

DIRECT AND INVERSE SCATTERING FOR SELFADJOINT HAMILTONIAN SYSTEMS ON THE LINE

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Dedicated to Israel Gohberg on the Occasion of his 70th Birthday

A direct and inverse scattering theory on the full line is developed for a class of first-order selfadjoint $2n \times 2n$ systems of differential equations with integrable potential matrices. Various properties of the corresponding scattering matrices including unitarity and canonical Wiener-Hopf factorization are established. The Marchenko integral equations are derived and their unique solvability is proved. The unique recovery of the potential from the solutions of the Marchenko equations is shown. In the case of rational scattering matrices, state space methods are employed to construct the scattering matrix from a reflection coefficient and to recover the potential explicitly.

1. INTRODUCTION

Consider the selfadjoint Hamiltonian system of differential equations

$$-iJ_{2n} \frac{dX(x, \lambda)}{dx} - V(x) X(x, \lambda) = \lambda X(x, \lambda), \quad x \in \mathbf{R}, \quad (1.1)$$

where

$$J_{2n} = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad V(x) = \begin{bmatrix} 0 & k(x) \\ k(x)^\dagger & 0 \end{bmatrix}, \quad (1.2)$$

with I_n the identity matrix of order n , the $n \times n$ matrix function k has complex-valued entries belonging to $L^1(\mathbf{R})$, $\lambda \in \mathbf{R}$ is an eigenvalue parameter, and \dagger denotes the matrix conjugate transpose. We call the function k the potential and the parameter λ the wavenumber. Note that $V(x)$ is a selfadjoint $2n \times 2n$ matrix and satisfies

$$J_{2n} V(x) = -V(x) J_{2n}.$$

We can think of $X(x, \lambda)$ in (1.1) as either a column vector of $2n$ entries or as a $2n \times 2n$ matrix. For $\lambda \in \mathbf{R}$, we define the Jost solution from the left, $F_l(x, \lambda)$, and the Jost solution

from the right, $F_r(x, \lambda)$, as the $2n \times 2n$ matrix solutions of (1.1) satisfying the boundary conditions

$$F_l(x, \lambda) = e^{i\lambda J_{2n}x} [I_{2n} + o(1)], \quad x \rightarrow +\infty, \quad (1.3)$$

$$F_r(x, \lambda) = e^{i\lambda J_{2n}x} [I_{2n} + o(1)], \quad x \rightarrow -\infty. \quad (1.4)$$

Using (1.1), (1.3), and (1.4), we obtain

$$F_l(x, \lambda) = e^{i\lambda J_{2n}x} - iJ_{2n} \int_x^\infty dy e^{-i\lambda J_{2n}(y-x)} V(y) F_l(y, \lambda), \quad (1.5)$$

$$F_r(x, \lambda) = e^{i\lambda J_{2n}x} + iJ_{2n} \int_{-\infty}^x dy e^{i\lambda J_{2n}(x-y)} V(y) F_r(y, \lambda). \quad (1.6)$$

For a given square matrix function $E(x)$, let us use $\|E\|_1$ to denote $\int_{-\infty}^\infty dx \|E(x)\|$, where $\|\cdot\|$ stands for the matrix norm defined by $\|A\| = \sup\{\|Av\|_2 : \|v\|_2 = 1\}$ and $\|\cdot\|_2$ is the Euclidean vector norm. Since the entries of $k(x)$ belong to $L^1(\mathbf{R})$, for each fixed $\lambda \in \mathbf{R}$ it follows by iteration that (1.5) and (1.6) are uniquely solvable and that $\|F_l(x, \lambda)\|$ and $\|F_r(x, \lambda)\|$ are bounded above by $e^{\|k\|_1}$. From (1.3)-(1.6) we get

$$F_l(x, \lambda) = e^{i\lambda J_{2n}x} [a_l(\lambda) + o(1)], \quad x \rightarrow -\infty, \quad (1.7)$$

$$F_r(x, \lambda) = e^{i\lambda J_{2n}x} [a_r(\lambda) + o(1)], \quad x \rightarrow +\infty, \quad (1.8)$$

where

$$a_l(\lambda) = I_{2n} - iJ_{2n} \int_{-\infty}^\infty dy e^{-i\lambda J_{2n}y} V(y) F_l(y, \lambda),$$

$$a_r(\lambda) = I_{2n} + iJ_{2n} \int_{-\infty}^\infty dy e^{-i\lambda J_{2n}y} V(y) F_r(y, \lambda).$$

The term “canonical differential equations” for the system (1.1) has been used by Melik-Adamyanyan [32-34], L. A. Sakhnovich [39,40], and A. L. Sakhnovich [38], who have studied the direct and inverse scattering problems for (1.1) on the half line. Under minor restrictions on the given so-called reflection function, a characterization of the scattering data corresponding to an L^1 -potential on the half line was given by Melik-Adamyanyan [34], who also supplied a method to reduce the inverse spectral problem on the full line for a canonical equation of order $2n$ to an inverse spectral problem on the half line for a canonical equation of order $4n$ [32] (see also [41]). We will comment on that characterization result at the end of Section 6. More recently, Alpay and Gohberg [3-6] have applied state space methods to derive explicit expressions for the solution of the inverse scattering problem for

(1.1) on the half line from the general theory in [34] when the scattering data are rational functions and consist of either the spectral function of the differential operator

$$H = -iJ_{2n} \frac{d}{dx} - V(x), \quad (1.9)$$

or a reflection function. A more self-contained treatment of these results was given by Alpay et al. [7]. Gohberg et al. have solved a similar inverse problem when the scattering data consist of the spectral function of H and this function is rational, both on the half line [24,25] and on the full line [26].

Let us mention that there are other, more general first-order systems for which the direct and inverse scattering problems have been analyzed. Shabat [42] and Beals and Coifman [10,11] considered the $n \times n$ system $d\varphi/dx = \lambda \mathbf{J}\varphi + q(x)\varphi$, where $\mathbf{J} = \text{diag}\{\alpha_1, \dots, \alpha_n\}$ with distinct complex α_j and $q(x)$ an $n \times n$ off-diagonal matrix with entries belonging to $L^1(\mathbf{R})$ or more restrictive classes, without requiring $q(x)$ to be selfadjoint. As indicated in [16], the distinctness of α_j is not an essential restriction. It has been proved that the inverse problem has a unique solution within a certain class of potentials for an open and dense set of scattering data. The solution of the inverse scattering problem for such linear systems is useful in solving the Cauchy problem for various nonlinear evolution equations; for details and further references, we refer the interested reader to [2,12,16] and the references therein.

Note that by putting $Z(x, \lambda) = \frac{1}{\sqrt{2}}[I_{2n} + i\mathbf{q}_{2n}]X(x, \lambda)$, where

$$\mathbf{q}_{2n} = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad (1.10)$$

we can convert (1.1) into the massless Dirac equation of order $2n$ given by

$$\frac{dZ(x, \lambda)}{dx} = \begin{bmatrix} p(x) & \lambda I_n - v(x) \\ -\lambda I_n - v(x) & -p(x) \end{bmatrix} Z(x, \lambda),$$

where $p(x) = \frac{1}{2}[k(x) + k(x)^\dagger]$ and $v(x) = -\frac{1}{2}i[k(x) - k(x)^\dagger]$ are the real and imaginary parts of $k(x)$, respectively. The direct and inverse scattering problems for the Dirac system on the half line were studied in [22]. The interested reader is referred to [22,28,29] and the references therein for more information on the Dirac system.

The direct scattering problem for (1.1) consists of the determination of the scattering matrix $\mathbf{S}(\lambda)$ defined in (3.11) when the potential $k(x)$ is given, whereas the inverse scattering problem is the determination of $k(x)$ from $\mathbf{S}(\lambda)$ or, equivalently, from either of the reflection coefficients $R(\lambda)$ and $L(\lambda)$, which are defined in (3.7) in terms of the matrices

$a_l(\lambda)$ and $a_r(\lambda)$. In this article we develop a direct and inverse scattering theory for (1.1) when $k(x)$ has entries belonging to $L^1(\mathbf{R})$. Working within the framework established by Faddeev [21] and Deift and Trubowitz [20] for the Schrödinger equation on the line, we derive the analyticity and asymptotic properties of the Faddeev matrices and the scattering coefficients, employ them to derive a Riemann-Hilbert problem and various Marchenko integral equations, and recover the potential in terms of the solutions of the Marchenko equations. We prove the unitarity of the scattering matrix and exploit this property to prove the unique solvability of the Marchenko equations. We also establish the unique canonical Wiener-Hopf factorization of the (unitarily dilated) scattering matrix and show how the potential is obtained once the factors in the factorization are known. We then give a rather general sufficient condition on the reflection coefficient to lead to a potential whose entries belong to $L^1(\mathbf{R})$. After that, for rational reflection coefficients we present a procedure to compute explicitly the scattering matrix from a reflection coefficient. This is no longer as elementary as in the case of the (scalar) Schrödinger equation [20,21] and involves a suitable extension of a contractive $n \times n$ matrix function to a unitary $2n \times 2n$ matrix function (cf. [8,27]). When the reflection coefficients are rational, we apply state space methods [13] to solve the Marchenko equations and the inverse problem explicitly. For rational reflection coefficients, this approach provides us with a systematic inversion method for inverse scattering problems on the line, which is different from previous methods such as those used in [9].

This article is organized as follows. In Section 2 we introduce the Faddeev matrices, obtain their analyticity properties, and analyze some other properties of the Faddeev matrices and the Jost solutions of (1.1). In Section 3 we define the scattering matrix $S(\lambda)$ in terms of the spatial asymptotics of the Jost solutions, prove the unitarity of $S(\lambda)$, and obtain various properties of the scattering coefficients. In Section 4 we analyze the Fourier transforms of the Faddeev matrices and the scattering coefficients. We then go on, in Section 5, to derive a Riemann-Hilbert problem for the Faddeev matrices and show that the (unitarily dilated) scattering matrix has a canonical Wiener-Hopf factorization. We also show that this factorization can be used to solve the inverse scattering problem. In Section 6, we convert the Riemann-Hilbert problem into both coupled and uncoupled Marchenko integral equations, prove their unique solvability by a contraction mapping argument, and give a partial characterization of the scattering data corresponding to potentials with entries in $L^1(\mathbf{R})$. In Section 7 we show how the scattering matrix can be constructed from a reflection coefficient, and we also construct $S(\lambda)$ explicitly when one of the reflection coefficients is a rational function. Finally, in Section 8 we give an explicit solution of the inverse scattering problem with rational reflection coefficients; this is done

by using the minimal realization of the reflection coefficients as the input to the Marchenko equations.

