}^+$ of multiplicity $n_j^\pm$. Then, in the Taylor series of $c^\pm(k,x)$ and $d^\pm(k,x)$ at $k_j^\pm$, the first $n_j^\pm$ coefficients do not depend on x.*

PROOF. From (0.5) and (4.1) we get

$$\frac{\partial c^{\pm}(k,x)}{\partial x} = e^{\pm p}\frac{[f_l^{\pm}(k,x); f_r^{\pm}(k,x)]}{f_l^{\pm}(k,x)^2} = \frac{-2ik}{e^{\mp p}T^{\pm}(k)}\,\frac{1}{f_l^{\pm}(k,x)^2}.$$

Hence, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial k}\right)^s c^{\pm}(k,x) = \sum_{j=0}^{s}\binom{s}{j}\left[\left(\frac{d}{dk}\right)^j \frac{-2ik}{e^{\mp p}T^{\pm}(k)}\right]\left[\left(\frac{\partial}{\partial k}\right)^{s-j}\frac{1}{f_l^{\pm}(k,x)^2}\right]. \tag{4.2}$$

Since the value of $-2ik/T^{\pm}(k)$ and its first $n_j^{\pm}-1$ derivatives at $k_j^{\pm}$ vanish, from (4.2) we see that $\partial^s c^{\pm}(k_j^{\pm},x)/\partial k^s$ vanish for $s = 0, 1, \cdots, n_j^{\pm}-1$. Thus,

$$c^{\pm}(k,x) = \sum_{s=0}^{\infty} c_{j,s}^{\pm}(x)\,(k-k_j^{\pm})^s, \tag{4.3}$$

where $c_{j,0}^{\pm}, \cdots, c_{j,n_j^{\pm}-1}^{\pm}$ do not depend on $x \in \mathbf{R}$. Note that in the expansion

$$d^{\pm}(k,x) = \sum_{s=0}^{\infty} d_{j,s}^{\pm}(x)\,(k-k_j^{\pm})^s,$$

each coefficient $d_{j,s}^{\pm}(x)$ can be expressed in terms of $c_{j,0}^{\pm}(x), \cdots, c_{j,s}^{\pm}(x)$ because we have

$$\sum_{m=0}^{s} c_{j,m}^{\pm}(x)\,d_{j,s-m}^{\pm}(x) = \delta_{s,0}, \qquad s = 0, 1, \cdots, n_j^{\pm}-1, \tag{4.4}$$

where $\delta_{j,m}$ is the Kronecker delta. Hence, the first $n_j^{\pm}$ coefficients in the expansion (4.4) are independent of x if and only if the same is true for the expansion (4.3). ∎

Note that we can construct $e^{-p}T^{+}(k)$ and $e^{p}T^{-}(k)$ uniquely in terms of $R^{+}(k)$, $R^{-}(k)$, k_j^{+}, k_s^{-}, n_j^{+}, and n_s^{-}, where $j = 1, \cdots, N^{+}$ and $s = 1, \cdots, N^{-}$. Let us write (0.9) in the form

$$[e^{\mp p}T^{\pm}(k)]\,[e^{\pm p}T^{\mp}(-k)] = 1 - R^{\pm}(k)\,R^{\mp}(-k), \qquad k \in \mathbf{R}. \tag{4.5}$$

Recall that $k\,e^{\pm p}/[(k+i)\,T^{\pm}(k)]$ is analytic in $\mathbf{C}^{+}$, is continuous in $\overline{\mathbf{C}^{+}}$, and approaches 1 as $k \to \infty$ in $\overline{\mathbf{C}^{+}}$. The construction of $e^{\mp p}T^{\pm}(k)$ can be carried out by solving the scalar Riemann-Hilbert problem (4.5). Having constructed $e^{-p}T^{+}(k)$ and $e^{p}T^{-}(k)$, one can also construct $e^{\mp 2p}L^{\pm}(k)$ using (0.8). As we will see later, the constants $\{c_{j,0}^{\pm}, \cdots, c_{j,n_j^{\pm}-1}^{\pm}\}$ for $j = 1, \cdots, N^{\pm}$ play the role of the bound-state norming constants in the inverse scattering problem for the usual Schrödinger equation [Fa64,DT79,Ne80,CS89,AKV93]. As seen in

the proof of Proposition 4.1, knowledge of these constants is equivalent to knowledge of the set $\{d^{\pm}_{j,0}, \cdots, d^{\pm}_{j,n^{\pm}_j-1}\}$ for $j = 1, \cdots, N^{\pm}$.

Let us define the reduced transmission coefficients $T^{+}_{0}(k)$ and $T^{-}_{0}(k)$ by

$$T^{\pm}(k) = T^{\pm}_{0}(k) \prod_{j=1}^{N^{\pm}} \left(\frac{k - \overline{k^{\pm}_j}}{k - k^{\pm}_j} \right)^{n^{\pm}_j}. \tag{4.6}$$

Note that $T^{\pm}_{0}(k)$ is analytic in $\mathbf{C}^{+}$. We have

$$e^{\mp p} T^{\pm}(k) = \sum_{j=1}^{N^{\pm}} \sum_{s=0}^{n^{\pm}_j - 1} \frac{t^{\pm}_{j,s}}{(k - k^{\pm}_j)^{s+1}} + F^{\pm}(k), \tag{4.7}$$

where

$$t^{\pm}_{j,s} = \frac{1}{(n^{\pm}_j - 1 - s)!} \left[\frac{d^{n^{\pm}_j - 1 - s}}{dk^{n^{\pm}_j - 1 - s}} \left[e^{\mp p}\, T^{\pm}(k)\, (k - k^{\pm}_j)^{n^{\pm}_j} \right] \right]_{k = k^{\pm}_j},$$

and $F^{\pm}(k)$ is analytic in $\mathbf{C}^{+}$, is continuous in $\overline{\mathbf{C}^{+}}$, and tends to 1 as $k \to \infty$ in $\overline{\mathbf{C}^{+}}$. Note that the parameters $t^{+}_{j,s}$ and $t^{-}_{m,u}$ are uniquely determined when one knows $R^{+}(k)$, $R^{-}(k)$, k^{+}_j, k^{-}_m, n^{+}_j, and n^{-}_m, where $j = 1, \cdots, N^{+}$; $m = 1, \cdots, N^{-}$; $s = 0, \cdots, n^{+}_j - 1$; and $u = 0, \cdots, n^{-}_m - 1$.

Recall that for each $x \in \mathbf{R}$, the functions $\eta^{\pm}_{l}(k, x)$ and $\eta^{\pm}_{r}(k, x)$ are analytic in $\mathbf{C}^{+}$, and hence we have the convergent expansions

$$\eta^{\pm}_{l}(k, x) = \sum_{s=0}^{\infty} \eta^{\pm}_{l;j,s}(x)\, (k - k^{\pm}_j)^s, \qquad \eta^{\pm}_{r}(k, x) = \sum_{s=0}^{\infty} \eta^{\pm}_{r;j,s}(x)\, (k - k^{\pm}_j)^s, \tag{4.8}$$

valid for $|k - k^{\pm}_j| < \operatorname{Im} k^{\pm}_j$. Using (1.1), (2.1), (4.1), and (4.3) in (4.8), we get

$$\eta^{\pm}_{r;j,s}(x) = e^{\mp 2\zeta(x)} e^{2ik^{\pm}_j x} \sum_{m=0}^{s} \sum_{n=0}^{s-m} \frac{(2ix)^{s-m-n}}{(s-m-n)!}\, \eta^{\pm}_{l;j,m}(x)\, c^{\pm}_{j,n}. \tag{4.9}$$

Using (2.2) and (4.8), we also have

$$\eta^{\pm}_{l;j,s}(x) = \delta_{s,0} + \int_0^{\infty} dz\, \frac{(iz)^s}{s!} e^{ik^{\pm}_j z} B^{\pm}_{l}(x, z), \tag{4.10}$$

$$\eta^{\pm}_{r;j,s}(x) = \delta_{s,0} + \int_0^{\infty} dz\, \frac{(iz)^s}{s!} e^{ik^{\pm}_j z} B^{\pm}_{r}(x, z).$$

Using (4.7) and (4.8) we get

$$e^{\mp p}T^{\pm}(k)\,\eta_r^{\pm}(k,x)-1=H_r^{\pm}(k,x)+\sum_{j=1}^{N^{\pm}}\sum_{s=0}^{n_j^{\pm}-1}\sum_{m=0}^{n_j^{\pm}-1-s}\frac{\eta_{r;j,m}^{\pm}(x)\,t_{j,s+m}^{\pm}}{(k-k_j^{\pm})^{s+1}}, \tag{4.11}$$

$$e^{\mp p}T^{\pm}(k)\,\eta_l^{\pm}(k,x)-1=H_l^{\pm}(k,x)+\sum_{j=1}^{N^{\pm}}\sum_{s=0}^{n_j^{\pm}-1}\sum_{m=0}^{n_j^{\pm}-1-s}\frac{\eta_{l;j,m}^{\pm}(x)\,t_{j,s+m}^{\pm}}{(k-k_j^{\pm})^{s+1}},$$

where for each x the functions $H_l^{\pm}(k,x)$ and $H_r^{\pm}(k,x)$ belong to the Hardy space $\mathbf{H}_+^2(\mathbf{R})$. Let us define

$$A_l^{\pm}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{iky}\,[e^{\mp p}T^{\pm}(k)\,\eta_r^{\pm}(k,x)-1], \tag{4.12}$$

$$A_r^{\pm}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{iky}\,[e^{\mp p}T^{\pm}(k)\,\eta_l^{\pm}(k,x)-1].$$

From (0.6) and (1.3) it follows that $A_l^{\pm}(x,y)$ and $A_r^{\pm}(x,y)$ are real valued. Using (4.11) in (4.12), for $y>0$ we obtain

$$A_l^{\pm}(x,y)=\sum_{j=1}^{N^{\pm}}\sum_{s=0}^{n_j^{\pm}-1}\sum_{m=0}^{n_j^{\pm}-1-s}\frac{i}{s!}e^{ik_j^{\pm}y}(iy)^s\,\eta_{r;j,m}^{\pm}(x)\,t_{j,s+m}^{\pm}. \tag{4.13}$$

Using (4.9) in (4.13), for $y>0$ we can write $A_l^{\pm}(x,y)$ as

$$A_l{}^{\pm}(x,y)$$
$$=ie^{\mp 2\zeta(x)}\sum_{j=1}^{N^{\pm}}e^{ik_j^{\pm}(2x+y)}\sum_{s=0}^{n_j^{\pm}-1}\sum_{m=0}^{n_j^{\pm}-1-s}\sum_{u=0}^{m}\sum_{n=0}^{m-u}\frac{(iy)^s}{s!}\frac{(2ix)^{m-u-n}}{(m-u-n)!}t_{j,s+m}^{\pm}\,\eta_{l;j,u}^{\pm}(x)\,c_{j,n}^{\pm}. \tag{4.14}$$

Note that we can write (4.14) as

$$A_l^{\pm}(x,y)$$
$$=ie^{\mp 2\zeta(x)}\sum_{j=1}^{N^{\pm}}e^{ik_j^{\pm}(2x+y)}\sum_{s=0}^{n_j^{\pm}-1}\sum_{w=s}^{n_j^{\pm}-1}\sum_{u=0}^{w-s}\sum_{n=0}^{w-s-u}\frac{(iy)^s}{s!}\frac{(2ix)^{w-s-u-n}}{(w-s-u-n)!}t_{j,w}^{\pm}\,\eta_{l;j,u}^{\pm}(x)\,c_{j,n}^{\pm}. \tag{4.15}$$

Using

$$\sum_{s=0}^{n_j^{\pm}-1}\sum_{w=s}^{n_j^{\pm}-1}\sum_{u=0}^{w-s}\sum_{n=0}^{w-s-u}=\sum_{w=0}^{n_j^{\pm}-1}\sum_{s=0}^{w}\sum_{n=0}^{w-s}\sum_{u=0}^{w-s-n}=\sum_{w=0}^{n_j^{\pm}-1}\sum_{n=0}^{w}\sum_{s=0}^{w-n}\sum_{u=0}^{w-s-n}=\sum_{w=0}^{n_j^{\pm}-1}\sum_{n=0}^{w}\sum_{u=0}^{w-n}\sum_{s=0}^{w-u-n},$$

we can simplify (4.15) to

$$A_l^{\pm}(x,y) = ie^{\mp 2\zeta(x)} \sum_{j=1}^{N^{\pm}} e^{ik_j^{\pm}(2x+y)} \sum_{w=0}^{n_j^{\pm}-1} t_{j,w}^{\pm} \sum_{n=0}^{w} c_{j,n}^{\pm} \sum_{u=0}^{w-n} \eta_{l;j,u}^{\pm}(x) \frac{[i(2x+y)]^{w-u-n}}{(w-u-n)!}, \quad (4.16)$$

where $y > 0$. In a similar manner, we obtain

$$A_r^{\pm}(x,y) = ie^{\pm 2\zeta(x)} \sum_{j=1}^{N^{\pm}} e^{ik_j^{\pm}(-2x+y)} \sum_{w=0}^{n_j^{\pm}-1} t_{j,w}^{\pm} \sum_{n=0}^{w} d_{j,n}^{\pm} \sum_{u=0}^{w-n} \eta_{r;j,u}^{\pm}(x) \frac{[i(-2x+y)]^{w-u-n}}{(w-u-n)!},$$

where $y > 0$ irrespective of the sign of x. Note that $e^{\pm 2\zeta(x)} A_l^{\pm}(x,y)$ and $e^{\mp 2\zeta(x)} A_r^{\pm}(x,y)$ are functions of $2x+y$ and $-2x+y$, respectively.

When $P(x) \leq 0$ in (0.1), there are notable simplifications. In this special case, $1/T^+(k)$ cannot have any real zeros and hence $\mathbf{S}^+(k)$ exists for all $k \in \mathbf{R}$. We have

$$|T^+(k)|^2 + |L^+(k)|^2 \leq 1, \quad |T^+(k)|^2 + |R^+(k)|^2 \leq 1, \qquad k \in \mathbf{R}.$$

The number of poles of $T^+(k)$ in $\mathbf{C}^+$ is equal to $N(0,Q)$, and each such pole is simple and located on the imaginary axis. Let us order these poles such that $\operatorname{Im} k_j^+ < \operatorname{Im} k_{j+1}^+$. Then the Jost solutions $f_l^+(k_j^+,x)$ and $f_r^+(k_j^+,x)$ each have exactly $N(0,Q)-j$ zeros, and hence from Theorem 10.4 of [AKV97] and (1.1) we conclude that the common sign of $\eta_l^+(k_j^+,-\infty)$ and $\eta_r^+(k_j^+,+\infty)$ is the same as the sign of $(-1)^{N(0,Q)-j}$. Therefore, in this case the quantity $c^+(k_j^+,x)$ is a nonzero real constant, usually called a bound-state norming constant, whose sign agrees with that of $(-1)^{N(0,Q)-j}$.

5. Marchenko equations and the inverse problem

In this section, we derive the two uncoupled Marchenko integral equations (5.14) and show that the corresponding integral operators are compact and have the same nonzero eigenvalues. We also describe the recovery of $P(x)$ and $Q(x)$ from the solutions of these Marchenko equations.

When $k \in \mathbf{R}$, the quantities $f_l^-(-k,x)$ and $f_r^-(-k,x)$ are also solutions of (0.1), and hence, they can be expressed as linear combinations of $f_l^+(k,x)$ and $f_r^+(k,x)$, unless the latter functions are linearly dependent. Using (0.3) and (0.4) we obtain

$$\begin{bmatrix} f_l^{\mp}(-k,x) \\ f_r^{\mp}(-k,x) \end{bmatrix} = \begin{bmatrix} T^{\pm}(k) & -R^{\pm}(k) \\ -L^{\pm}(k) & T^{\pm}(k) \end{bmatrix} \begin{bmatrix} f_r^{\pm}(k,x) \\ f_l^{\pm}(k,x) \end{bmatrix}, \qquad k \in \mathbf{R}. \quad (5.1)$$

Using (1.1), we can write (5.1) in the form

$$\begin{bmatrix} \eta_l^{\mp}(-k,x) \\ \eta_r^{\mp}(-k,x) \end{bmatrix} = \begin{bmatrix} e^{\mp p}\, T^{\pm}(k) & -R^{\pm}(k)\, e^{2ikx \mp 2\zeta(x)} \\ -L^{\pm}(k)\, e^{-2ikx \mp 2p \pm 2\zeta(x)} & e^{\mp p}\, T^{\pm}(k) \end{bmatrix} \begin{bmatrix} \eta_r^{\pm}(k,x) \\ \eta_l^{\pm}(k,x) \end{bmatrix}. \quad (5.2)$$

For $z \in \mathbf{R}$, define

$$\hat{R}^{\pm}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz}\, R^{\pm}(k), \qquad \hat{L}^{\pm}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikz}\, L^{\pm}(k), \tag{5.3}$$

$$\hat{S}_l^{\pm}(z) = \hat{R}^{\pm}(z) - i\sum_{j=1}^{N^{\pm}} e^{ik_j^{\pm}z} \sum_{m=0}^{n_j^{\pm}-1} t_{j,m}^{\pm} \sum_{s=0}^{m} \frac{(iz)^{m-s}}{(m-s)!}\, c_{j,s}^{\pm}, \tag{5.4}$$

$$\hat{S}_r^{\pm}(z) = e^{\mp 2p}\hat{L}^{\pm}(z) - i\sum_{j=1}^{N^{\pm}} e^{ik_j^{\pm}z} \sum_{m=0}^{n_j^{\pm}-1} t_{j,m}^{\pm} \sum_{s=0}^{m} \frac{(iz)^{m-s}}{(m-s)!}\, d_{j,s}^{\pm}. \tag{5.5}$$

Note that $\hat{S}_l^{\pm}(z)$ and $\hat{S}_r^{\pm}(z)$ are real valued because of (0.6) and the real-valuedness of $A_l^{\pm}(x,y)$ and $A_r^{\pm}(x,y)$. Let us write (5.2) in the form

$$\eta_l^{\mp}(-k,x) - 1 = e^{\mp p}\, T^{\pm}(k)\, \eta_r^{\pm}(k,x) - 1 - R^{\pm}(k)\, e^{2ikx \mp 2\zeta(x)}\, \eta_l^{\pm}(k,x), \tag{5.6}$$

$$\eta_r^{\mp}(-k,x) - 1 = e^{\mp p}\, T^{\pm}(k)\, \eta_l^{\pm}(k,x) - 1 - L^{\pm}(k)\, e^{-2ikx \mp 2p \pm 2\zeta(x)}\, \eta_r^{\pm}(k,x). \tag{5.7}$$

With the help of (2.1), (4.12), and (5.3), the Fourier transform of (5.6) gives us for $y > 0$

$$B_l^{\mp}(x,y) = A_l^{\pm}(x,y) - e^{\mp 2\zeta(x)}\left[\hat{R}_l^{\pm}(2x+y) + \int_0^{\infty} dz\, \hat{R}_l^{\pm}(2x+y+z)\, B_l^{\pm}(x,z)\right]. \tag{5.8}$$

Using (4.10) and (4.16) in (5.8), we obtain the coupled Marchenko equations

$$B_l^{\mp}(x,y) = -e^{\mp 2\zeta(x)}\left[\hat{S}_l^{\pm}(2x+y) + \int_0^{\infty} dz\, \hat{S}_l^{\pm}(2x+y+z)\, B_l^{\pm}(x,z)\right], \qquad y > 0. \tag{5.9}$$

In a similar way, applying the Fourier transform to (5.7) we obtain the coupled Marchenko equations

$$B_r^{\mp}(x,y) = -e^{\pm 2\zeta(x)}\left[\hat{S}_r^{\pm}(-2x+y) + \int_0^{\infty} dz\, \hat{S}_r^{\pm}(-2x+y+z)\, B_r^{\pm}(x,z)\right], \qquad y > 0, \tag{5.10}$$

where $\hat{S}_r^{\pm}(x,y)$ and $B_r^{\pm}(x,y)$ are the quantities defined in (5.5) and (2.1), respectively. In (5.9) and (5.10) the coupling refers to the fact that the quantities pertaining to (0.1) and (0.2) appear in the same equation; for example, both $B_l^{+}(x,y)$ related to (0.1) and $B_l^{-}(x,y)$ related to (0.2) appear in the same equation.

We will only analyze the pair of coupled integral equations (5.9). The analysis of (5.10) is similar. Note that $\hat{S}_l^{\pm}(z)$ and $\hat{S}_r^{\pm}(z)$ given in (5.4) and (5.5), respectively, can be constructed uniquely in terms of the scattering data consisting of the reflection coefficients $R^{+}(k)$ and $R^{-}(k)$, the bound-state energies corresponding to the k_j^{+} each with multiplicity

n_j^+ and the k_m^- each with multiplicity n_m^-, and the corresponding bound-state constants $c_{j,s}^+$ and $c_{m,u}^-$, where $j = 1, \cdots, N^+$; $m = 1, \cdots, N^-$; $s = 0, \cdots, n_j^+ - 1$; and $u = 0, \cdots, n_m^- - 1$.

PROPOSITION 5.1 *Let $g \in L^1_\alpha(\mathbf{R}^+)$ for some $\alpha \in [0,1]$. Then, for every $p \in [1, +\infty)$ and for every $\beta \in [0, \alpha]$, the integral operator $\mathcal{O}_g$ defined by*

$$(\mathcal{O}_g f)(x) = \int_0^\infty dy\, g(x+y)\, f(y)$$

is a compact operator on $L^p(\mathbf{R}^+)$ and on $L^1_\beta(\mathbf{R}^+)$. In all of these cases, the operator norm of $\mathcal{O}_g$ is bounded above by $||g||_{1,\beta}$.