2. SCATTERING SOLUTIONS

In this section we introduce the Faddeev matrices and study some of their properties. The results obtained here will be used later to establish various properties of the scattering matrix and to solve the inverse scattering problem by the Marchenko method.

PROPOSITION 2.1. *Let $X(x, \lambda)$ and $Y(x, \lambda)$ be any two solutions of (1.1). Then, for real λ , $X(x, \lambda)^\dagger J_{2n} Y(x, \lambda)$ is independent of x .*

PROOF. The result follows by differentiating $X(x, \lambda)^\dagger J_{2n} Y(x, \lambda)$ and using (1.1) together with the selfadjointness of $V(x)$ and J_{2n} . ■

PROPOSITION 2.2. *For $\lambda \in \mathbf{R}$, either of the Jost solutions $F_l(x, \lambda)$ and $F_r(x, \lambda)$ forms a fundamental matrix of (1.1) and has determinant equal to one, and the matrices $a_l(\lambda)$ and $a_r(\lambda)$ appearing in (1.7) and (1.8), respectively, satisfy*

$$\det a_l(\lambda) = \det a_r(\lambda) = 1. \quad (2.1)$$

Moreover, for $\lambda \in \mathbf{R}$, the Jost solutions satisfy

$$F_l(x, \lambda) = F_r(x, \lambda) a_l(\lambda), \quad (2.2)$$

$$F_r(x, \lambda)^\dagger J_{2n} F_l(x, \lambda) = a_r(\lambda)^\dagger J_{2n} = J_{2n} a_l(\lambda), \quad (2.3)$$

$$F_l(x, \lambda)^\dagger J_{2n} F_l(x, \lambda) = a_l(\lambda)^\dagger J_{2n} a_l(\lambda) = J_{2n}, \quad (2.4)$$

$$F_r(x, \lambda)^\dagger J_{2n} F_r(x, \lambda) = a_r(\lambda)^\dagger J_{2n} a_r(\lambda) = J_{2n}, \quad (2.5)$$

and hence

$$a_l(\lambda) a_r(\lambda) = a_r(\lambda) a_l(\lambda) = I_{2n}, \quad (2.6)$$

$$a_l(\lambda)^{-1} = J_{2n} a_l(\lambda)^\dagger J_{2n}, \quad a_r(\lambda)^{-1} = J_{2n} a_r(\lambda)^\dagger J_{2n}. \quad (2.7)$$

PROOF. From (1.1) it follows [35] that

$$\frac{d[\det F_l(x, \lambda)]}{dx} = (\operatorname{tr} \{iJ_{2n} V(x) + i\lambda J_{2n}\}) (\det F_l(x, \lambda)),$$

where tr denotes the matrix trace. By (1.2), $iJ_{2n} V(x) + i\lambda J_{2n}$ has zero trace, and hence $\det F_l(x, \lambda)$ is independent of x and its value can be evaluated as $x \rightarrow +\infty$. Thus, we get $\det F_l(x, \lambda) = 1$, from which we also conclude that $F_l(x, \lambda)$ is a fundamental matrix

of (1.1). Similarly, we find that $\det F_r(x, \lambda) = 1$ and $F_r(x, \lambda)$ is a fundamental matrix of (1.1). Then, from (1.2), (1.7), and (1.8) we obtain (2.1). Since either of $F_l(x, \lambda)$ and $F_r(x, \lambda)$ is a fundamental matrix of (1.1), with the help of (1.3) and (1.7), we get (2.2). Using Proposition 2.1, we obtain (2.3)-(2.5) by evaluating $F_r(x, \lambda)^\dagger J_{2n} F_l(x, \lambda)$, $F_l(x, \lambda)^\dagger J_{2n} F_l(x, \lambda)$, and $F_r(x, \lambda)^\dagger J_{2n} F_r(x, \lambda)$ as $x \rightarrow \pm\infty$. Then (2.6) and (2.7) readily follow. ■

In terms of the Jost solutions, we define the Faddeev matrices $M_l(x, \lambda)$ and $M_r(x, \lambda)$ as

$$M_l(x, \lambda) = F_l(x, \lambda) e^{-i\lambda J_{2n} x}, \quad M_r(x, \lambda) = F_r(x, \lambda) e^{-i\lambda J_{2n} x}. \quad (2.8)$$

From (1.3) and (1.4) we get

$$M_l(x, \lambda) = I_{2n} + o(1), \quad x \rightarrow +\infty,$$

$$M_r(x, \lambda) = I_{2n} + o(1), \quad x \rightarrow -\infty.$$

Let us partition the Jost solutions and Faddeev matrices into $n \times n$ blocks as follows:

$$F_l(x, \lambda) = \begin{bmatrix} F_{l1}(x, \lambda) & F_{l2}(x, \lambda) \\ F_{l3}(x, \lambda) & F_{l4}(x, \lambda) \end{bmatrix}, \quad F_r(x, \lambda) = \begin{bmatrix} F_{r1}(x, \lambda) & F_{r2}(x, \lambda) \\ F_{r3}(x, \lambda) & F_{r4}(x, \lambda) \end{bmatrix}, \quad (2.9)$$

$$M_l(x, \lambda) = \begin{bmatrix} M_{l1}(x, \lambda) & M_{l2}(x, \lambda) \\ M_{l3}(x, \lambda) & M_{l4}(x, \lambda) \end{bmatrix}, \quad M_r(x, \lambda) = \begin{bmatrix} M_{r1}(x, \lambda) & M_{r2}(x, \lambda) \\ M_{r3}(x, \lambda) & M_{r4}(x, \lambda) \end{bmatrix}. \quad (2.10)$$

By \mathbf{C}^+ and \mathbf{C}^- we denote the open upper half and lower half complex planes, respectively. We also define

$$\sigma_\pm(x) = \pm \int_x^{\pm\infty} dy \|k(y)\|. \quad (2.11)$$

PROPOSITION 2.3. *Assume that the entries of $k(x)$ belong to $L^1(\mathbf{R})$. Then:*

- (i) *For each fixed $x \in \mathbf{R}$, $\begin{bmatrix} M_{l1}(x, \lambda) \\ M_{l3}(x, \lambda) \end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda \in \overline{\mathbf{C}^+}$ and analytic in $\lambda \in \mathbf{C}^+$ and tends to $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.*
- (ii) *For all $\lambda \in \overline{\mathbf{C}^+}$, $M_{l1}(x, \lambda)$ and $M_{l3}(x, \lambda)$ are bounded by $e^{\sigma_+(x)}$ in the norm.*
- (iii) *For each fixed $x \in \mathbf{R}$, $\begin{bmatrix} M_{l2}(x, \lambda) \\ M_{l4}(x, \lambda) \end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda \in \overline{\mathbf{C}^-}$ and analytic in $\lambda \in \mathbf{C}^-$ and tends to $\begin{bmatrix} 0 \\ I_n \end{bmatrix}$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.*
- (iv) *For all $\lambda \in \overline{\mathbf{C}^-}$, $M_{l2}(x, \lambda)$ and $M_{l4}(x, \lambda)$ are bounded by $e^{\sigma_+(x)}$ in the norm.*

PROOF. Using (2.8) in (1.5), we obtain

$$M_l(x, \lambda) = I_{2n} - iJ_{2n} \int_x^\infty dy e^{-i\lambda J_{2n}(y-x)} V(y) M_l(y, \lambda) e^{i\lambda J_{2n}(y-x)}. \quad (2.12)$$

Iterating (2.12) once, we get the uncoupled systems

$$M_{l1}(x, \lambda) = I_n + \int_x^\infty dy \int_y^\infty dz e^{2i\lambda(z-y)} k(y) k(z)^\dagger M_{l1}(z, \lambda), \quad (2.13)$$

$$M_{l2}(x, \lambda) = -i \int_x^\infty dy e^{-2i\lambda(y-x)} k(y) + \int_x^\infty dy \int_y^\infty dz e^{-2i\lambda(y-x)} k(y) k(z)^\dagger M_{l2}(z, \lambda), \quad (2.14)$$

$$M_{l3}(x, \lambda) = i \int_x^\infty dy e^{2i\lambda(y-x)} k(y)^\dagger + \int_x^\infty dy \int_y^\infty dz e^{2i\lambda(y-x)} k(y)^\dagger k(z) M_{l3}(z, \lambda), \quad (2.15)$$

$$M_{l4}(x, \lambda) = I_n + \int_x^\infty dy \int_y^\infty dz e^{-2i\lambda(z-y)} k(y)^\dagger k(z) M_{l4}(z, \lambda). \quad (2.16)$$

Iterating the Volterra integral equations (2.13) and (2.15), we prove that the series of iterates converge absolutely and uniformly in $\lambda \in \overline{\mathbf{C}}^+$, and we also get the estimate in (ii). Similarly, we prove that the series of iterates of (2.14) and (2.16) converge absolutely and uniformly in $\lambda \in \overline{\mathbf{C}}^-$ and that the estimate in (iv) holds. To prove the assertions concerning the large- λ limit we first consider $M_{l3}(x, \lambda)$. To deal with the first term on the right-hand side of (2.15) we define

$$\omega(\lambda) = \sup_{x \in \mathbf{R}} \left\| \int_x^\infty dy e^{2i\lambda(y-x)} k(y)^\dagger \right\|.$$

By approximating $k(y)$ by infinitely differentiable matrix functions of compact support (as in the proof of the Riemann-Lebesgue lemma) it follows that $\omega(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}}^+$. Iterating (2.15) we get $\|M_{l3}(x, \lambda)\| \leq \omega(\lambda) e^{\sigma_+(x)}$, which implies that $\|M_{l3}(x, \lambda)\| \rightarrow 0$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}}^+$. Next we consider $M_{l1}(x, \lambda)$. Let $G_{l1}(x, \lambda) = M_{l1}(x, \lambda) - I_n$ and consider the following integral equation for $G_{l1}(x, \lambda)$ which follows from (2.13):

$$G_{l1}(x, \lambda) = H_{l1}(x, \lambda) + \int_x^\infty dy \int_y^\infty dz e^{2i\lambda(z-y)} k(y) k(z)^\dagger G_{l1}(z, \lambda),$$

where

$$H_{l1}(x, \lambda) = \int_x^\infty dy \int_y^\infty dz e^{2i\lambda(z-y)} k(y) k(z)^\dagger.$$

Since $\|H_{l1}(x, \lambda)\| \leq \nu(\lambda) \sigma_+(x)$, we conclude that $\|G_{l1}(x, \lambda)\| \rightarrow 0$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}}^+$. This proves the assertion of (i) regarding the limit $\lambda \rightarrow \infty$. The proof of the corresponding statement in (iii) is similar. ■

As in Proposition 2.3, we have a similar result for the Faddeev matrix $M_r(x, \lambda)$:

PROPOSITION 2.4. *Assume that the entries of $k(x)$ belong to $L^1(\mathbf{R})$. Then:*

- (i) *For each fixed $x \in \mathbf{R}$, $\begin{bmatrix} M_{r1}(x, \lambda) \\ M_{r3}(x, \lambda) \end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda \in \overline{\mathbf{C}^-}$ and analytic in $\lambda \in \mathbf{C}^-$ and tends to $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.*
- (ii) *For all $\lambda \in \overline{\mathbf{C}^-}$, $M_{r1}(x, \lambda)$ and $M_{r3}(x, \lambda)$ are bounded by $e^{\sigma^-(x)}$ in the norm.*
- (iii) *For each fixed $x \in \mathbf{R}$, $\begin{bmatrix} M_{r2}(x, \lambda) \\ M_{r4}(x, \lambda) \end{bmatrix}$ can be extended to a matrix function that is continuous in $\lambda \in \overline{\mathbf{C}^+}$ and analytic in $\lambda \in \mathbf{C}^+$ and tends to $\begin{bmatrix} 0 \\ I_n \end{bmatrix}$ as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.*
- (iv) *For all $\lambda \in \overline{\mathbf{C}^+}$, $M_{r2}(x, \lambda)$ and $M_{r4}(x, \lambda)$ are bounded by $e^{\sigma^-(x)}$ in the norm.*

PROOF. Using (2.8) in (1.6), we obtain

$$M_r(x, \lambda) = I_{2n} + iJ_{2n} \int_{-\infty}^x dy e^{i\lambda J_{2n}(x-y)} V(y) M_r(y, \lambda) e^{-i\lambda J_{2n}(x-y)}. \quad (2.17)$$

Iterating (2.17) once, we obtain the four systems given by

$$M_{r1}(x, \lambda) = I_n + \int_{-\infty}^x dy \int_{-\infty}^y dz e^{-2i\lambda(y-z)} k(y) k(z)^\dagger M_{r1}(z, \lambda), \quad (2.18)$$

$$M_{r2}(x, \lambda) = i \int_{-\infty}^x dy e^{2i\lambda(x-y)} k(y) + \int_{-\infty}^x dy \int_{-\infty}^y dz e^{2i\lambda(x-y)} k(y) k(z)^\dagger M_{r2}(z, \lambda), \quad (2.19)$$

$$M_{r3}(x, \lambda) = -i \int_{-\infty}^x dy e^{-2i\lambda(x-y)} k(y)^\dagger + \int_{-\infty}^x dy \int_{-\infty}^y dz e^{-2i\lambda(x-y)} k(y)^\dagger k(z) M_{r3}(z, \lambda), \quad (2.20)$$

$$M_{r4}(x, \lambda) = I_n + \int_{-\infty}^x dy \int_{-\infty}^y dz e^{2i\lambda(y-z)} k(y)^\dagger k(z) M_{r4}(z, \lambda). \quad (2.21)$$

Iterating (2.18)-(2.21) as in the proof of Proposition 2.3, we complete the proof. ■

Let us write

$$a_l(\lambda) = \begin{bmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{bmatrix}, \quad a_r(\lambda) = \begin{bmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{bmatrix}. \quad (2.22)$$

From (1.7), (1.8), and (2.8) we see that

$$\begin{bmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{bmatrix} = \lim_{x \rightarrow -\infty} \begin{bmatrix} M_{l1}(x, \lambda) & e^{-2i\lambda x} M_{l2}(x, \lambda) \\ e^{2i\lambda x} M_{l3}(x, \lambda) & M_{l4}(x, \lambda) \end{bmatrix}, \quad (2.23)$$

$$\begin{bmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{bmatrix} = \lim_{x \rightarrow +\infty} \begin{bmatrix} M_{r1}(x, \lambda) & e^{-2i\lambda x} M_{r2}(x, \lambda) \\ e^{2i\lambda x} M_{r3}(x, \lambda) & M_{r4}(x, \lambda) \end{bmatrix}. \quad (2.24)$$

Using (2.12), (2.17), (2.23), and (2.24) we find the integral representations

$$a_{l1}(\lambda) = I_n - i \int_{-\infty}^{\infty} dy k(y) M_{l3}(y, \lambda), \quad (2.25)$$

$$a_{l2}(\lambda) = -i \int_{-\infty}^{\infty} dy e^{-2i\lambda y} k(y) M_{l4}(y, \lambda), \quad (2.26)$$

$$a_{l3}(\lambda) = i \int_{-\infty}^{\infty} dy e^{2i\lambda y} k(y)^\dagger M_{l1}(y, \lambda), \quad (2.27)$$

$$a_{l4}(\lambda) = I_n + i \int_{-\infty}^{\infty} dy k(y)^\dagger M_{l2}(y, \lambda), \quad (2.28)$$

$$a_{r1}(\lambda) = I_n + i \int_{-\infty}^{\infty} dy k(y) M_{r3}(y, \lambda), \quad (2.29)$$

$$a_{r2}(\lambda) = i \int_{-\infty}^{\infty} dy e^{-2i\lambda y} k(y) M_{r4}(y, \lambda), \quad (2.30)$$

$$a_{r3}(\lambda) = -i \int_{-\infty}^{\infty} dy e^{2i\lambda y} k(y)^\dagger M_{r1}(y, \lambda), \quad (2.31)$$

$$a_{r4}(\lambda) = I_n - i \int_{-\infty}^{\infty} dy k(y)^\dagger M_{r2}(y, \lambda). \quad (2.32)$$

PROPOSITION 2.5. *Assume that the entries of $k(x)$ belong to $L^1(\mathbf{R})$. Then:*

- (i) *The matrices $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbf{C}^+}$ and analytic in $\lambda \in \mathbf{C}^+$ and tend to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.*
- (ii) *The matrices $a_{l4}(\lambda)$ and $a_{r1}(\lambda)$ are continuous in $\lambda \in \overline{\mathbf{C}^-}$ and analytic in $\lambda \in \mathbf{C}^-$ and tend to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.*
- (iii) *The matrices $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ are continuous in $\lambda \in \mathbf{R}$ and vanish as $\lambda \rightarrow \pm\infty$.*
- (iv) *The matrices $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ satisfy*

$$a_{r2}(\lambda) = -a_{l3}(\lambda)^\dagger, \quad a_{r3}(\lambda) = -a_{l2}(\lambda)^\dagger, \quad \lambda \in \mathbf{R}. \quad (2.33)$$

PROOF. Using Propositions 2.3 and 2.4 in (2.25)-(2.32), we get (i), (ii), and (iii). We obtain (iv) from (2.3). ■

Using the notations of (2.9), let us form the following matrices:

$$f_+(x, \lambda) = \begin{bmatrix} F_{l1}(x, \lambda) & F_{r2}(x, \lambda) \\ F_{l3}(x, \lambda) & F_{r4}(x, \lambda) \end{bmatrix}, \quad f_-(x, \lambda) = \begin{bmatrix} F_{r1}(x, \lambda) & F_{l2}(x, \lambda) \\ F_{r3}(x, \lambda) & F_{l4}(x, \lambda) \end{bmatrix}. \quad (2.34)$$

Let an asterisk denote complex conjugation. From Propositions 2.3 and 2.4, it follows that $f_+(x, \lambda)$ is a solution of (1.1) that is continuous in $\lambda \in \overline{\mathbf{C}^+}$ and analytic in $\lambda \in \mathbf{C}^+$; similarly, $f_-(x, \lambda)$ is a solution of (1.1) that is continuous in $\lambda \in \overline{\mathbf{C}^-}$ and analytic in $\lambda \in \mathbf{C}^-$.

PROPOSITION 2.6. *The $2n \times 2n$ matrix $f_-(x, \lambda^*)^\dagger J_{2n} f_+(x, \lambda)$ is independent of x for all $\lambda \in \overline{\mathbf{C}^+}$. Similarly, $f_+(x, \lambda^*)^\dagger J_{2n} f_-(x, \lambda)$ is independent of x for all $\lambda \in \overline{\mathbf{C}^-}$. We have*

$$f_-(x, \lambda^*)^\dagger J_{2n} f_+(x, \lambda) = \begin{bmatrix} a_{l1}(\lambda) & 0 \\ 0 & -a_{r4}(\lambda) \end{bmatrix}, \quad \lambda \in \overline{\mathbf{C}^+}. \quad (2.35)$$

Further, $a_{l1}(\lambda)^\dagger$ and $a_{r4}(\lambda)^\dagger$ have analytic extensions to \mathbf{C}^- , $a_{r1}(\lambda)^\dagger$ and $a_{l4}(\lambda)^\dagger$ have analytic extensions to \mathbf{C}^+ , and

$$a_{l1}(\lambda) = a_{r1}(\lambda^*)^\dagger, \quad a_{r4}(\lambda) = a_{l4}(\lambda^*)^\dagger, \quad \lambda \in \overline{\mathbf{C}^+}, \quad (2.36)$$

$$a_{r1}(\lambda) = a_{l1}(\lambda^*)^\dagger, \quad a_{l4}(\lambda) = a_{r4}(\lambda^*)^\dagger, \quad \lambda \in \overline{\mathbf{C}^-}. \quad (2.37)$$

PROOF. Using (1.1), one can show that $f_\mp(x, \lambda^*)^\dagger J_{2n} f_\pm(x, \lambda)$ is independent of x for $\lambda \in \overline{\mathbf{C}^\pm}$. Evaluating it as $x \rightarrow \pm\infty$ and using (1.7) and (1.8) we get (2.35)-(2.37). ■

PROPOSITION 2.7. *For $\lambda \in \mathbf{R}$, either of $f_+(x, \lambda)$ and $f_-(x, \lambda)$ is nonsingular and hence is a fundamental matrix for (1.1).*