PROOF. Since the Fourier transform $\hat{g}(k)$ of $g(x)$ is continuous and vanishes as $k \to +\infty$ in $\mathbf{R}$, the compactness on $L^p(\mathbf{R}^+)$ follows from the Hartman-Wintner theorem on the compactness of Hankel operators (Theorem 1.4 and the discussion following (1.1) of [Po82]; Corollary 4.7 of [Pa88]). Here we give an independent proof. The convolution product of a function in $L^1(\mathbf{R}^+)$ and a function in $L^p(\mathbf{R}^+)$ belongs to $L^p(\mathbf{R}^+)$, and hence the norm of $\mathcal{O}_g$ is bounded above by $||g||_1$. The convolution product of two functions in $L^1_\beta(\mathbf{R}^+)$ is again in $L^1_\beta(\mathbf{R}^+)$, and hence the norm of $\mathcal{O}_g$ is bounded above by $||g||_{1,\beta}$. Approximating g by integrable step functions in the norm of either $L^p(\mathbf{R}^+)$ or $L^1_\alpha(\mathbf{R}^+)$, we approximate $\mathcal{O}_g$ in the operator norm by compact operators, which implies the compactness of $\mathcal{O}_g$. ∎

Let us introduce the integral operators $\mathbf{M}_l^\pm(x)$ and $\mathbf{K}_l^\pm(x)$:

$$[\mathbf{M}_l^\pm(x)\, h](y) = \int_0^\infty dz\, \hat{S}_l^\pm(2x+y+z)\, h(z), \qquad y > 0, \tag{5.11}$$

$$\begin{cases} K_l^\pm(x;y,z) = \displaystyle\int_0^\infty du\, \hat{S}_l^\pm(2x+y+u)\, \hat{S}_l^\mp(2x+u+z), & y,z > 0, \\ [\mathbf{K}_l^\pm(x)\, h](y) = \displaystyle\int_0^\infty dz\, K_l^\pm(x;y,z)\, h(z), & y > 0. \end{cases} \tag{5.12}$$

We may then decouple the system of equations (5.9) to obtain the two uncoupled equations

$$B_l^\mp(x,y) - \int_0^\infty dz\, K_l^\pm(x;y,z) B_l^\mp(x,z) = -e^{\mp 2\zeta(x)} \hat{S}_l^\pm(2x+y) + K_l^\pm(x;y,0), \qquad y > 0.$$

These equations are not convenient for solving the inverse scattering problem, because their right-hand sides contain the unknown quantity $e^{\zeta(x)}$. However, letting

$$B_l^\pm(x,y) = -e^{\pm 2\zeta(x)} a_l^\mp(x,y) + b_l^\mp(x,y), \tag{5.13}$$

we can obtain $B_l^\pm(x,y)$ by using $e^{2\zeta(x)}$ and the solutions of the equations

$$a_l^\pm(x,y) - \int_0^\infty dz\, K_l^\pm(x;y,z)\, a_l^\pm(x,z) = \hat{S}_l^\pm(2x+y), \qquad y > 0, \tag{5.14}$$

$$b_l^{\pm}(x,y) - \int_0^{\infty} dz\, K_l^{\pm}(x;y,z)\, b_l^{\pm}(x,z) = K_l^{\pm}(x;y,0), \qquad y>0, \tag{5.15}$$

where the right-hand sides are now known in terms of the scattering data. Note that the coupled Marchenko equations (5.9) are equivalent to the uncoupled equations (5.14) and (5.15). We will refer to (5.14) and (5.15) as the uncoupled Marchenko equations.

PROPOSITION 5.2 *In the generic case, assume that* $P \in L^1(\mathbf{R})$, $Q \in L_1^1(\mathbf{R})$, $1/T^+(k)$ *and* $1/T^-(k)$ *do not vanish for* $k \in \mathbf{R} \setminus \{0\}$, *and* $W^+, W^- \in L_{1+\alpha}^1(\mathbf{R})$ *for some* $\alpha \geq 0$. *In the exceptional case, assume that* $P, Q \in L_1^1(\mathbf{R})$, $1/T^+(k)$ *and* $1/T^-(k)$ *do not vanish for* $k \in \mathbf{R}$, and $W^+, W^- \in L_{2+\alpha}^1(\mathbf{R})$ *for some* $\alpha \geq 0$. *Then, for each* $x \in \mathbf{R}$, *the integral operators* $\mathbf{M}_l^+(x)$ *and* $\mathbf{M}_l^-(x)$ *defined in* (5.11) *are compact on* $L_\alpha^1(\mathbf{R}^+)$.

PROOF. From Proposition 5.1 and Theorems 3.5 and 3.6, it follows that the operators corresponding to the kernels $\hat{R}^{\pm}(z)$ and $\hat{L}^{\pm}(z)$ are compact on $L_\alpha^1(\mathbf{R}^+)$. Since $\hat{S}_l^{\pm}(z) - \hat{R}^{\pm}(z)$ and $\hat{S}_r^{\pm}(z) - e^{\mp 2p}\hat{L}^{\pm}(z)$ correspond to degenerate kernels, it follows that the integral operators in (5.9) and (5.10) are compact on $L_\alpha^1(\mathbf{R}^+)$. ∎

In the next theorem we show that the Marchenko integral operators in (5.14) and (5.15) are compact perturbations of the identity. Thus, the uncoupled Marchenko integral equations (5.14) and (5.15) are uniquely solvable if the corresponding homogeneous equations do not have any nontrivial solutions.

THEOREM 5.3 *Under the assumptions in Proposition* 5.2, *the kernels* $K_l^+(x;y,z)$ *and* $K_l^-(x;y,z)$ *defined in* (5.12) *are real valued and satisfy*

$$K_l^{\pm}(x;y,z) = K_l^{\mp}(x;z,y), \qquad y,z>0. \tag{5.16}$$

Moreover, the operators $\mathbf{K}_l^+(x)$ and $\mathbf{K}_l^-(x)$ *are compact on* $L_\beta^1(\mathbf{R}^+)$ *for* $\beta \in [0,\alpha]$ *and have the same nonzero eigenvalues, and these eigenvalues are real.*

PROOF. Since $\hat{S}_l^+(z)$ and $\hat{S}_l^-(z)$ are real valued, it follows that $K_l^+(x;y,z)$ and $K_l^-(x;y,z)$ are real, and from (5.12) we get (5.16). The compactness of $\mathbf{M}_l^+(x)$ and $\mathbf{M}_l^-(x)$ on $L_\beta^1(\mathbf{R}^+)$ follows from Propositions 5.1 and 5.2. From (5.11) and (5.12) we have

$$\mathbf{K}_l^{\pm}(x) = \mathbf{M}_l^{\pm}(x)\,\mathbf{M}_l^{\mp}(x), \tag{5.17}$$

and hence $\mathbf{K}_l^+(x)$ and $\mathbf{K}_l^-(x)$ are compact operators on $L_\beta^1(\mathbf{R}^+)$ having the same nonzero eigenvalues. By Theorems 3.5 and 3.6 we have $\hat{R}^+, \hat{R}^- \in L_\alpha^1(\mathbf{R}^+)$; thus, using Proposition 5.1 and an argument as in the proof of Proposition 5.2, we can conclude that $\mathbf{M}^+(x)$ and $\mathbf{M}^-(x)$ and hence $\mathbf{K}^+(x)$ and $\mathbf{K}^-(x)$ are compact operators on both $L_\beta^1(\mathbf{R}^+)$ and $L^2(\mathbf{R}^+)$. Then a simple Fredholm argument implies that the nonzero eigenvalues of $\mathbf{K}^{\pm}(x)$ on $L_\beta^1(\mathbf{R}^+)$ and on $L^2(\mathbf{R}^+)$ are identical. However, as a result of (5.16) and the realness of

$K^+(x;y,z)$ and $K^-(x;y,z)$, it follows that $\mathbf{K}^+(x)$ and $\mathbf{K}^-(x)$ are selfadjoint operators on $L^2(\mathbf{R}^+)$ that are each other's adjoints. Thus the nonzero eigenvalues of $\mathbf{K}^+(x)$ and $\mathbf{K}^-(x)$ are all real. ∎

PROPOSITION 5.4 *Under the assumptions in Proposition* 5.2, *if the integral equations* (5.14) *are uniquely solvable in* $L^1(\mathbf{R}^+)$, *then so are* (5.15).

PROOF. This follows from Theorem 5.3 and the fact that (5.14) and (5.15) are uniquely solvable if and only if 1 is not an eigenvalue of $\mathbf{K}_l^{\pm}(x)$. ∎

From (5.17) we have

$$\mathbf{M}_l^{\mp}(x)\,\mathbf{K}_l^{\pm}(x) = \mathbf{K}_l^{\mp}(x)\,\mathbf{M}_l^{\mp}(x). \tag{5.18}$$

From Proposition 5.2 it follows that $\mathbf{M}_l^{\mp}(x)$ is a bounded operator on $L^1(\mathbf{R}^+)$, and hence $\mathbf{M}_l^{\mp}(x)\,a_l^{\pm}(x,\cdot)$ belongs to $L^1(\mathbf{R}^+)$ whenever $a_l^{\pm}(x,\cdot) \in L^1(\mathbf{R}^+)$. Applying $\mathbf{M}_l^{\mp}(x)$ to (5.14) and using (5.18), we see that $\mathbf{M}_l^{\mp}(x)\,a_l^{\pm}(x,\cdot)$ satisfies (5.15), and hence the unique solution $a_l^{\pm}(x,y)$ of (5.14) leads to the unique solution $b_l^{\pm}(x,y)$ of (5.15) given by

$$b_l^{\pm}(x,y) = [\mathbf{M}_l^{\mp}(x)\,a_l^{\pm}(x,\cdot)](y) = \int_0^{\infty} dz\,\hat{S}_l^{\mp}(2x+y+z)\,a_l^{\pm}(x,z), \qquad y>0. \tag{5.19}$$

Note that (5.14) and (5.15) are equivalent to the linear system of coupled Marchenko equations

$$\begin{cases} a_l^{\pm}(x,y) - \displaystyle\int_0^{\infty} dz\,\hat{S}_l^{\pm}(2x+y+z)\,b_l^{\pm}(x,z) = \hat{S}_l^{\pm}(2x+y), & y>0, \\ b_l^{\pm}(x,y) - \displaystyle\int_0^{\infty} dz\,\hat{S}_l^{\mp}(2x+y+z)\,a_l^{\pm}(x,z) = 0, & y>0. \end{cases} \tag{5.20}$$

Under the general assumptions of this section, we have $\hat{S}_l^{+}, \hat{S}_l^{-} \in L_1^1(\mathbf{R})$. This means that the integral terms in (5.20) are continuous in $y \in (0,+\infty)$ whenever $a_l^{\pm}(x,\cdot)$ and $b_l^{\pm}(x,\cdot)$ belong to $L^1(\mathbf{R}^+)$. Thus, from (5.20) we see that for each $x \in \mathbf{R}$, the discontinuities of $a_l^{\pm}(x,y)$ and $\hat{S}_l^{\pm}(2x+y)$ coincide for $y>0$ and that $b_l^{\pm}(x,y)$ is continuous in $y>0$.