PROOF. As in the proof of Proposition 2.2 we find that $\det f_+(x, \lambda)$ is independent of x , and evaluating that determinant as $x \rightarrow \pm\infty$ we obtain

$$\det f_+(x, \lambda) = \det a_{l1}(\lambda) = \det a_{r4}(\lambda). \quad (2.38)$$

From (2.4) it follows that

$$a_{l1}(\lambda)^\dagger a_{l1}(\lambda) = I_n + a_{l3}(\lambda)^\dagger a_{l3}(\lambda), \quad \lambda \in \mathbf{R}, \quad (2.39)$$

and hence $a_{l1}(\lambda)$ is invertible for all $\lambda \in \mathbf{R}$. Thus, $f_+(x, \lambda)$ is nonsingular and forms a fundamental matrix for (1.1). Similarly, we get

$$\det f_-(x, \lambda) = \det a_{r1}(\lambda) = \det a_{l4}(\lambda), \quad (2.40)$$

and hence with the help of (2.36) and (2.39), we conclude that $f_-(x, \lambda)$ is nonsingular and forms a fundamental matrix for (1.1). ■

Next we will prove that $f_+(x, \lambda)$ is nonsingular for $\lambda \in \overline{\mathbf{C}^+}$ and $f_-(x, \lambda)$ is nonsingular for $\lambda \in \overline{\mathbf{C}^-}$. First, using (2.10) and (2.34), let us define

$$m_+(x, \lambda) = \begin{bmatrix} M_{l1}(x, \lambda) & M_{r2}(x, \lambda) \\ M_{l3}(x, \lambda) & M_{r4}(x, \lambda) \end{bmatrix} = f_+(x, \lambda) e^{-i\lambda J_{2n}x}, \quad (2.41)$$

$$m_-(x, \lambda) = \begin{bmatrix} M_{r1}(x, \lambda) & M_{l2}(x, \lambda) \\ M_{r3}(x, \lambda) & M_{l4}(x, \lambda) \end{bmatrix} = f_-(x, \lambda) e^{-i\lambda J_{2n}x}. \quad (2.42)$$

PROPOSITION 2.8. *For each $\lambda \in \overline{\mathbf{C}^+}$, $f_+(x, \lambda)$ is a fundamental matrix for (1.1). Similarly, for each $\lambda \in \overline{\mathbf{C}^-}$, $f_-(x, \lambda)$ is a fundamental matrix for (1.1).*

PROOF. The proof for $f_-(x, \lambda)$ is similar to the proof for $f_+(x, \lambda)$, and hence we will only present the latter. From Proposition 2.7, we already know that $f_+(x, \lambda)$ is a fundamental matrix for (1.1) when $\lambda \in \mathbf{R}$. Thus, we need only prove that the columns of $f_+(x, \lambda)$ are linearly independent for $\lambda \in \mathbf{C}^+$. From (2.34) and Proposition 2.3 it follows that, for each fixed $x \in \mathbf{R}$, $f_+(x, \lambda)$ has an analytic extension to $\lambda \in \mathbf{C}^+$. Because of (1.3), (1.4) and (2.8), the first n columns of $f_+(x, \lambda)$ are linearly independent for $\lambda \in \mathbf{C}^+$, and also the last n columns of $f_+(x, \lambda)$ are linearly independent for $\lambda \in \mathbf{C}^+$. It is sufficient to prove that an arbitrary nontrivial linear combination of the first n columns cannot be written as a linear combination of the last n columns of $f_+(x, \lambda)$. Otherwise, we would have

$$\sum_{s=1}^n c_s(\lambda) e^{i\lambda x} C_s(x, \lambda) = \sum_{q=n+1}^{2n} c_q(\lambda) e^{-i\lambda x} C_q(x, \lambda), \quad (2.43)$$

where $C_j(x, \lambda)$ represents the j th column of the matrix $m_+(x, \lambda)$ defined in (2.41), and $c_j(\lambda)$ are independent of x . From (2.8), Proposition 2.3 (ii), and Proposition 2.4 (ii), it follows that each entry of $C_j(x, \lambda)$ is uniformly bounded in $x \in \mathbf{R}$ for each $\lambda \in \mathbf{C}^+$. The left-hand side in (2.43) decreases exponentially as $x \rightarrow +\infty$ while the right-hand side decreases exponentially as $x \rightarrow -\infty$; this would turn either side into a nontrivial L^2 -solution of (1.1), which is a contradiction because the selfadjoint differential operator of (1.1) cannot have nonreal eigenvalues. ■

When $\text{Im } \lambda \neq 0$, a result similar to the one in Proposition 2.8 was proved in [30].

From Propositions 2.3, 2.4, and 2.8 we obtain the following result.

COROLLARY 2.9. *For each $x \in \mathbf{R}$, the $2n \times 2n$ matrix $m_+(x, \lambda)$ and its inverse $m_+(x, \lambda)^{-1}$ are continuous in $\lambda \in \overline{\mathbf{C}^+}$, are analytic in $\lambda \in \mathbf{C}^+$, and converge to I_{2n} as*

$\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Similarly, for each $x \in \mathbf{R}$, the $2n \times 2n$ matrix $m_-(x, \lambda)$ and its inverse $m_-(x, \lambda)^{-1}$ are continuous in $\lambda \in \overline{\mathbf{C}^-}$, are analytic in $\lambda \in \mathbf{C}^-$, and converge to I_{2n} as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.

Next we present certain properties of the matrices $a_l(\lambda)$ and $a_r(\lambda)$ appearing in (2.22).

PROPOSITION 2.10. *Assume that the entries of $k(x)$ belong to $L^1(\mathbf{R})$. Then:*

- (i) *The matrices $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are invertible for each $\lambda \in \overline{\mathbf{C}^+}$, and $a_{l4}(\lambda)$ and $a_{r1}(\lambda)$ are invertible for each $\lambda \in \overline{\mathbf{C}^-}$.*
- (ii) *The matrix functions $a_{l1}(\lambda)^{-1}$ and $a_{r4}(\lambda)^{-1}$ are continuous in $\overline{\mathbf{C}^+}$ and analytic in \mathbf{C}^+ and tend to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.*
- (iii) *The matrix functions $a_{l4}(\lambda)^{-1}$ and $a_{r1}(\lambda)^{-1}$ are continuous in $\overline{\mathbf{C}^-}$ and analytic in \mathbf{C}^- and tend to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.*

PROOF. From Proposition 2.8 it follows that $f_+(x, \lambda)$ and $f_-(x, \lambda^*)^\dagger$ are nonsingular for $\lambda \in \overline{\mathbf{C}^+}$. Hence, (2.35) implies that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are invertible for each $\lambda \in \overline{\mathbf{C}^+}$. Then, using (2.36), we can conclude that $a_{l4}(\lambda)$ and $a_{r1}(\lambda)$ are invertible for each $\lambda \in \overline{\mathbf{C}^-}$. The proof of (ii) and (iii) follows from (i) and Proposition 2.5. ■

3. THE SCATTERING MATRIX

In this section we define and analyze the properties of the scattering coefficients of (1.1) when the entries of the potential $k(x)$ belong to $L^1(\mathbf{R})$.

We can write (2.6) as

$$a_{l1}(\lambda) a_{r1}(\lambda) + a_{l2}(\lambda) a_{r3}(\lambda) = I_n = a_{r1}(\lambda) a_{l1}(\lambda) + a_{r2}(\lambda) a_{l3}(\lambda), \quad (3.1)$$

$$a_{l1}(\lambda) a_{r2}(\lambda) + a_{l2}(\lambda) a_{r4}(\lambda) = 0 = a_{r1}(\lambda) a_{l2}(\lambda) + a_{r2}(\lambda) a_{l4}(\lambda), \quad (3.2)$$

$$a_{l3}(\lambda) a_{r1}(\lambda) + a_{l4}(\lambda) a_{r3}(\lambda) = 0 = a_{r3}(\lambda) a_{l1}(\lambda) + a_{r4}(\lambda) a_{l3}(\lambda), \quad (3.3)$$

$$a_{l3}(\lambda) a_{r2}(\lambda) + a_{l4}(\lambda) a_{r4}(\lambda) = I_n = a_{r3}(\lambda) a_{l2}(\lambda) + a_{r4}(\lambda) a_{l4}(\lambda). \quad (3.4)$$

Let us define the transmission coefficients $T_l(\lambda)$ from the left and $T_r(\lambda)$ from the right, and the reflection coefficients $R(\lambda)$ from the right and $L(\lambda)$ from the left, as follows:

$$T_l(\lambda) = a_{l1}(\lambda)^{-1}, \quad T_r(\lambda) = a_{r4}(\lambda)^{-1}, \quad (3.5)$$

$$R(\lambda) = a_{r2}(\lambda) a_{r4}(\lambda)^{-1}, \quad L(\lambda) = a_{l3}(\lambda) a_{l1}(\lambda)^{-1}. \quad (3.6)$$

From (3.2), (3.3), and (3.6) we get

$$R(\lambda) = -a_{l1}(\lambda)^{-1} a_{l2}(\lambda), \quad L(\lambda) = -a_{r4}(\lambda)^{-1} a_{r3}(\lambda). \quad (3.7)$$

Note that using (2.3) and (3.1)-(3.7), we can express the matrices in (2.22) in terms of the scattering coefficients as

$$a_l(\lambda) = \begin{bmatrix} T_l(\lambda)^{-1} & -T_l(\lambda)^{-1} R(\lambda) \\ L(\lambda) T_l(\lambda)^{-1} & [T_r(\lambda)^\dagger]^{-1} \end{bmatrix}, \quad (3.8)$$

$$a_r(\lambda) = \begin{bmatrix} [T_l(\lambda)^\dagger]^{-1} & R(\lambda) T_r(\lambda)^{-1} \\ -T_r(\lambda)^{-1} L(\lambda) & T_r(\lambda)^{-1} \end{bmatrix}, \quad (3.9)$$

where the off-diagonal entries can be expressed in terms of $L(\lambda)$ or $R(\lambda)$ by using

$$L(\lambda) T_l(\lambda)^{-1} = -[R(\lambda) T_r(\lambda)^{-1}]^\dagger, \quad (3.10)$$

which is immediate from (2.33).

The scattering matrix $\mathbf{S}(\lambda)$ associated with (1.1) is defined as

$$\mathbf{S}(\lambda) = \begin{bmatrix} T_l(\lambda) & R(\lambda) \\ L(\lambda) & T_r(\lambda) \end{bmatrix}. \quad (3.11)$$

THEOREM 3.1. *The scattering matrix $\mathbf{S}(\lambda)$ is continuous for $\lambda \in \mathbf{R}$ and converges to I_{2n} as $\lambda \rightarrow \pm\infty$. It is unitary for each $\lambda \in \mathbf{R}$, and hence the scattering coefficients satisfy*

$$T_l(\lambda) T_l(\lambda)^\dagger + R(\lambda) R(\lambda)^\dagger = I_n = T_r(\lambda)^\dagger T_r(\lambda) + R(\lambda)^\dagger R(\lambda), \quad (3.12)$$

$$T_l(\lambda)^\dagger T_l(\lambda) + L(\lambda)^\dagger L(\lambda) = I_n = T_r(\lambda) T_r(\lambda)^\dagger + L(\lambda) L(\lambda)^\dagger, \quad (3.13)$$

$$T_r(\lambda) R(\lambda)^\dagger + L(\lambda) T_l(\lambda)^\dagger = 0 = T_r(\lambda)^\dagger L(\lambda) + R(\lambda)^\dagger T_l(\lambda). \quad (3.14)$$

Moreover, for $\lambda \in \mathbf{R}$ we have

$$\det T_l(\lambda) = \det T_r(\lambda), \quad (3.15)$$

$$\det \begin{bmatrix} I_n & R(\lambda) \\ R(\lambda)^\dagger & I_n \end{bmatrix} = \det \begin{bmatrix} I_n & L(\lambda) \\ L(\lambda)^\dagger & I_n \end{bmatrix} = |\det T_l(\lambda)|^2, \quad (3.16)$$

$$\det \mathbf{S}(\lambda) = \frac{\det T_l(\lambda)}{[\det T_l(\lambda)]^*}. \quad (3.17)$$

PROOF. The continuity and the large- λ asymptotics follow from Propositions 2.5 and 2.10. Using (3.5)-(3.7) in (2.7), we get $\mathbf{S}(\lambda) \mathbf{S}(\lambda)^\dagger = I_{2n}$, from which (3.12)-(3.14) follow. Furthermore, from (2.38), (3.8), and (3.9) we obtain (3.15). Using (3.10), we can write (3.8) and (3.9) as

$$a_l(\lambda) = \begin{bmatrix} T_l(\lambda)^{-1} & 0 \\ 0 & [T_r(\lambda)^\dagger]^{-1} \end{bmatrix} \begin{bmatrix} I_n & -R(\lambda) \\ -R(\lambda)^\dagger & I_n \end{bmatrix}, \quad (3.18)$$

$$a_r(\lambda) = \begin{bmatrix} [T_l(\lambda)^\dagger]^{-1} & 0 \\ 0 & T_r(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} I_n & -L(\lambda)^\dagger \\ -L(\lambda) & I_n \end{bmatrix}, \quad (3.19)$$

and hence, using (2.1), (3.15), (3.18), (3.19), and $\det J_{2n} = (-1)^n$, we get (3.16). Using (2.2), (2.34), (3.5), and (3.6) it follows that

$$f_-(x, \lambda) = f_+(x, \lambda) J_{2n} \mathbf{S}(\lambda) J_{2n}, \quad \lambda \in \mathbf{R}. \quad (3.20)$$

Thus, from (3.5), (2.38), (2.40), (3.20), and $\det J_{2n} = (-1)^n$, we obtain (3.17). ■

In Proposition 2.10 we have seen that $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ have invertible, continuous, and analytic extensions from the real axis to \mathbf{C}^+ . Thus, from (3.5) and Proposition 2.10, we obtain the following result.

COROLLARY 3.2. *The transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ and their inverses $T_l(\lambda)^{-1}$ and $T_r(\lambda)^{-1}$ are continuous in $\lambda \in \overline{\mathbf{C}^+}$ and analytic in $\lambda \in \mathbf{C}^+$; these four matrices all converge to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Similarly, the matrices $T_l(\lambda^*)^\dagger$ and $T_r(\lambda^*)^\dagger$ and their inverses $[T_l(\lambda^*)^\dagger]^{-1}$ and $[T_r(\lambda^*)^\dagger]^{-1}$ are continuous in $\lambda \in \overline{\mathbf{C}^-}$ and analytic in $\lambda \in \mathbf{C}^-$; these four matrices all converge to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.*

In general, $R(\lambda)$ and $L(\lambda)$ do not have analytic continuations off the real axis. In the special case when $k(x)$ vanishes on a half line, we have the following.

PROPOSITION 3.3. *If $k(x)$ is supported in the right half line \mathbf{R}^+ , then $L(\lambda)$ extends to a function that is continuous on $\overline{\mathbf{C}^+}$, is analytic on \mathbf{C}^+ , and vanishes as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Similarly, if $k(x)$ is supported in the left half line \mathbf{R}^- , then $R(\lambda)$ extends to a function that is continuous on $\overline{\mathbf{C}^+}$, is analytic on \mathbf{C}^+ , and vanishes as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.*

PROOF. If k has support in \mathbf{R}^+ , then from (2.27) and Proposition 2.3 we see that $a_{l3}(\lambda)$ has an extension that is continuous in $\lambda \in \overline{\mathbf{C}^+}$, is analytic in $\lambda \in \mathbf{C}^+$, and converges to 0 as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Thus, using (3.6) and Corollary 3.2, we can conclude that $L(\lambda)$ extends to a function that is continuous on $\overline{\mathbf{C}^+}$, is analytic on \mathbf{C}^+ , and vanishes as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. In a similar manner, if k is supported in \mathbf{R}^- , using (2.30), (3.6), Proposition 2.3, and Corollary 3.2, we obtain that $R(\lambda)$ extends to a function that is continuous on $\overline{\mathbf{C}^+}$, is analytic on \mathbf{C}^+ , and vanishes as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. ■

In the sequel we will use the notation $L^j(I; \mathbf{C}^{p \times q})$ to denote the Banach space of all complex $p \times q$ matrix functions $z(\alpha)$ whose entries belong to $L^j(I)$, endowed with the norm $\int_I d\alpha \|z(\alpha)\|^j$; if $q = 1$, we simply write $L^j(I; \mathbf{C}^p)$.

Considering the operator H defined in (1.9) and $H_0 = -iJ_{2n}(d/dx)$ as the perturbed and free Hamiltonians, respectively, one can prove the existence of the Møller wave oper-

ators

$$W_{\pm} = \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$$

as (partial) isometries and construct [1,37] the scattering operator $S = W_+^* W_-$. Using the integral transform

$$(\mathcal{F}\phi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha e^{-i\lambda J_{2n}\alpha} \phi(\alpha), \quad \phi \in L^2(\mathbf{R}; \mathbf{C}^{2n}),$$

one can prove that $(\mathcal{F}S\mathcal{F}^{-1}\phi)(\lambda) = \mathbf{S}(\lambda)\phi(\lambda)$, where $\mathbf{S}(\lambda)$ is given by (3.11). In other words, $\mathbf{S}(\lambda)$ coincides with the scattering matrix obtained from time-dependent scattering theory.

4. FOURIER TRANSFORMS

Let \mathcal{W}^q denote the Wiener algebra of all $q \times q$ matrix functions of the form

$$Z(\lambda) = Z_{\infty} + \int_{-\infty}^{\infty} d\alpha z(\alpha) e^{i\lambda\alpha}, \quad (4.1)$$

where $z(\alpha)$ is a $q \times q$ matrix function whose entries belong to $L^1(\mathbf{R})$ and $Z_{\infty} = Z(\pm\infty)$. Then \mathcal{W}^q is a Banach algebra with a unit element and endowed with the norm

$$\|Z\|_{\mathcal{W}^q} = \|Z_{\infty}\| + \int_{-\infty}^{\infty} d\alpha \|z(\alpha)\|,$$

and its invertible elements are those $Z(\lambda)$ as in (4.1) for which Z_{∞} and $Z(\lambda)$ are nonsingular matrices for all $\lambda \in \mathbf{R}$ (see e.g. [23]). We will use \mathcal{W}_{\pm}^q to denote the subalgebra of those functions $Z(\lambda)$ for which $z(\alpha)$ has support in \mathbf{R}^{\pm} and $\mathcal{W}_{\pm,0}^q$ to denote the subalgebra of those functions $Z(\lambda)$ for which $Z_{\infty} = 0$ and $z(\alpha)$ has support in \mathbf{R}^{\pm} . Then, $\mathcal{W}^q = \mathcal{W}_+^q \oplus \mathcal{W}_{-,0}^q = \mathcal{W}_{+,0}^q \oplus \mathcal{W}_-^q$.

In this section we prove that the matrix functions $M_l(x, \cdot)$, $M_r(x, \cdot)$, and $\mathbf{S}(\cdot)$ belong to \mathcal{W}^{2n} , and that $m_{\pm}(x, \cdot)$ belongs to \mathcal{W}_{\pm}^{2n} .