The functions $a_l^{\pm}(x,y)$ and $b_l^{\pm}(x,y)$ appear as elements of the 2×2 resolvent kernel matrix $\Gamma(x;y,z)$ of the linear system of integral equations

$$\begin{cases} c_l(x,y) - \displaystyle\int_0^{\infty} dz\,\hat{S}_l^{+}(2x+y+z)\,d_l(x,z) = \mathrm{RHS}_1(y), & y>0, \\ d_l(x,y) - \displaystyle\int_0^{\infty} dz\,\hat{S}_l^{-}(2x+y+z)\,c_l(x,z) = \mathrm{RHS}_2(y), & y>0, \end{cases} \tag{5.21}$$

where $\mathrm{RHS}_1(y)$ and $\mathrm{RHS}_2(y)$ denote nonhomogeneous terms. In fact, if (5.20) are uniquely solvable, then the unique solution of (5.21) is given by

$$\begin{bmatrix} c_l(x,y) \\ d_l(x,y) \end{bmatrix} = \begin{bmatrix} \mathrm{RHS}_1(y) \\ \mathrm{RHS}_2(y) \end{bmatrix} + \int_0^{\infty} dz\,\Gamma(x;y,z) \begin{bmatrix} \mathrm{RHS}_1(z) \\ \mathrm{RHS}_2(z) \end{bmatrix}. \tag{5.22}$$

Then (5.20) implies

$$\Gamma(x;y,0) = \begin{bmatrix} b_l^-(x,y) & a_l^+(x,y) \\ a_l^-(x,y) & b_l^+(x,y) \end{bmatrix}. \tag{5.23}$$

Indeed, using the short-hand notations $\mathbf{N}(x)$ and $\Gamma(x)$ for the integral operators on the half-line with respective kernels $\begin{bmatrix} 0 & \hat{S}_l^+(2x+y+z) \\ \hat{S}_l^-(2x+y+z) & 0 \end{bmatrix}$ and $\Gamma(x;y,z)$, one gets from (5.21) and (5.22)

$$[\mathbf{I} - \mathbf{N}(x)]^{-1} = \mathbf{I} + \Gamma(x),$$

and hence

$$[\mathbf{I} - \mathbf{N}(x)]\,\Gamma(x) = \mathbf{N}(x). \tag{5.24}$$

Let $[M]_{jm}$ stand for the (j,m)-entry of a matrix M. Rewriting (5.24) in the form

$$\begin{cases} [\Gamma(x;y,w)]_{1j} - \displaystyle\int_0^\infty dz\, \hat{S}_l^+(2x+y+z)\, [\Gamma(x;z,w)]_{2j} = [\mathbf{N}(2x+y+w)]_{1j}\,, \\ [\Gamma(x;y,w)]_{2j} - \displaystyle\int_0^\infty dz\, \hat{S}_l^-(2x+y+z)\, [\Gamma(x;z,w)]_{1j} = [\mathbf{N}(2x+y+w)]_{2j}\,, \end{cases}$$

where $j = 1,2$, $y > 0$, $[\mathbf{N}(2x+y+w)]_{11} = [\mathbf{N}(2x+y+w)]_{22} = 0$, $[\mathbf{N}(2x+y+w)]_{12} = \hat{S}_l^+(2x+y+w)$, and $[\mathbf{N}(2x+y+w)]_{21} = \hat{S}_l^-(2x+y+w)$, and comparing the latter systems for $w = 0$ with (5.20), we obtain (5.23). Finally, we remark that the resolvent kernels $\Gamma^+(x;y,z)$ and $\Gamma^-(x;y,z)$ of the integral equations (5.14) are given by $[\Gamma(x;y,z)]_{11}$ and $[\Gamma(x;y,z)]_{22}$, respectively. Moreover,

$$\Gamma(x;y,z) = \begin{bmatrix} \Gamma^+(x;y,z) & \displaystyle\int_0^\infty du\, \hat{S}_l^+(2x+y+u)\, \Gamma^-(x;u,z) \\ \displaystyle\int_0^\infty du\, \hat{S}_l^-(2x+y+u)\, \Gamma^+(x;u,z) & \Gamma^-(x;y,z) \end{bmatrix}.$$

In the next theorem we show how the unique solutions of the pair of uncoupled equations (5.14) lead to the solution of the inverse scattering problem.

THEOREM 5.5 *Suppose that, for each $x \in \mathbf{R}$, the two integral equations* (5.14) *have unique solutions $a_l^+(x,y)$ and $a_l^-(x,y)$ belonging to $L^1(\mathbf{R}^+)$. Then $e^{2\zeta(x)}$, $P(x)$, and $Q(x)$ can be obtained from $a_l^+(x,y)$ and $a_l^-(x,y)$ as follows:*

$$e^{2\zeta(x)} = \frac{1 + < a_l^+(x,\cdot) > + < b_l^-(x,\cdot) >}{1 + < a_l^-(x,\cdot) > + < b_l^+(x,\cdot) >}, \tag{5.25}$$

where $b_l^\pm(x,y)$ is the quantity in (5.19) *and $< f >= \int_0^\infty dy\, f(y)$,*

$$P(x) = -2\zeta'(x) = -\frac{1}{e^{2\zeta(x)}} \frac{de^{2\zeta(x)}}{dx}, \tag{5.26}$$

$$Q(x) = 2\frac{d}{dx}\left[e^{\pm 2\zeta(x)}\, a_l^{\mp}(x,0+) - b_l^{\mp}(x,0+)\right] \pm \frac{1}{2}P'(x) + \frac{1}{4}P(x)^2. \tag{5.27}$$

PROOF. From (0.11), (1.1), and (2.2) it follows that

$$e^{2\zeta(x)} = \frac{e^{\zeta(x)} f_l(0,x)}{e^{-\zeta(x)} f_l(0,x)} = \frac{\eta_l^+(0,x)}{\eta_l^-(0,x)} = \frac{1+ < B_l^+(x,\cdot) >}{1+ < B_l^-(x,\cdot) >}. \tag{5.28}$$

Furthermore, from (5.13) we have

$$< B_l^{\pm}(x,\cdot) >= -e^{\pm 2\zeta(x)} < a_l^{\mp}(x,\cdot) > + < b_l^{\mp}(x,\cdot) > . \tag{5.29}$$

Eliminating $< B_l^{\pm}(x,\cdot) >$ from (5.28) and (5.29), we get (5.25). Next, from (2.3) we get (5.26), and from (1.6), (2.14), and (5.29) we get (5.27). ∎

In order for the inversion algorithm contained in Theorem 5.5 to lead to real potentials $P \in L^1(\mathbf{R})$ and $Q \in L_1^1(\mathbf{R})$, the right-hand side of (5.25) should be positive for every $x \in \mathbf{R}$, the derivative in (5.26) should exist for all but finitely many $x \in \mathbf{R}$, and the derivative in (5.27) should exist almost everywhere. Even then, restrictions on the scattering data are usually necessary to assure that the potentials $P(x)$ and $Q(x)$ thus obtained satisfy the general conditions of this article.

6. UNIQUE SOLVABILITY OF THE MARCHENKO EQUATIONS

In this section we present some conditions on the scattering data for the unique solvability of the Marchenko integral equations (5.14). The assumptions for the results in this section to hold are the same as those stated in Proposition 5.2. That is, in the generic case, we assume that $P \in L^1(\mathbf{R})$, $Q \in L_1^1(\mathbf{R})$, $W^+, W^- \in L_{1+\alpha}^1(\mathbf{R})$ for some $\alpha \geq 0$, and $1/T^+(k)$ and $1/T^-(k)$ do not vanish for $k \in \mathbf{R} \setminus \{0\}$; in the exceptional case, we assume that $P, Q \in L_1^1(\mathbf{R})$, $W^+, W^- \in L_{2+\alpha}^1(\mathbf{R})$ for some $\alpha \geq 0$, and $1/T^+(k)$ and $1/T^-(k)$ do not vanish for $k \in \mathbf{R}$. Under these conditions, as we have seen in Theorem 5.3, the Marchenko integral operators are compact perturbations of the identity acting on $L_\beta^1(\mathbf{R}^+)$ for any $\beta \in [0,\alpha]$, and hence the unique solvability of the two uncoupled Marchenko equations (5.14) follows from the nonexistence of nontrivial solutions of the corresponding homogeneous equations.

Using (5.4) let us define

$$S_l^{\pm}(k) = \int_{-\infty}^{\infty} dz\, e^{-ikz} \hat{S}_l^{\pm}(z) = R^{\pm}(k) - \sum_{j=1}^{N^{\pm}} \sum_{m=0}^{n_j^{\pm}-1} t_{j,m}^{\pm} \sum_{s=0}^{m} \frac{c_{j,s}^{\pm}}{(k-k_j^{\pm})^{m-s+1}}. \tag{6.1}$$

Because $\hat{S}_l^{\pm}(y)$ is real valued, we have the symmetry relation

$$\overline{S_l^{\pm}(k)} = S_l^{\pm}(-k), \qquad k \in \mathbf{R}. \tag{6.2}$$

The Banach space $\mathcal{W}_\beta^0$ of Fourier transforms of functions in $L_\beta^1(\mathbf{R})$ can be decomposed in a natural way as $\mathcal{W}_\beta^0 = \mathcal{W}_\beta^{0,+} \oplus \mathcal{W}_\beta^{0,-}$, where $\mathcal{W}_\beta^{0,\pm}$ denotes the Banach space of Fourier transforms of functions in $L_\beta^1(\mathbf{R}^\pm)$. The projection $\Pi_\pm$ of $\mathcal{W}_\beta^0$ onto $\mathcal{W}_\beta^{0,\pm}$ is bounded with unit norm if we endow this space with the norm $\|\hat{h}\|_{\mathcal{W}_\beta^0} := \|h\|_{1,\beta}$. We write

$$\hat{h}_\pm = \Pi_\pm \hat{h}, \qquad \hat{h} \in \mathcal{W}_\beta^0. \tag{6.3}$$

In our analysis of the uncoupled Marchenko equations (5.14) and (5.15), h will be one of $a^\pm(x,\cdot)$ and $b^\pm(x,\cdot)$ extended to the full line.

Let $Z(k)$ be a continuous matrix function for $k \in \mathbf{R}$ tending to the identity as $|k| \to +\infty$ such that its entries belong to $\mathcal{W}_\beta$ for some $\beta \geq 0$. Then by a *right canonical factorization* of $Z(k)$ we mean a representation of $Z(k)$ in the form $Z(k) = Z_+(k)\, Z_-(k)$, such that for $j, m \in \{1,2\}$ we have

$$[Z_\pm(\cdot)]_{jm} - \delta_{jm} \in \mathcal{W}_\beta^{0,\pm}, \qquad [Z_\pm(\cdot)^{-1}]_{jm} - \delta_{jm} \in \mathcal{W}_\beta^{0,\pm}. \tag{6.4}$$

By a *left canonical factorization* of $Z(k)$ we mean a representation of $Z(k)$ in the form $Z(k) = Z_-(k)\, Z_+(k)$, where $Z_\pm(k)$ satisfies (6.4). Right and left canonical factorizations are unique when they exist.

The next theorem will be proved by converting (5.20) into a Riemann-Hilbert problem whose unique solvability can be reduced to the existence of a canonical Wiener-Hopf factorization following a procedure given in [Fe61,FGK94].

THEOREM 6.1 *Under the assumptions of Proposition* 5.2, *the scalar* 1 *is not an eigenvalue of the integral operator* $\mathbf{K}_l^\pm(x)$ *defined in* (5.12) *if and only if the matrix function* $Z^\pm(\cdot,x)$ *defined by*

$$Z^\pm(k,x) = \begin{bmatrix} 1 & S_l^\mp(k)\, e^{2ikx} \\ -S_l^\pm(-k)\, e^{-2ikx} & 1 - S_l^\pm(-k)\, S_l^\mp(k) \end{bmatrix} \tag{6.5}$$

has both a right canonical factorization and a left canonical factorization.