Let us construct the L^1 -matrix functions $b_{\pm}(x, \cdot)$, $B_l(x, \cdot)$, and $B_r(x, \cdot)$ such that

$$m_{\pm}(x, \lambda) = I_{2n} + \int_0^{\infty} d\alpha b_{\pm}(x, \alpha) e^{\pm i\lambda\alpha}, \quad (4.2)$$

$$M_l(x, \lambda) = I_{2n} + \int_0^{\infty} d\alpha B_l(x, \alpha) e^{i\lambda J_{2n}\alpha}, \quad M_r(x, \lambda) = I_{2n} + \int_0^{\infty} d\alpha B_r(x, \alpha) e^{-i\lambda J_{2n}\alpha}. \quad (4.3)$$

Indeed, partitioning the matrix functions $B_l(x, \alpha)$ and $B_r(x, \alpha)$ in (4.3) into $n \times n$ blocks as

$$B_l(x, \alpha) = \begin{bmatrix} B_{l1}(x, \alpha) & B_{l2}(x, \alpha) \\ B_{l3}(x, \alpha) & B_{l4}(x, \alpha) \end{bmatrix}, \quad B_r(x, \alpha) = \begin{bmatrix} B_{r1}(x, \alpha) & B_{r2}(x, \alpha) \\ B_{r3}(x, \alpha) & B_{r4}(x, \alpha) \end{bmatrix},$$

so that

$$b_+(x, \alpha) = \begin{bmatrix} B_{l1}(x, \alpha) & B_{r2}(x, \alpha) \\ B_{l3}(x, \alpha) & B_{r4}(x, \alpha) \end{bmatrix}, \quad b_-(x, \alpha) = \begin{bmatrix} B_{r1}(x, \alpha) & B_{l2}(x, \alpha) \\ B_{r3}(x, \alpha) & B_{l4}(x, \alpha) \end{bmatrix}, \quad (4.4)$$

we apply (4.3) to (2.12) and (2.17), and derive the coupled integral equations for $\alpha > 0$

$$B_{l1}(x, \alpha) = -i \int_x^\infty dy k(y) B_{l3}(y, \alpha), \quad (4.5)$$

$$B_{l2}(x, \alpha) = -\frac{i}{2}k(x + \alpha/2) - i \int_x^{x+\alpha/2} dy k(y) B_{l4}(y, \alpha + 2x - 2y), \quad (4.6)$$

$$B_{l3}(x, \alpha) = \frac{i}{2}k(x + \alpha/2)^\dagger + i \int_x^{x+\alpha/2} dy k(y)^\dagger B_{l1}(y, \alpha + 2x - 2y), \quad (4.7)$$

$$B_{l4}(x, \alpha) = i \int_x^\infty dy k(y)^\dagger B_{l2}(y, \alpha), \quad (4.8)$$

$$B_{r1}(x, \alpha) = i \int_{-\infty}^x dy k(y) B_{r3}(y, \alpha), \quad (4.9)$$

$$B_{r2}(x, \alpha) = \frac{i}{2}k(x - \alpha/2) + i \int_{x-\alpha/2}^x dy k(y) B_{r4}(y, \alpha + 2y - 2x), \quad (4.10)$$

$$B_{r3}(x, \alpha) = -\frac{i}{2}k(x - \alpha/2)^\dagger - i \int_{x-\alpha/2}^x dy k(y)^\dagger B_{r1}(y, \alpha + 2y - 2x), \quad (4.11)$$

$$B_{r4}(x, \alpha) = -i \int_{-\infty}^x dy k(y)^\dagger B_{r2}(y, \alpha). \quad (4.12)$$

We first prove that, for each $x \in \mathbf{R}$, the four systems of integral equations (4.5) and (4.7), (4.6) and (4.8), (4.9) and (4.11), (4.10) and (4.12) have unique solutions with entries in $L^1(\mathbf{R}^+)$. Then for the matrix functions $m_\pm(x, \lambda)$, $M_l(x, \lambda)$, and $M_r(x, \lambda)$ defined in (4.2) and (4.3), we derive the integral relations (2.13)-(2.16) and (2.18)-(2.21). In this way we will have proved that $M_l(x, \cdot)$ and $M_r(x, \cdot)$ belong to \mathcal{W}^{2n} and $m_\pm(x, \cdot)$ belongs to \mathcal{W}_\pm^{2n} .

Let us introduce the following mixed norm on the $2n \times 2n$ matrix functions $B(x, \alpha)$ depending on $(x, \alpha) \in \mathbf{R} \times \mathbf{R}^+$:

$$\|B(\cdot, \cdot)\|_{\infty, 1} = \sup_{x \in \mathbf{R}} \|B(x, \cdot)\|_1. \quad (4.13)$$

THEOREM 4.1. *Assume that the entries of $k(x)$ belong to $L^1(\mathbf{R})$. Then, for each $x \in \mathbf{R}$, the four pairs of integral equations (4.5) and (4.7), (4.6) and (4.8), (4.9) and (4.11), (4.10) and (4.12) have unique solutions with finite mixed norm as defined in (4.13). Consequently, $m_+(x, \cdot)$ belongs to \mathcal{W}_+^{2n} , $m_-(x, \cdot)$ belongs to \mathcal{W}_-^{2n} , and $M_l(x, \cdot)$ and $M_r(x, \cdot)$ belong to \mathcal{W}^{2n} .*

PROOF. Consider (4.5) and (4.7). We can solve this system by iteration as follows. Define

$$\begin{aligned} B_{l3}^{(0)}(x, \alpha) &= 0, \\ B_{l1}^{(j)}(x, \alpha) &= -i \int_x^\infty dy k(y) B_{l3}^{(j)}(y, \alpha), \quad j \geq 0, \end{aligned} \quad (4.14)$$

$$B_{l3}^{(j+1)}(x, \alpha) = \frac{i}{2} k(x + \alpha/2)^\dagger + i \int_x^{x+\alpha/2} dy k(y)^\dagger B_{l1}^{(j)}(y, \alpha + 2x - 2y), \quad j \geq 0. \quad (4.15)$$

Taking operator norms we obtain from (4.14) and (4.15)

$$\begin{aligned} \|B_{l1}^{(j)}(x, \alpha)\| &\leq \int_x^\infty dy \|k(y)\| \|B_{l3}^{(j)}(y, \alpha)\|, \\ \|B_{l3}^{(j+1)}(x, \alpha)\| &\leq \frac{1}{2} \|k(x + \alpha/2)\| + \int_x^{x+\alpha/2} dy \|k(y)\| \|B_{l1}^{(j)}(y, \alpha + 2x - 2y)\|. \end{aligned}$$

Then the norms of $B_{l1}^{(j)}(x, \cdot)$ and $B_{l3}^{(j)}(x, \cdot)$ in $L^1(\mathbf{R}^+)$ satisfy

$$\|B_{l1}^{(j)}(x, \cdot)\|_1 \leq \int_x^\infty dy \|k(y)\| \|B_{l3}^{(j)}(y, \cdot)\|_1, \quad (4.16)$$

$$\begin{aligned} \|B_{l3}^{(j+1)}(x, \cdot)\|_1 &\leq \sigma_+(x) + \int_0^\infty d\alpha \int_x^{x+\alpha/2} dy \|k(y)\| \|B_{l1}^{(j)}(y, \alpha + 2x - 2y)\| \\ &= \sigma_+(x) + \int_x^\infty dy \|k(y)\| \|B_{l1}^{(j)}(y, \cdot)\|_1, \end{aligned} \quad (4.17)$$

where $\sigma_+(x)$ is the quantity defined in (2.11), the order of integration has been changed, and the change of variable $\gamma = \alpha + 2x - 2y$ has been applied. From (4.16) and (4.17) we obtain by induction

$$\|B_{l1}^{(j)}(x, \cdot)\|_1 \leq \sum_{s=1}^j \frac{\sigma_+(x)^{2s}}{(2s)!}, \quad \|B_{l3}^{(j)}(x, \cdot)\|_1 \leq \sum_{s=1}^j \frac{\sigma_+(x)^{2s-1}}{(2s-1)!}, \quad j \geq 1.$$

Consequently,

$$\|B_{l1}(x, \cdot)\|_1 + \|B_{l3}(x, \cdot)\|_1 \leq \sum_{s=1}^\infty \frac{\sigma_+(x)^s}{s!} = e^{\sigma_+(x)} - 1.$$

At the same time we have proved that, for each $x \in \mathbf{R}$, the system of equations (4.5) and (4.7) has a unique solution with entries belonging to $L^1(\mathbf{R}^+)$.

The proofs for the three other systems of equations, namely (4.6) and (4.8), (4.9) and (4.11), (4.10) and (4.12) are analogous. We are led to the estimates

$$\|B_{lj}(x, \cdot)\|_1 \leq e^{\sigma^+(x)} - 1, \quad \|B_{rj}(x, \cdot)\|_1 \leq e^{\sigma^-(x)} - 1, \quad (4.18)$$

where $j = 1, 2, 3, 4$. ■

The integral equations (4.5)-(4.12) allow us to derive the following relations for the potential $k(x)$:

$$k(x) = 2i B_{l2}(x, 0^+) = -2i B_{r2}(x, 0^+) = 2i B_{l3}(x, 0^+)^\dagger = -2i B_{r3}(x, 0^+)^\dagger. \quad (4.19)$$

To justify (4.19), let us fix $\alpha > 0$ and integrate the norm of the left-hand side in (4.7) with respect to $x \in \mathbf{R}$. We obtain

$$\begin{aligned} \|B_{l3}(\cdot, \alpha)\|_1 &\leq \frac{1}{2} \|k\|_1 + \int_{-\infty}^{\infty} ds \int_{y-\alpha/2}^y dx \|k(y)\| \|B_{l1}(y, \alpha + 2x - 2y)\| \\ &= \frac{1}{2} \left[\|k\|_1 + \int_{-\infty}^{\infty} dy \int_0^{\alpha} dz \|k(y)\| \|B_{l1}(y, z)\| \right] \\ &\leq \frac{1}{2} \left[\|k\|_1 + \int_{-\infty}^{\infty} dy \|k(y)\| \|B_{l1}(y, \cdot)\|_1 \right] \leq \frac{1}{2} \left[e^{\|k\|_1} - 1 \right], \end{aligned}$$

where we have used (4.18). Hence, for each $\alpha > 0$, $B_{l3}(\cdot, \alpha)$ is a matrix function with entries in $L^1(\mathbf{R})$. We now easily derive the estimate

$$\left\| B_{l3}(\cdot, \alpha) - \frac{i}{2} k(\cdot + \alpha/2)^\dagger \right\|_1 \leq \frac{1}{2} \int_{-\infty}^{\infty} dy \|k(y)\| \int_0^{\alpha} dz \|B_{l1}(y, z)\| = o(1), \quad \alpha \rightarrow 0^+,$$

which justifies the identity $k(x) = 2i B_{l3}(x, 0^+)^\dagger$. In an analogous way one proves the similar result for $B_{l2}(\cdot, \alpha)$, $B_{r2}(\cdot, \alpha)$, and $B_{r3}(\cdot, \alpha)$.

THEOREM 4.2. *The reflection coefficients $R(\lambda)$ and $L(\lambda)$ belong to \mathcal{W}^n , and $R(\pm\infty) = L(\pm\infty) = 0$. The transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ belong to \mathcal{W}_+^n , and they converge to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$.*

PROOF. Using (2.3) and (2.6)-(2.8) we get

$$a_l(\lambda) = e^{-i\lambda J_{2n}x} J_{2n} M_r(x, \lambda)^\dagger J_{2n} M_l(x, \lambda) e^{i\lambda J_{2n}x}, \quad (4.20)$$

$$a_r(\lambda) = e^{-i\lambda J_{2n}x} J_{2n} M_l(x, \lambda)^\dagger J_{2n} M_r(x, \lambda) e^{i\lambda J_{2n}x}. \quad (4.21)$$

From Theorem 4.1 we see that $M_l(x, \lambda)$ and $M_r(x, \lambda)$ belong to \mathcal{W}^{2n} . Using (4.20) and (4.21) at $x = 0$, we can show that $a_l(\lambda)$ and $a_r(\lambda)$ are products of elements of \mathcal{W}^{2n} and hence belong to \mathcal{W}^{2n} . Using (3.5)-(3.7) and Proposition 2.10, we complete the proof of the theorem. ■

5. WIENER-HOPF FACTORIZATION

Using (2.41), (2.42), and (3.20), we obtain the Riemann-Hilbert problem

$$m_-(x, \lambda) = m_+(x, \lambda) \mathbf{G}(x, \lambda), \quad \lambda \in \mathbf{R}, \quad (5.1)$$

where $\mathbf{G}(x, \lambda)$ is the unitarily dilated scattering matrix given by

$$\mathbf{G}(x, \lambda) = e^{i\lambda J_{2n}x} J_{2n} \mathbf{S}(\lambda) J_{2n} e^{-i\lambda J_{2n}x} = \begin{bmatrix} T_l(\lambda) & -R(\lambda) e^{2i\lambda x} \\ -L(\lambda) e^{-2i\lambda x} & T_r(\lambda) \end{bmatrix}. \quad (5.2)$$

Equation (5.1) can in principle be used to compute the potential from a reflection matrix. To do so, we first construct the scattering matrix $\mathbf{S}(\lambda)$ in terms of $L(\lambda)$ or $R(\lambda)$ alone. Given $R(\lambda)$ for $\lambda \in \mathbf{R}$, we can obtain $T_l(\lambda)$ by performing the factorization

$$T_l(\lambda) T_l(\lambda)^\dagger = I_n - R(\lambda) R(\lambda)^\dagger, \quad \lambda \in \mathbf{R}, \quad (5.3)$$

which follows from (3.12). Here $T_l(\lambda)$ is continuous in $\overline{\mathbf{C}^+}$, is analytic in \mathbf{C}^+ , and converges to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$; similarly, $T_l(\lambda^*)^\dagger$ is continuous for $\lambda \in \overline{\mathbf{C}^-}$, is analytic for $\lambda \in \mathbf{C}^-$, and converges to I_n as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$. In a similar way, $T_r(\lambda)$ can be constructed by performing the factorization

$$T_r(\lambda)^\dagger T_r(\lambda) = I_n - R(\lambda)^\dagger R(\lambda), \quad \lambda \in \mathbf{R}, \quad (5.4)$$

which is found from (3.12). With the help of Proposition 2.6 we then get

$$L(\lambda) = -T_r(\lambda) R(\lambda)^\dagger [T_l(\lambda)^\dagger]^{-1}, \quad \lambda \in \mathbf{R}. \quad (5.5)$$

Note that from Theorem 4.2, it follows that the $n \times n$ matrices on the right-hand sides in (5.3) and (5.4) both belong to the Wiener algebra \mathcal{W}^n . Furthermore, from Corollary 3.2 and Theorem 4.2, it follows that $T_l(\lambda)$ and $T_r(\lambda)$ belong to the subalgebra \mathcal{W}_+^n , and $T_l(\lambda^*)^\dagger$ and $T_r(\lambda^*)^\dagger$ belong to \mathcal{W}_-^n . Hence, (5.3) and (5.4) lead to a left and a right canonical Wiener-Hopf factorization in \mathcal{W}^n , respectively. One now builds the matrix function $\mathbf{G}(x, \lambda)$ as in (5.2).

We have the following result.

THEOREM 5.1. *Suppose that $k \in L^1(\mathbf{R}; \mathbf{C}^n)$ and that the reflection coefficient $R(\lambda)$ of (1.1) belongs to \mathcal{W}^n and satisfies $\sup_{\lambda \in \mathbf{R}} \|R(\lambda)\| < 1$. Then, for each $x \in \mathbf{R}$, the matrix function $\mathbf{G}(x, \cdot)$ given by (5.2) has a unique left canonical Wiener-Hopf factorization*

$$\mathbf{G}(x, \lambda) = m_+(x, \lambda)^{-1} m_-(x, \lambda), \quad \lambda \in \mathbf{R}, \quad (5.6)$$

where $m_+(x, \cdot)$ and $m_+(x, \cdot)^{-1}$ belong to \mathcal{W}_+^{2n} , $m_-(x, \cdot)$ and $m_-(x, \cdot)^{-1}$ belong to \mathcal{W}_-^{2n} , $m_+(x, \lambda)$ and $m_+(x, \lambda)^{-1}$ tend to I_{2n} as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$, and $m_-(x, \lambda)$ and $m_-(x, \lambda)^{-1}$ tend to I_{2n} as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^-}$.

PROOF. From (3.8), (3.9), and (5.2), we get

$$\mathbf{G}(x, \lambda) = \begin{bmatrix} T_l(\lambda) & 0 \\ 0 & T_r(\lambda) \end{bmatrix} \begin{bmatrix} I_n & a_{l2}(\lambda) e^{2i\lambda x} \\ a_{r3}(\lambda) e^{-2i\lambda x} & I_n \end{bmatrix},$$

where the right factor is easily seen to have a positive selfadjoint real part when $\lambda \in \mathbf{R}$. The left factor and its inverse belong to \mathcal{W}_+^{2n} and tend to I_{2n} as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^+}$. Thus the right factor has [19] a unique left canonical Wiener-Hopf factorization of the form

$$\begin{bmatrix} I_n & a_{l2}(\lambda) e^{2i\lambda x} \\ a_{r3}(\lambda) e^{-2i\lambda x} & I_n \end{bmatrix} = \mathbf{W}_+(\lambda) \mathbf{W}_-(\lambda), \quad \lambda \in \mathbf{R},$$

where $\mathbf{W}_\pm(\lambda)$ and $\mathbf{W}_\pm(\lambda)^{-1}$ are continuous in $\lambda \in \overline{\mathbf{C}^\pm}$, are analytic in $\lambda \in \mathbf{C}^\pm$, and tend to I_{2n} as $\lambda \rightarrow \infty$ in $\overline{\mathbf{C}^\pm}$. Putting

$$m_+(x, \lambda) = \mathbf{W}_+(\lambda)^{-1} \begin{bmatrix} T_l(\lambda)^{-1} & 0 \\ 0 & T_r(\lambda)^{-1} \end{bmatrix}, \quad m_-(x, \lambda) = \mathbf{W}_-(\lambda),$$

completes the proof. ■

The matrix $V(x)$ appearing in (1.1) can be recovered from the scattering matrix $\mathbf{S}(\lambda)$ as follows. First, construct the matrix function $\mathbf{G}(x, \lambda)$ as in (5.2) and compute the Wiener-Hopf factors of $\mathbf{G}(x, \lambda)$ as in (5.6). Then $V(x)$ is given by

$$V(x) = -iJ_{2n} m'_\pm(x, 0) m_\pm(x, 0)^{-1}. \quad (5.7)$$

Indeed, from (1.1) using (2.41) and (2.42) we easily derive

$$-iJ_{2n} m'_\pm(x, \lambda) - V(x) m_\pm(x, \lambda) = \lambda [m_\pm(x, \lambda) - J_{2n} m_\pm(x, \lambda) J_{2n}],$$

which implies (5.7). To make (5.7) a viable way of computing the potential $k(x)$ from the Wiener-Hopf factors of (5.2), one still needs to prove that $m_{\pm}(x, 0)$ are absolutely continuous and that the entries of the $2n \times 2n$ matrix on the right-hand side in (5.7) belong to $L^1(\mathbf{R})$.

6. THE MARCHENKO METHOD

In order to establish the connection between the Riemann-Hilbert problem (5.1) and the Marchenko integral equations we first express the scattering coefficients in terms of their Fourier transforms as

$$R(\lambda) = \int_{-\infty}^{\infty} d\alpha \hat{R}(\alpha) e^{-i\lambda\alpha}, \quad L(\lambda) = \int_{-\infty}^{\infty} d\alpha \hat{L}(\alpha) e^{-i\lambda\alpha}, \quad (6.1)$$

$$T_l(\lambda) = I_n + \int_{-\infty}^{\infty} d\alpha \nu_l(\alpha) e^{i\lambda\alpha}, \quad T_r(\lambda) = I_n + \int_{-\infty}^{\infty} d\alpha \nu_r(\alpha) e^{i\lambda\alpha}. \quad (6.2)$$

Note that by Theorem 4.3, $\nu_l(\alpha)$ and $\nu_r(\alpha)$ vanish for $\alpha < 0$ and their entries belong to $L^1(\mathbf{R}^+)$, while the entries of $\hat{R}(\cdot)$ and $\hat{L}(\cdot)$ belong to $L^1(\mathbf{R})$. Let us define

$$g(x, \alpha) = \begin{bmatrix} 0 & -\hat{R}(2x + \alpha) \\ -\hat{L}(-2x + \alpha) & 0 \end{bmatrix}, \quad \alpha > 0. \quad (6.3)$$

THEOREM 6.1. *For each $x \in \mathbf{R}$ the matrices $b_-(x, \cdot)$ and $b_+(x, \cdot)$ defined in (4.4) satisfy the $2n \times 2n$ systems of coupled Marchenko equations*

$$b_-(x, \alpha) = g(x, \alpha) + \int_0^{\infty} d\beta b_+(x, \beta) g(x, \alpha + \beta), \quad \alpha > 0, \quad (6.4)$$

$$b_+(x, \alpha) = g(x, \alpha)^\dagger + \int_0^{\infty} d\beta b_-(x, \beta) g(x, \alpha + \beta)^\dagger, \quad \alpha > 0. \quad (6.5)$$

PROOF. Using (4.2), (5.1), and the fact that $b_+(x, \alpha) = b_-(x, \alpha) = 0$ for $\alpha < 0$, we get

$$m_+(x, \lambda) [\mathbf{G}(x, \lambda) - I_{2n}] = \int_{-\infty}^{\infty} d\alpha [b_-(x, \alpha) - b_+(x, -\alpha)] e^{-i\lambda\alpha}, \quad \lambda \in \mathbf{R}. \quad (6.6)$$