PROOF. Note that (5.20) is valid only when $y > 0$. Let us extend (5.20) to $y \in \mathbf{R}$ by letting

$$a_l^\pm(x,y) = a_+^\pm(x,y) + a_-^\pm(x,y), \qquad b_l^\pm(x,y) = b_+^\pm(x,y) + b_-^\pm(x,y),$$

where $a_+^\pm(x,y) = b_+^\pm(x,y) = 0$ for $y < 0$ and $a_-^\pm(x,y) = b_-^\pm(x,y) = 0$ for $y > 0$. Thus, we have

$$\begin{cases} a_+^\pm(x,y) + a_-^\pm(x,y) - \displaystyle\int_{-\infty}^{\infty} dz\, \hat{S}_l^\pm(2x+y+z)\, b_+^\pm(x,z) = \hat{S}_l^\pm(2x+y), & y \in \mathbf{R}, \\ b_+^\pm(x,y) + b_-^\pm(x,y) - \displaystyle\int_{-\infty}^{\infty} dz\, \hat{S}_l^\mp(2x+y+z)\, a_+^\pm(x,z) = 0, & y \in \mathbf{R}. \end{cases} \tag{6.6}$$

Let us define

$$\hat{a}_{\pm}^{\pm}(k,x) = \Pi_{\pm}\left[\int_{-\infty}^{\infty} dy\, a_l^{\pm}(x,y)\, e^{iky}\right] = \pm\int_0^{\pm\infty} dy\, a_l^{\pm}(x,y)\, e^{iky}, \tag{6.7}$$

$$\hat{b}_{\pm}^{\pm}(k,x) = \Pi_{\pm}\left[\int_{-\infty}^{\infty} dy\, b_l^{\pm}(x,y)\, e^{iky}\right] = \pm\int_0^{\pm\infty} dy\, b_l^{\pm}(x,y)\, e^{iky}, \tag{6.8}$$

$$\hat{a}_{\mp}^{\pm}(k,x) = \Pi_{\mp}\left[\int_{-\infty}^{\infty} dy\, a_l^{\pm}(x,y)\, e^{iky}\right] = \mp\int_0^{\mp\infty} dy\, a_l^{\pm}(x,y)\, e^{iky}, \tag{6.9}$$

$$\hat{b}_{\mp}^{\pm}(k,x) = \Pi_{\mp}\left[\int_{-\infty}^{\infty} dy\, b_l^{\pm}(x,y)\, e^{iky}\right] = \mp\int_0^{\mp\infty} dy\, b_l^{\pm}(x,y)\, e^{iky}, \tag{6.10}$$

where Π_+ and Π_- are the projections given in (6.3). Applying the Fourier transform to (6.6) and using (6.7)-(6.10) we obtain

$$\begin{cases} \hat{a}_+^{\pm}(k,x) + \hat{a}_-^{\pm}(k,x) - S_l^{\pm}(-k)\, e^{-2ikx}\, b_+^{\pm}(-k,x) = S_l^{\pm}(-k)\, e^{-2ikx}, \\ \hat{b}_+^{\pm}(k,x) + \hat{b}_-^{\pm}(k,x) - S_l^{\mp}(-k)\, e^{-2ikx}\, \hat{a}_+^{\pm}(-k,x) = 0. \end{cases} \tag{6.11}$$

Substituting $-k$ for k in (6.11) we get

$$\begin{cases} \hat{a}_+^{\pm}(-k,x) + \hat{a}_-^{\pm}(-k,x) - S_l^{\pm}(k)\, e^{2ikx}\, b_+^{\pm}(k,x) = S_l^{\pm}(k)\, e^{2ikx}, \\ \hat{b}_+^{\pm}(-k,x) + \hat{b}_-^{\pm}(-k,x) - S_l^{\mp}(k)\, e^{2ikx}\, \hat{a}_+^{\pm}(k,x) = 0. \end{cases} \tag{6.12}$$

Letting

$$X^{\pm}(k,x) = \begin{bmatrix} \hat{a}_+^{\pm}(k,x) \\ \hat{b}_+^{\pm}(k,x) \\ \hat{a}_-^{\pm}(-k,x) \\ \hat{b}_-^{\pm}(-k,x) \end{bmatrix}, \qquad Y^{\pm}(k,x) = \begin{bmatrix} S_l^{\pm}(k)\, e^{2ikx} \\ 0 \\ S_l^{\pm}(-k)\, e^{-2ikx} \\ 0 \end{bmatrix},$$

$$\mathcal{L}^{\pm}(k,x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -S_l^{\pm}(-k)\, e^{-2ikx} & 1 & 0 \\ -S_l^{\mp}(-k)\, e^{-2ikx} & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{R}^{\pm}(k,x) = \begin{bmatrix} 0 & -S_l^{\pm}(k)\, e^{2ikx} & 1 & 0 \\ -S_l^{\mp}(k)\, e^{2ikx} & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

we can write (6.11) and (6.12) together as the single system

$$\mathcal{L}^{\pm}(k,x)\, X^{\pm}(-k,x) + \mathcal{R}^{\pm}(k,x)\, X^{\pm}(k,x) = Y^{\pm}(k,x), \qquad k \in \mathbf{R}. \tag{6.13}$$

From Theorem 5.3 it follows that, for each fixed $x \in \mathbf{R}$, the entries of the vector function $X^{\pm}(\cdot, x)$ belong to $\mathcal{W}_{\beta}^{0,+}$, where $\beta \in [0, \alpha]$. Thus, finding $X^{\pm}(k, x)$ in terms of the scattering data constitutes a Riemann-Hilbert problem. Let us define

$$M_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that

$$M_1 \, \mathcal{L}^{\pm}(k, x)^{-1} \, \mathcal{R}^{\pm}(k, x) \, M_2 = Z^{\pm}(k, x) \oplus Z^{\pm}(k, x)^*, \tag{6.14}$$

where $Z^{\pm}(k, x)^*$ denotes the conjugate transpose of $Z^{\pm}(k, x)$. Multiplying (6.13) on the left by $M_1 \, \mathcal{L}^{\pm}(k, x)^{-1}$ and using (6.14), we can write (6.13) as the pair of Riemann-Hilbert problems

$$\begin{bmatrix} -\hat{b}_{+}^{\pm}(-k, x) \\ \hat{a}_{-}^{\pm}(k, x) \end{bmatrix} + Z^{\pm}(k, x) \begin{bmatrix} -\hat{b}_{-}^{\pm}(-k, x) \\ \hat{a}_{+}^{\pm}(k, x) \end{bmatrix} = \begin{bmatrix} 0 \\ S_l^{\pm}(-k) e^{-2ikx} \end{bmatrix}, \qquad k \in \mathbf{R}, \tag{6.15}$$

$$\begin{bmatrix} \hat{a}_{+}^{\pm}(-k, x) \\ \hat{b}_{-}^{\pm}(k, x) \end{bmatrix} + Z^{\pm}(k, x)^* \begin{bmatrix} \hat{a}_{-}^{\pm}(-k, x) \\ \hat{b}_{+}^{\pm}(k, x) \end{bmatrix} = \begin{bmatrix} S_l^{\pm}(k) \, e^{2ikx} \\ S_l^{\mp}(-k) \, S_l^{\pm}(k) \end{bmatrix}, \qquad k \in \mathbf{R}. \tag{6.16}$$

It is known [GF71] that the two problems described by (6.15) and (6.16) are uniquely solvable if $Z^{\pm}(k, x)$ and $Z^{\pm}(k, x)^*$ both have a left canonical factorization, or equivalently, if $Z^{\pm}(k, x)$ has both right and left canonical factorizations. Thus, we have shown that (5.20) is uniquely solvable if $Z^{\pm}(k, x)$ has both right and left canonical factorizations.

We will next prove that if (5.20) is uniquely solvable, then $Z^{\pm}(k, x)$ has both right and left canonical factorizations. Let us replace (6.6) with the system

$$\begin{cases} a_{+}^{\pm}(x, y) + a_{-}^{\pm}(x, y) - \displaystyle\int_{-\infty}^{\infty} dz \, \hat{S}_l^{\pm}(2x + y + z) \, b_{+}^{\pm}(x, z) = \hat{r}_1^{\pm}(y), & y \in \mathbf{R}, \\ b_{+}^{\pm}(x, y) + b_{-}^{\pm}(x, y) - \displaystyle\int_{-\infty}^{\infty} dz \, \hat{S}_l^{\mp}(2x + y + z) \, a_{+}^{\pm}(x, z) = \hat{r}_2^{\pm}(y), & y \in \mathbf{R}, \end{cases} \tag{6.17}$$

where $\hat{r}_1^{\pm}, \hat{r}_2^{\pm} \in L_{\beta}^1(\mathbf{R})$. Writing $r_j^{\pm}(k) = \int_{-\infty}^{\infty} dy \, e^{-iky} \hat{r}_j(y)$ for $j = 1, 2$ and following the above procedure, instead of (6.15) and (6.16) we get for $k \in \mathbf{R}$

$$\begin{bmatrix} -\hat{b}_{+}^{\pm}(-k, x) \\ \hat{a}_{-}^{\pm}(k, x) \end{bmatrix} + Z^{\pm}(k, x) \begin{bmatrix} -\hat{b}_{-}^{\pm}(-k, x) \\ \hat{a}_{+}^{\pm}(k, x) \end{bmatrix} = \begin{bmatrix} -r_2^{\pm}(-k) \\ r_1^{\pm}(k) + S_l^{\pm}(-k) \, e^{-2ikx} \, r_2^{\pm}(-k) \end{bmatrix}, \tag{6.18}$$

$$\begin{bmatrix} \hat{a}_{+}^{\pm}(-k, x) \\ \hat{b}_{-}^{\pm}(k, x) \end{bmatrix} + Z^{\pm}(k, x)^* \begin{bmatrix} \hat{a}_{-}^{\pm}(-k, x) \\ \hat{b}_{+}^{\pm}(k, x) \end{bmatrix} = \begin{bmatrix} r_1^{\pm}(-k) \\ r_2^{\pm}(k) + S_l^{\mp}(-k) \, e^{-2ikx} r_1^{\pm}(-k) \end{bmatrix}. \tag{6.19}$$

Analogously, replacing (6.17) by the linear system

$$\begin{cases} a_+^{\pm}(x,y) + a_-^{\pm}(x,y) + \displaystyle\int_{-\infty}^{\infty} dz\, \hat{S}_l^{\pm}(2x+y+z)\, b_+^{\pm}(x,z) = \hat{r}_3^{\pm}(y), & y \in \mathbf{R}, \\ b_+^{\pm}(x,y) + b_-^{\pm}(x,y) + \displaystyle\int_{-\infty}^{\infty} dz\, \hat{S}_l^{\mp}(2x+y+z)\, a_+^{\pm}(x,z) = \hat{r}_4^{\pm}(y), & y \in \mathbf{R}, \end{cases} \tag{6.20}$$

where $\hat{r}_3^{\pm}, \hat{r}_4^{\pm} \in L_\beta^1(\mathbf{R})$, we get instead of (6.18) and (6.19) for $k \in \mathbf{R}$

$$\begin{bmatrix} \hat{b}_+^{\pm}(-k,x) \\ \hat{a}_-^{\pm}(k,x) \end{bmatrix} + Z^{\pm}(k,x) \begin{bmatrix} \hat{b}_-^{\pm}(-k,x) \\ \hat{a}_+^{\pm}(k,x) \end{bmatrix} = \begin{bmatrix} r_4^{\pm}(-k) \\ r_3^{\pm}(k) - S_l^{\pm}(-k)\, e^{-2ikx}\, r_4^{\pm}(-k) \end{bmatrix},$$

$$\begin{bmatrix} \hat{a}_+^{\pm}(-k,x) \\ -\hat{b}_-^{\pm}(k,x) \end{bmatrix} + Z^{\pm}(k,x)^* \begin{bmatrix} \hat{a}_-^{\pm}(-k,x) \\ -\hat{b}_+^{\pm}(k,x) \end{bmatrix} = \begin{bmatrix} r_3^{\pm}(-k) \\ -r_4^{\pm}(k) + S_l^{\mp}(-k)\, e^{-2ikx}\, r_3^{\pm}(-k) \end{bmatrix}.$$

The systems (6.17) and (6.20) are uniquely solvable if and only if $Z^{\pm}(k,x)$ has both right and left canonical factorizations. Thus, we see that the existence of both right and left canonical factorizations of $Z^{\pm}(k,x)$ is equivalent to $+1$ and -1 not being eigenvalues of the linear operators [cf. (5.11)]

$$\begin{bmatrix} 0 & \mathbf{M}_l^{\pm}(x) \\ \mathbf{M}_l^{\mp}(x) & 0 \end{bmatrix} \tag{6.21}$$

defined on $L_\beta^1(\mathbf{R}^+) \oplus L_\beta^1(\mathbf{R}^+)$. From (5.11) and (5.12) we have

$$\begin{bmatrix} \mathbf{I} - \lambda^2 \mathbf{K}_l^{\pm}(x) & 0 \\ 0 & \mathbf{I} - \lambda^2 \mathbf{K}_l^{\mp}(x) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \lambda \mathbf{M}_l^{\pm}(x) \\ \lambda \mathbf{M}_l^{\mp}(x) & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\lambda \mathbf{M}_l^{\pm}(x) \\ -\lambda \mathbf{M}_l^{\mp}(x) & \mathbf{I} \end{bmatrix}. \tag{6.22}$$

Note that none of the three matrices in (6.22) can be one-sided invertible unless they are two-sided invertible. Hence, from (6.22) we see that the nonzero eigenvalues of $\mathbf{K}_l^{\pm}(x)$, which coincide with the nonzero eigenvalues of $\mathbf{K}_l^{\mp}(x)$, are exactly the squares of the nonzero eigenvalues of the two operators defined by (6.21). Hence the existence of both right and left canonical factorizations of $Z^{\pm}(k,x)$ is equivalent to 1 not being an eigenvalue of $\mathbf{K}_l^{\pm}(x)$. ∎

PROPOSITION 6.2 *Suppose*

$$\sup_{k \in \mathbf{R}} |S_l^+(k) + S_l^-(k)| < 2, \tag{6.23}$$

where $S_l^+(k)$ and $S_l^-(k)$ are the quantities defined in (6.1). *Then the matrix function $Z^{\pm}(k,x)$ given in* (6.5) *has both right and left canonical factorizations.*

PROOF. Let

$$C^{\pm}(k,x,\varepsilon) = \frac{1}{2}\left[Z^{\pm}(k,x) + Z^{\pm}(k,x)^*\right] - \varepsilon \mathbf{I},$$

where ε is a positive parameter. From Proposition III 1.2 and Theorem II 6.3 in [CG81], it follows that the matrix $Z^{\pm}(k,x)$ has both right and left canonical factorizations if and only if for some small ε the matrix $C^{\pm}(k,x,\varepsilon)$ is selfadjoint and nonnegative for all $k \in \mathbf{R}$. Note that $C^{\pm}(k,x,\varepsilon)$ is selfadjoint for any real ε. To show that $C^{\pm}(k,x,\varepsilon)$ is nonnegative, it is enough to show that the determinant and the trace of $C^{\pm}(k,x,0)$ are positive. Using (6.2), from (6.5) we get

$$\det C^{\pm}(k,x,0) = 1 - \frac{1}{4}|S_l^{\pm}(k) + S_l^{\mp}(k)|^2, \tag{6.24}$$

$$\operatorname{tr} C^{\pm}(k,x,0) = 1 + \frac{1}{4}|S_l^{\pm}(k) - S_l^{\mp}(k)|^2 + \det C^{\pm}(k,x,0). \tag{6.25}$$

From (6.2), (6.24), and (6.25) we see that $C^{\pm}(k,x,0)$ has positive trace and positive determinant if and only if $|S_l^{\pm}(k) + S_l^{\mp}(k)| < 2$ for $k \in \mathbf{R}$. Hence, if (6.23) is satisfied, then $C^{\pm}(k,x,\varepsilon)$ is selfadjoint and nonnegative for sufficiently small positive ε. ∎

From Theorem 6.1 and Proposition 6.2 we have the following.

COROLLARY 6.3 *Assume that* (6.23) *is satisfied. Then, under the assumptions of Proposition* 5.2, *the uncoupled Marchenko equations* (5.14) *and* (5.15) *are uniquely solvable in* $L^1_\beta(\mathbf{R}^+)$ *for* $\beta \in [0,\alpha]$.

Consider the function

$$\left(\frac{k^2+1}{k^2}\right)^d \left[1 - R^+(k)\,R^-(-k)\right], \qquad k \in \mathbf{R}, \tag{6.26}$$

where $d = 0$ in the exceptional case and $d = 1$ in the generic case. Let w denote the winding number of the graph of the function in (6.26) as k varies from $-\infty$ to $+\infty$ on the real axis.

THEOREM 6.4 *Suppose* $R^+(k)$ *and* $R^-(k)$ *are continuous in* $k \in \mathbf{R}$. *Then, the winding number* w *of the graph of the function defined in* (6.26) *is given by*

$$w = N(-P,Q) - N(P,Q), \tag{6.27}$$

where $N(P,Q)$ *and* $N(-P,Q)$ *are the number of bound states of* (0.1) *and* (0.2), *respectively.*

PROOF. Using (0.9) and (4.6) we have

$$1 - R^+(k)R^-(-k) = \left[T_0^+(k)\prod_{j=1}^{N^+}\left(\frac{k-\overline{k_j^+}}{k-k_j^+}\right)^{n_j^+}\right]\left[T_0^-(-k)\prod_{j=1}^{N^-}\left(\frac{k+k_j^-}{k+\overline{k_j^-}}\right)^{n_j^-}\right], \qquad k \in \mathbf{R},$$

where $[(k+i)/k]^d T_0^{\pm}(k)$ are continuous in $k \in \overline{\mathbf{C}^+}$, are analytic in $k \in \mathbf{C}^+$, do not vanish in $\overline{\mathbf{C}^+}$, and have nonzero limits as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Thus the winding number w of (6.26) is equal to $\sum_{j=1}^{N^-} n_j^- - \sum_{j=1}^{N^+} n_j^+$. Note that $N(\pm P, Q)$ coincides with the number of poles of $T^{\pm}(k)$ in $\mathbf{C}^+$ including multiplicities, and hence $N(\pm P, Q) = \sum_{j=1}^{N^{\pm}} n_j^{\pm}$. Thus, (6.27) holds. ∎

From Theorem 6.4 we see that (0.1) and (0.2) must have the same number of bound states if the winding number of the function in (6.26) is zero. Conversely, if neither (0.1) nor (0.2) have any bound states, the winding number must be zero.

7. Examples

In this section we present some examples illustrating the recovery of $P(x)$ and $Q(x)$ by the Marchenko method outlined in Section 5.

Example 7.1 Consider the inverse scattering problem with no bound states, where $R^-(k) = 0$ and $R^+(k)$ satisfies some mild technical conditions that will be apparent as we proceed. From (4.5) we get $T^{\pm}(k) = e^{\pm p}$. Thus we are in the exceptional case. Using (5.12) we see that $K_l^{\pm}(x; y, z) = 0$. Solving (5.14) and using (5.19) we get

$$a_l^+(x,y) = \hat{R}^+(2x+y), \qquad a_l^-(x,y) = b_l^+(x,y) = b_l^-(x,y) = 0,$$

so that

$$\langle a_l^+(x,\cdot)\rangle = \int_{2x}^{\infty} dy\, \hat{R}^+(y), \qquad \langle a_l^-(x,\cdot)\rangle = \langle b_l^+(x,\cdot)\rangle = \langle b_l^-(x,\cdot)\rangle = 0.$$

Then from (5.25) we get $e^{2\zeta(x)} = 1 + \int_{2x}^{\infty} dy\, \hat{R}^+(y)$, from which we see that, in order to have $P \in L^1(\mathbf{R})$, our scattering data need to satisfy $\inf_{x\in\mathbf{R}} \int_{2x}^{\infty} dy\, \hat{R}^+(y) > -1$. When this condition is satisfied, as outlined in Theorem 5.5, we obtain

$$P(x) = \frac{2\hat{R}^+(2x)}{1 + \int_{2x}^{\infty} dy\, \hat{R}^+(y)}, \qquad Q(x) = \frac{2\hat{R}^{+\prime}(2x)}{1 + \int_{2x}^{\infty} dy\, \hat{R}^+(y)} + \frac{3\,\hat{R}^+(2x)^2}{\left[1 + \int_{2x}^{\infty} dy\, \hat{R}^+(y)\right]^2}.$$

Note that if we further require that $\hat{R}^+ \in L^1(\mathbf{R})$ and $(\hat{R}^+)^2, \hat{R}^{+\prime} \in L_1^1(\mathbf{R})$, then we have $P \in L^1(\mathbf{R})$ and $Q \in L_1^1(\mathbf{R})$.

Example 7.2 Let us consider the inverse scattering problem with the scattering data consisting of the reflection coefficients

$$R^+(k) = \frac{ic\,(k+i\alpha)}{(k-i\beta)(k+i\varepsilon)}, \qquad R^-(k) = \frac{i(\varepsilon-\alpha)(\beta+\varepsilon)(k+i\beta)}{c(k-i\alpha)[k+i(\beta-\alpha+\varepsilon)]},$$

where c is a nonzero real constant and $\alpha, \beta, \varepsilon$ are positive parameters such that $\beta - \alpha + \varepsilon > 0$. Let us also assume that neither (0.1) nor (0.2) have any bound states. We construct $e^{\mp p} T^{\pm}(k)$ by using (4.5) and obtain

$$e^{-p} T^{+}(k) = \frac{k + i\alpha}{k + i\varepsilon}, \qquad e^{p} T^{-}(k) = \frac{k + i\beta}{k + i(\beta - \alpha + \varepsilon)}.$$

In fact, using (0.8) we also construct $e^{\mp 2p} L^{\pm}(k)$ and get

$$e^{-2p} L^{+}(k) = \frac{i(\varepsilon - \alpha)(\beta + \varepsilon)}{c\,(k + i\varepsilon)}, \qquad e^{2p} L^{-}(k) = \frac{ic}{k + i(\beta - \alpha + \varepsilon)}. \tag{7.1}$$

From (7.1) we see that $e^{\mp 2p} L^{\pm}(k)$ belongs to the Hardy space $\mathbf{H}^2_+(\mathbf{R})$, and hence from (5.10) we conclude that $B^{\pm}_r(x, y) = 0$ for $x < 0$, and thus $P(x) = Q(x) = 0$ for $x < 0$. Now let us find $P(x)$ and $Q(x)$ for $x > 0$. We will do so by using the procedure outlined in Theorem 5.5. With the help of (5.3) and (5.4) we obtain

$$\hat{S}^{+}_l(z) = \hat{R}^{+}(z) = -\frac{c(\alpha + \beta)}{\beta + \varepsilon} e^{-\beta z}, \qquad z > 0, \tag{7.2}$$

$$\hat{S}^{-}_l(z) = \hat{R}^{-}(z) = -c^{-1}(\alpha + \beta)(\varepsilon - \alpha)\, e^{-\alpha z}, \qquad z > 0. \tag{7.3}$$

Using (7.2) and (7.3) in (5.12) we get

$$K^{+}_l(x; y, z) = \frac{(\alpha + \beta)(\varepsilon - \alpha)}{\beta + \varepsilon} e^{-2(\alpha + \beta)x} e^{-\beta y - \alpha z}, \qquad y, z > 0,$$

$$K^{-}_l(x; y, z) = \frac{(\alpha + \beta)(\varepsilon - \alpha)}{\beta + \varepsilon} e^{-2(\alpha + \beta)x} e^{-\alpha y - \beta z}, \qquad y, z > 0.$$

Thus we have the two uncoupled Marchenko equations (5.14) for $x > 0$ given by

$$\begin{aligned} a^{+}_l(x, y) - \frac{(\alpha + \beta)(\varepsilon - \alpha)}{\beta + \varepsilon} e^{-2(\alpha + \beta)x - \beta y} \int_0^{\infty} dz\, e^{-\alpha z}\, a^{+}_l(x, z) \\ = -\frac{c(\alpha + \beta)}{\beta + \varepsilon} e^{-\beta(2x + y)}, \qquad y > 0, \end{aligned} \tag{7.4}$$

$$\begin{aligned} a^{-}_l(x, y) - \frac{(\alpha + \beta)(\varepsilon - \alpha)}{\beta + \varepsilon} e^{-2(\alpha + \beta)x - \alpha y} \int_0^{\infty} dz\, e^{-\beta z}\, a^{-}_l(x, z) \\ = -c^{-1}\,(\alpha + \beta)(\varepsilon - \alpha)\, e^{-\alpha(2x + y)}, \qquad y > 0. \end{aligned} \tag{7.5}$$

The solutions of the Marchenko equations (7.4) and (7.5) are given by

$$a^{+}_l(x, y) = \frac{-c(\alpha + \beta) e^{-\beta(2x + y)}}{(\beta + \varepsilon) - (\varepsilon - \alpha) e^{-2(\alpha + \beta)x}}, \qquad y > 0, \tag{7.6}$$

$$a_l^-(x,y) = \frac{-c^{-1}(\alpha+\beta)(\varepsilon-\alpha)(\beta+\varepsilon)\,e^{-\alpha(2x+y)}}{(\beta+\varepsilon)-(\varepsilon-\alpha)e^{-2(\alpha+\beta)x}}, \qquad y>0. \tag{7.7}$$

Substituting (7.6) and (7.7) in (5.19), we obtain

$$b_l^+(x,y) = \frac{(\alpha+\beta)(\varepsilon-\alpha)\,e^{-2(\alpha+\beta)x-\alpha y}}{(\beta+\varepsilon)-(\varepsilon-\alpha)e^{-2(\alpha+\beta)x}}, \qquad y>0, \tag{7.8}$$

$$b_l^-(x,y) = \frac{(\alpha+\beta)(\varepsilon-\alpha)\,e^{-2(\alpha+\beta)x-\beta y}}{(\beta+\varepsilon)-(\varepsilon-\alpha)e^{-2(\alpha+\beta)x}}, \qquad y>0. \tag{7.9}$$

Using (7.6)-(7.9) in (5.25) we get for $x>0$

$$e^{2\zeta(x)} = \frac{\alpha\beta(\beta+\varepsilon)-c\,\alpha(\alpha+\beta)\,e^{-2\beta x}+\alpha^2(\varepsilon-\alpha)\,e^{-2(\alpha+\beta)x}}{\alpha\beta(\beta+\varepsilon)-c^{-1}\beta(\alpha+\beta)(\varepsilon-\alpha)(\beta+\varepsilon)\,e^{-2\alpha x}+\beta^2(\varepsilon-\alpha)\,e^{-2(\alpha+\beta)x}}. \tag{7.10}$$

Note that the right-hand side in (7.10) must be positive for $x>0$, and this forces us to impose certain additional restrictions on the parameters $\alpha,\beta,\varepsilon,c$ if we want $P\in L^1(\mathbf{R})$. For example, by choosing $c<0$ and $\varepsilon>\alpha$, we see that both the numerator and the denominator of the right-hand side in (7.10) are positive. On the other hand, the choice $\alpha=\varepsilon$ results in the scattering data $e^{\mp p}T^{\pm}(k)=1$, $R^-(k)=0$, and $R^+(k)=c/(k-i\beta)$. Note also that since $P(x)=0$ when $x<0$, from (7.10) we see that

$$e^{2p} = \frac{\alpha\beta(\beta+\varepsilon)-c\,\alpha(\alpha+\beta)+\alpha^2(\varepsilon-\alpha)}{\alpha\beta(\beta+\varepsilon)-c^{-1}\beta(\alpha+\beta)(\varepsilon-\alpha)(\beta+\varepsilon)+\beta^2(\varepsilon-\alpha)}. \tag{7.11}$$

Since $R^{\pm}(k)$ is $O(1/k)$ but not $o(1/k)$ as $k\to+\infty$, from the argument leading to (3.13) we see that $Q(x)$ and $P'(x)$ contain delta-functions at $x=0$. In fact, from (3.13) we get

$$q_0 = -c\,e^{-2p}-c^{-1}(\varepsilon-\alpha)(\beta+\varepsilon)\,e^{2p}, \qquad p_0' = -2c\,e^{-2p}+2c^{-1}(\varepsilon-\alpha)(\beta+\varepsilon)\,e^{2p}, \tag{7.12}$$

where e^{2p} is the constant given in (7.11). Hence q_0 and p_0' given in (7.12) are completely determined by the scattering data. Thus, using (5.26), (5.27), (7.10), and (7.12), we get $P(x)$ and $Q(x)$ also for $x\geq 0$.

EXAMPLE 7.3 Let us consider the inverse scattering problem with $R^+(k)=R^-(k)=0$, when (0.1) has a bound state at $k=i\alpha^+$ with the bound-state constant c^+ and (0.2) has a bound state at $k=i\alpha^-$ with the bound-state constant c^-, where α^+ and α^- are some positive constants. Using (4.5) and (4.6) we obtain $e^{\mp p}T^{\pm}(k)=(k+i\alpha^{\mp})/(k-i\alpha^{\pm})$. With the help of (4.7), from (5.4) we obtain $\hat{S}_l^{\pm}(z)=(\alpha^++\alpha^-)\,c^{\pm}e^{-\alpha^{\pm}z}$. From (5.12), for $x\in\mathbf{R}$, we obtain

$$K_l^{\pm}(x;y,z) = c^+c^-(\alpha^++\alpha^-)\,e^{-2(\alpha^++\alpha^-)x-\alpha^{\pm}y-\alpha^{\mp}z}, \qquad y,z>0.$$

The two uncoupled Marchenko equations (5.14) are given by

$$a_l^{\pm}(x,y) - c^+c^-(\alpha^+ + \alpha^-)e^{-2(\alpha^+ + \alpha^-)x - \alpha^{\pm}y} \int_0^{\infty} dz\, e^{-\alpha^{\mp}z}\, a_l^{\pm}(x,z) = c^{\pm}(\alpha^+ + \alpha^-)\, e^{-\alpha^{\pm}(2x+y)}. \tag{7.13}$$

The solutions of the Marchenko equations (7.13) are

$$a_l^{\pm}(x,y) = \frac{c^{\pm}(\alpha^+ + \alpha^-)\, e^{-\alpha^{\pm}(2x+y)}}{1 - c^+c^-\, e^{-2(\alpha^+ + \alpha^-)x}}, \qquad y \in \mathbf{R}^+,\ x \in \mathbf{R}.$$

Using (5.19) we get

$$b_l^{\pm}(x,y) = \frac{c^+c^-(\alpha^+ + \alpha^-)e^{-2(\alpha^+ + \alpha^-)x - \alpha^{\mp}y}}{1 - c^+c^-\, e^{-2(\alpha^+ + \alpha^-)x}}, \qquad y \in \mathbf{R}^+,\ x \in \mathbf{R}.$$

From (5.25) we obtain

$$e^{2\zeta(x)} = \frac{\alpha^+\alpha^- + c^+\alpha^-(\alpha^+ + \alpha^-)\, e^{-2\alpha^+x} + c^+c^-(\alpha^+)^2 e^{-2(\alpha^+ + \alpha^-)x}}{\alpha^+\alpha^- + c^-\alpha^+(\alpha^+ + \alpha^-)\, e^{-2\alpha^-x} + c^+c^-(\alpha^-)^2 e^{-2(\alpha^+ + \alpha^-)x}}, \qquad x \in \mathbf{R}. \tag{7.14}$$

Proceeding as in Theorem 5.5, from (7.14) we compute $P(x)$ and $Q(x)$. Note that, when the bound-state constants c^+ and c^- are positive, we are assured that the numerator and the denominator in (7.14) are positive for all $x \in \mathbf{R}$. Then, letting $x \to -\infty$ in (7.14) we see that $P \in L^1(\mathbf{R})$ and $e^p = \alpha^+/\alpha^-$.

Note that the integral equations (7.13) are not uniquely solvable when $c^+c^- > 0$ and $x = x_0$, where $x_0 = \ln(c^+c^-)/[2(\alpha^+ + \alpha^-)]$. However, for $c^+, c^- > 0$ the right-hand side of (7.14) is well defined and $\zeta(x)$ remains continuous as $x \to x_0$. Moreover, tedious but straightforward analysis shows that $P(x)$ and $Q(x)$ are continuously differentiable everywhere (also at x_0) and exponentially decreasing as $x \to \pm\infty$.

EXAMPLE 7.4 Consider the inverse problem with $R^{\pm}(k) = i\alpha^{\pm}/(k - i\beta)$ with $\beta > 0$ and $\beta^2 > \alpha^+\alpha^-$. Assume that there are no bound states. Let $\varepsilon = \sqrt{\beta^2 - \alpha^+\alpha^-}$. From (4.5) we obtain $e^{\mp p}T^{\pm}(k) = (k+i\varepsilon)/(k+i\beta)$. Using (5.3) and (5.4) we get $\hat{S}_l^{\pm}(z) = -\alpha^{\pm}e^{-\beta z}\theta(z)$, where $\theta(z)$ is the Heaviside function.

When $x > 0$ we proceed as follows. From (5.12) we obtain

$$K_l^{\pm}(x;y,z) = \frac{\alpha^+\alpha^-}{2\beta} e^{-4\beta x - \beta(y+z)}, \qquad x,y,z > 0. \tag{7.15}$$

Using (7.15) we write the Marchenko equations (5.14) as

$$a_l^{\pm}(x,y) - \frac{\alpha^+\alpha^-}{2\beta} e^{-\beta(4x+y)} \int_0^{\infty} dz\, e^{-\beta z}\, a_l^{\pm}(x,z) = -\alpha^{\pm}\, e^{-\beta(2x+y)}, \qquad x,y > 0,$$

whose solutions are

$$a_l^{\pm}(x,y) = \frac{-4\beta^2\alpha^{\pm}\, e^{-\beta(2x+y)}}{4\beta^2 - \alpha^+\alpha^-\, e^{-4\beta x}}, \qquad x,y>0. \tag{7.16}$$

From (5.19) and (7.16) we get

$$b_l^{\pm}(x,y) = \frac{2\beta\alpha^+\alpha^-\, e^{-\beta(4x+y)}}{4\beta^2 - \alpha^+\alpha^-\, e^{-4\beta x}}, \qquad x,y>0. \tag{7.17}$$

Hence, using (7.16) and (7.17) in (5.25) we obtain

$$e^{2\zeta(x)} = \frac{4\beta^2 - 4\alpha^+\beta\, e^{-2\beta x} + \alpha^+\alpha^-\, e^{-4\beta x}}{4\beta^2 - 4\alpha^-\beta\, e^{-2\beta x} + \alpha^+\alpha^-\, e^{-4\beta x}}, \qquad x>0. \tag{7.18}$$

Note that the numerator and the denominator in (7.18) are positive when α^+ and α^- have the same sign and are each bounded in absolute value by β. As outlined in Theorem 5.5, we obtain $P(x)$ and $Q(x)$ for $x>0$ explicitly.

When $x<0$, using (5.19) we get

$$K_l^{\pm}(x;y,z) = \begin{cases} \dfrac{\alpha^+\alpha^-}{2\beta} e^{-\beta|y-z|}, & 0<\min\{y,z\}<-2x, \\ \dfrac{\alpha^+\alpha^-}{2\beta} e^{-4\beta x-\beta(y+z)}, & \min\{y,z\}>-2x. \end{cases} \tag{7.19}$$

Using (7.19) we write the Marchenko equations (5.14) as

$$a_l^{\pm}(x,y) - \frac{\alpha^+\alpha^-}{2\beta}\left[\int_0^y + \int_y^{-2x} + \int_{-2x}^{\infty}\right] dz\, e^{-\beta|y-z|}\, a_l^{\pm}(x,z) = 0, \qquad 0<y<-2x, \tag{7.20}$$

$$\begin{aligned} a_l^{\pm}(x,y) &- \frac{\alpha^+\alpha^-}{2\beta} e^{-\beta y}\int_0^{-2x} dz\, e^{\beta z}\, a_l^{\pm}(x,z) \\ &- \frac{\alpha^+\alpha^-}{2\beta} e^{-\beta(4x+y)}\int_{-2x}^{\infty} dz\, e^{-\beta z} a_l^{\pm}(x,z) = -\alpha^{\pm} e^{-\beta(2x+y)}, \qquad y>-2x. \end{aligned} \tag{7.21}$$

The solution $a_l^{\pm}(x,y)$ of (7.20) and (7.21) have the form

$$a_l^{\pm}(x,y) = \begin{cases} \rho^{\pm}(x)\left[e^{\varepsilon y} - \dfrac{\beta-\varepsilon}{\beta+\varepsilon} e^{-\varepsilon y}\right], & 0<y<-2x, \\ \omega^{\pm}(x)\, e^{-\beta y}, & y>-2x, \end{cases} \tag{7.22}$$

where $\rho^{\pm}(x)$ and $\omega^{\pm}(x)$ are to be determined. Using (7.22) in (7.20) and (7.21) we get

$$a_l^{\pm}(x,y) = \begin{cases} -\dfrac{(\alpha^{\pm})^2\alpha^{\mp}\, e^{2\varepsilon x}}{r(x)}\left[(\beta+\varepsilon)e^{\varepsilon y} - (\beta-\varepsilon)e^{-\varepsilon y}\right], & 0<y<-2x, \\ -\dfrac{2\beta\alpha^{\pm}\, e^{-\beta(2x+y)}}{r(x)}\left[(\beta+\varepsilon)^2 - (\beta-\varepsilon)^2 e^{4\varepsilon x}\right], & y>-2x, \end{cases} \tag{7.23}$$

where $r(x) = (\beta+\varepsilon)^3 - (\beta-\varepsilon)^3 e^{4\varepsilon x}$. Using (7.23) in (5.19) we get

$$b_l^{\pm}(x,y) = \begin{cases} \dfrac{\alpha^+\alpha^-}{r(x)}\left[(\beta+\varepsilon)^2 e^{-\varepsilon y} - (\beta-\varepsilon)^2 e^{\varepsilon(4x+y)}\right], & 0 < y < -2x, \\ \dfrac{4\beta\,\varepsilon\alpha^+\alpha^-\, e^{2\varepsilon x-\beta(2x+y)}}{r(x)}, & y > -2x. \end{cases} \tag{7.24}$$

Using (7.23) and (7.24) in (5.25) we obtain

$$e^{2\zeta(x)} = \frac{(\beta-\alpha^+)(\beta+\varepsilon)^3 + 2(\alpha^+)^2\alpha^-(\beta-\alpha^-)\,e^{2\varepsilon x} + (\beta-\alpha^+)(\beta-\varepsilon)^3\,e^{4\varepsilon x}}{(\beta-\alpha^-)(\beta+\varepsilon)^3 + 2\alpha^+(\alpha^-)^2(\beta-\alpha^+)\,e^{2\varepsilon x} + (\beta-\alpha^-)(\beta-\varepsilon)^3\,e^{4\varepsilon x}}, \qquad x < 0. \tag{7.25}$$

As in (7.21), the numerator and the denominator in (7.25) are positive when α^+ and α^- have the same sign and are each bounded in absolute value by β. Letting $x \to -\infty$ in (7.25) we see that $e^{2p} = (\beta-\alpha^+)/(\beta-\alpha^-)$. As outlined in Theorem 5.5, we can write $P(x)$ and $Q(x)$ for $x < 0$ using (7.23)-(7.25).

Since $2ikR^{\pm}(k)$ approaches $-2\alpha^{\pm}$ as $k \to +\infty$, from (3.13) we see that $Q(x)$ and $P'(x)$ contain Dirac-delta distributions at $x = 0$ with coefficients

$$q_0 = -\alpha^- e^{2\zeta(0)} - \alpha^+ e^{-2\zeta(0)}, \qquad p_0' = 2\alpha^- e^{2\zeta(0)} - 2\alpha^+ e^{-2\zeta(0)}, \tag{7.26}$$

where

$$e^{2\zeta(0)} = \frac{4\beta^2 - 4\alpha^+\beta + \alpha^+\alpha^-}{4\beta^2 - 4\alpha^-\beta + \alpha^+\alpha^-}.$$

Thus, having q_0 and p_0' in (7.26), we have completed the recovery of $Q(x)$ and $P(x)$ for all $x \in \mathbf{R}$. In the special case $\alpha^+ = \alpha^-$, from (7.18) and (7.25) we get $\zeta(x) = 0$ for $x \in \mathbf{R}$. Note that in the limiting case $\beta^2 = \alpha^+\alpha^-$, i.e. when $\varepsilon = 0$, we are in the generic case and we must have $\alpha^- = \alpha^+ = \beta$ in order to have $R^{\pm}(0) = -1$; in this case we get $Q(x) = -2\beta\,\delta(x)$ and $P(x) = 0$.

Finally, we remark that the solution of the inverse scattering problem in the last three examples can also be obtained by solving the Riemann-Hilbert problem in (5.2) directly. In fact, whenever the scattering coefficients are rational functions, the Riemann-Hilbert problem in (5.2) can be solved explicitly, leading to the recovery of $P(x)$ and $Q(x)$.

ACKNOWLEDGMENTS. This material is based upon work supported by the National Science Foundation under grant No. DMS-9501053. One of the authors (CvdM) is indebted to the Mathematics Departments at Virginia Tech and North Dakota State University for their hospitality and support during the visit in which a part of this research was performed and to C.N.R. and MURST for additional financial support.

REFERENCES

[AKV93] Aktosun, T., Klaus, M., and van der Mee, C.: *On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation.* J. Math. Phys. **34**, 2651–2690 (1993).

[AKV97] Aktosun, T., Klaus, M., and van der Mee, C.: *Wave scattering in one dimension with absorption.* J. Math. Phys., to appear.

[CS89] Chadan, K. and Sabatier, P.: *Inverse Problems in Quantum Scattering Theory.* 2nd ed., Springer, New York, 1989.

[CG81] Clancey, K. and Gohberg, I.: *Factorization of Matrix Functions and Singular Integral Operators.* OT **3**, Birkhäuser, Basel, 1981.

[DT79] Deift, P. and Trubowitz, E.: *Inverse scattering on the line.* Comm. Pure Appl. Math. **32**, 121–251 (1979).

[Fa64] Faddeev, L. D.: *Properties of the S-matrix of the one-dimensional Schrödinger equation.* Amer. Math. Soc. Transl. **2**, 139–166 (1964) [Trudy Mat. Inst. Steklova **73**, 314–336 (1964) (Russian)].

[Fe61] Feldman, I. A.: *On an effective solution of certain integral equations on the line and on the half-line.* Izv. Akad. Nauk Mold. SSR **10** (88), 16–26 (1961) (Russian).

[FGK94] Feldman, I., Gohberg, I., and Krupnik, N.: *A method of explicit factorization of matrix functions and applications.* Integral Equations Operator Theory **18**, 277–302 (1994).

[GRS64] Gel'fand, I. M., Raikov, D. A., and Shilov, G. E.: *Commutative Normed Rings.* Chelsea Publ. Co., New York, 1964.

[GF71] Gohberg, I. C. and Feldman, I. A.: *Convolution Equations and Projection Methods for their Solution.* Transl. Math. Monographs **41**, A.M.S., Providence, 1974 [Nauka, Moscow, 1971 (Russian)].

[GGK93] Gohberg, I., Goldberg, S., and Kaashoek, M. A.: *Classes of Linear Operators.* Vol. II, OT **63**, Birkhäuser, Basel, 1993.

[Ja76] Jaulent, M.: *Inverse scattering problems in absorbing media.* J. Math. Phys. **17**, 1351–1360 (1976).

[JJ72] Jaulent, M. and Jean, C.: *The inverse s-wave scattering problem for a class of potentials depending on energy.* Comm. Math. Phys. **28**, 177–220 (1972).

[JJ76a] Jaulent, M. and Jean, C.: *The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. I.* Ann. Inst. Henri Poincaré A **25**, 105–118 (1976).

[JJ76b] Jaulent, M. and Jean, C.: *The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential. II.* Ann. Inst. Henri Poincaré A **25**, 119–137 (1976).

[Ka75] Kaup, D. J.: *A higher-order water-wave equation and the method for solving it.* Progr. Theor. Phys. **54**, 396–408 (1975).

[Ne80] Newton, R. G.: *Inverse scattering. I. One dimension.* J. Math. Phys. **21**, 493–505 (1980).

[Pa88] Partington, J. R.: *An Introduction to Hankel Operators.* London Math. Soc. Student Texts, Vol. **13**, Cambridge Univ. Press, Cambridge, 1988.

[Po82] Power, S. C.: *Hankel Operators on Hilbert Space.* Research Notes in Mathematics, **64**, Pitman, Boston, 1982.

[SS95] Sattinger, D. H. and Szmigielski, J.: *Energy dependent scattering theory.* Differ. Integral Eqs. **8**, 945–959 (1995).

[SS96] Sattinger, D. H. and Szmigielski, J.: *A Riemann-Hilbert problem for an energy dependent Schrödinger operator.* Inverse Problems **12**, 1003–1025 (1996).

[Ts81] Tsutsumi, M.: *On the inverse scattering problem for the one-dimensional Schrödinger equation with an energy dependent potential.* J. Math. Anal. Appl. **83**, 316–350 (1981).

Tuncay Aktosun
Dept. of Mathematics
North Dakota State Univ.
Fargo, ND 58105
aktosun@plains.nodak.edu

Martin Klaus
Dept. of Mathematics
Virginia Polytechnic Inst.
and State Univ.
Blacksburg, VA 24061
klaus@math.vt.edu

Cornelis van der Mee
Dipartimento di Matematica
Università di Cagliari
Via Ospedale 72
09124 Cagliari, Italy
cornelis@krein.unica.it

MSC Primary 34A55, 81U40, Secondary 73D50

Submitted: August 25, 1997