Furthermore, from (5.2) and Theorem 4.2 we conclude that

$$\mathbf{G}(x, \lambda) - I_{2n} = \int_{-\infty}^{\infty} d\alpha \mathbf{H}(\alpha) e^{i\lambda\alpha}, \quad \lambda \in \mathbf{R}, \quad (6.7)$$

where

$$\mathbf{H}(\alpha) = \begin{bmatrix} \nu_l(\alpha) & -\hat{R}(2x - \alpha) \\ -\hat{L}(-2x - \alpha) & \nu_r(\alpha) \end{bmatrix}, \quad \alpha \in \mathbf{R}. \quad (6.8)$$

Upon writing

$$m_+(x, \lambda) [\mathbf{G}(x, \lambda) - I_{2n}] = [\mathbf{G}(x, \lambda) - I_{2n}] + [m_+(x, \lambda) - I_{2n}] [\mathbf{G}(x, \lambda) - I_{2n}],$$

by using (6.6) on the left-hand side, (4.2), (6.1)-(6.3), (6.7), and (6.8) on the right-hand side, together with the convolution theorem, we obtain (6.4). Similarly, using

$$m_-(x, \lambda) [\mathbf{G}(x, \lambda)^\dagger - I_{2n}] = [\mathbf{G}(x, \lambda)^\dagger - I_{2n}] + [m_-(x, \lambda) - I_{2n}] [\mathbf{G}(x, \lambda)^\dagger - I_{2n}],$$

we obtain (6.5). ■

Using (6.4) in (6.5) and vice versa, we can uncouple these $2n \times 2n$ systems. Using the notations in (4.4), this leads to the uncoupled $n \times n$ Marchenko equations for $\alpha > 0$ given by

$$B_{l2}(x, \alpha) = -\hat{R}(\alpha + 2x) + \int_0^\infty d\beta \int_0^\infty d\gamma B_{l2}(x, \gamma) \hat{R}(\beta + \gamma + 2x)^\dagger \hat{R}(\alpha + \beta + 2x), \quad (6.9)$$

$$B_{l3}(x, \alpha) = -\hat{R}(\alpha + 2x)^\dagger + \int_0^\infty d\beta \int_0^\infty d\gamma B_{l3}(x, \gamma) \hat{R}(\beta + \gamma + 2x) \hat{R}(\alpha + \beta + 2x)^\dagger, \quad (6.10)$$

$$B_{r2}(x, \alpha) = -\hat{L}(\alpha - 2x)^\dagger + \int_0^\infty d\beta \int_0^\infty d\gamma B_{r2}(x, \gamma) \hat{L}(\beta + \gamma - 2x) \hat{L}(\alpha + \beta - 2x)^\dagger, \quad (6.11)$$

$$B_{r3}(x, \alpha) = -\hat{L}(\alpha - 2x) + \int_0^\infty d\beta \int_0^\infty d\gamma B_{r3}(x, \gamma) \hat{L}(\beta + \gamma - 2x)^\dagger \hat{L}(\alpha + \beta - 2x), \quad (6.12)$$

$$B_{l1}(x, \alpha) = \int_0^\infty d\beta \hat{R}(\beta + 2x) \hat{R}(\alpha + \beta + 2x)^\dagger \quad (6.13)$$

$$+ \int_0^\infty d\beta \int_0^\infty d\gamma B_{l1}(x, \gamma) \hat{R}(\beta + \gamma + 2x) \hat{R}(\alpha + \beta + 2x)^\dagger,$$

$$B_{l4}(x, \alpha) = \int_0^\infty d\beta \hat{R}(\beta + 2x)^\dagger \hat{R}(\alpha + \beta + 2x) \quad (6.14)$$

$$+ \int_0^\infty d\beta \int_0^\infty d\gamma B_{l4}(x, \gamma) \hat{R}(\beta + \gamma + 2x)^\dagger \hat{R}(\alpha + \beta + 2x),$$

$$B_{r1}(x, \alpha) = \int_0^\infty d\beta \hat{L}(\beta - 2x)^\dagger \hat{L}(\alpha + \beta - 2x) \quad (6.15)$$

$$+ \int_0^\infty d\beta \int_0^\infty d\gamma B_{r1}(x, \gamma) \hat{L}(\beta + \gamma - 2x)^\dagger \hat{L}(\alpha + \beta - 2x),$$

$$\begin{aligned}
B_{r4}(x, \alpha) &= \int_0^\infty d\beta \hat{L}(\beta - 2x) \hat{L}(\alpha + \beta - 2x)^\dagger \\
&+ \int_0^\infty d\beta \int_0^\infty d\gamma B_{r4}(x, \gamma) \hat{L}(\beta + \gamma - 2x) \hat{L}(\alpha + \beta - 2x)^\dagger.
\end{aligned} \tag{6.16}$$

THEOREM 6.2. *The coupled system of Marchenko integral equations (6.4) and (6.5) is uniquely solvable in $L^1(\mathbf{R}^+; \mathbf{C}^{2n \times 2n})$. The integral operator in each of the eight uncoupled Marchenko equations (6.9)-(6.16) is selfadjoint, and each of these eight equations is uniquely solvable in $L^1(\mathbf{R}^+; \mathbf{C}^{n \times n})$.*

PROOF. The selfadjointness of the integral operators in (6.9)-(6.16) is clear. From (3.12), (3.13), and Corollary 3.2 it follows that

$$\sup_{\lambda \in \mathbf{R}} \|L(\lambda)\| < 1, \quad \sup_{\lambda \in \mathbf{R}} \|R(\lambda)\| < 1. \tag{6.17}$$

Now observe that, for fixed $x \in \mathbf{R}$, the action of the integral operators with kernels $\hat{R}(\alpha + \beta + 2x)$, $\hat{R}(\alpha + \beta + 2x)^\dagger$, $\hat{L}(\alpha + \beta - 2x)$, and $\hat{L}(\alpha + \beta - 2x)^\dagger$ on $L^2(\mathbf{R}^+; \mathbf{C}^n)$ can be interpreted as follows: one imbeds $L^2(\mathbf{R}^+; \mathbf{C}^n)$ into $L^2(\mathbf{R}; \mathbf{C}^n)$ isometrically, applies the sign flip $h(\beta) \mapsto h(-\beta)$, implements a convolution with an L^1 -matrix function, and then projects orthogonally onto $L^2(\mathbf{R}^+; \mathbf{C}^n)$. Since the Fourier transforms of these matrix functions have norms strictly less than one, this is also the case for the norms of these integral operators. Hence, the system of equations (6.4) and (6.5) as well as each of the eight equations (6.9)-(6.16) are uniquely solvable on the direct sum of a suitable number of copies of $L^2(\mathbf{R}^+)$. Since, as a result of the integrability of $\hat{L}(\cdot)$ and $\hat{R}(\cdot)$, the integral operators are compact on both L^2 and L^1 (cf. Lemma XII 2.4 of [23], the proof for the L^2 -case there can easily be adapted to cover the L^1 -case; also p. 401 of [18]), the system of equations (6.4) and (6.5) as well as each of the eight equations (6.9)-(6.16) are uniquely solvable on the direct sum of a suitable number of copies of $L^1(\mathbf{R}^+)$. ■

From (4.19), we see that we can recover the potential $k(x)$ by solving any one of the four Marchenko equations (6.9)-(6.12).

The unique solvability of the Marchenko equations (6.9)-(6.16) has a number of other consequences. For example, if $R(\lambda)$ is analytic on \mathbf{C}^+ , then $\hat{R}(\alpha)$ is supported on \mathbf{R}^- and hence the right-hand sides in (6.9), (6.10), (6.13), and (6.14) vanish for $x > 0$. Since these equations are uniquely solvable, their solutions vanish as well and therefore $k(x) = 0$ for $x > 0$. On the other hand, if $L(\lambda)$ is analytic on \mathbf{C}^+ , then $\hat{L}(\alpha)$ is supported on \mathbf{R}^- , and hence the right-hand sides in (6.11), (6.12), (6.15), and (6.16) vanish for $x < 0$. Since these equations are uniquely solvable, their solutions vanish as well, and therefore $k(x) = 0$ for $x < 0$. We have thus proved the converse of Proposition 3.3.

It remains to prove that the potential $k(x)$ obtained by the Marchenko method has entries in $L^1(\mathbf{R})$. To do so, we modify the inversion procedure as follows. We solve one of the Marchenko equations (6.9) and (6.10) for $x > 0$ and then employ (4.19) to compute $k(x)$ for $x > 0$. By the same token, we solve one of (6.11) and (6.12) for $x < 0$ and then use (4.19) to find $k(x)$ for $x < 0$. In fact, this procedure will be implemented in the case of rational reflection coefficients in Section 8.

We first derive the following partial characterization result.

THEOREM 6.3. *Let $R(\lambda)$ be a matrix function in \mathcal{W}^n such that*

$$\sup_{\lambda \in \mathbf{R}} \|R(\lambda)\| < 1, \quad \int_0^\infty d\alpha \left(\|\hat{R}(\alpha)\| + \alpha \|\hat{R}(\alpha)\|^2 \right) < +\infty, \quad (6.18)$$

where $\hat{R}(\alpha)$ is defined in (6.1). Then, for $x > 0$, the unique solutions $B_{l2}(x, \alpha)$ and $B_{l3}(x, \alpha)$ of (6.9) and (6.10), respectively, satisfy

$$\int_0^\infty dx \|B_{lj}(x, 0^+)\| < +\infty, \quad j = 2, 3.$$

In particular, the entries of $k(x) = 2i B_{l2}(x, 0^+)$ and $k(x) = 2i B_{l3}(x, 0^+)^{\dagger}$ belong to $L^1(\mathbf{R}^+)$. Similarly, let $L(\lambda)$ be a matrix function in \mathcal{W}^n such that

$$\sup_{\lambda \in \mathbf{R}} \|L(\lambda)\| < 1, \quad \int_{-\infty}^0 d\alpha \left(\|\hat{L}(\alpha)\| - \alpha \|\hat{L}(\alpha)\|^2 \right) < +\infty, \quad (6.19)$$

where $\hat{L}(\alpha)$ is defined in (6.1). Then for $x < 0$ the unique solutions $B_{r2}(x, \alpha)$ and $B_{r3}(x, \alpha)$ of (6.11) and (6.12), respectively, satisfy

$$\int_{-\infty}^0 dx \|B_{rj}(x, 0^+)\| < +\infty, \quad j = 2, 3.$$

In particular, the entries of $k(x) = -2i B_{r2}(x, 0^+)$ and $k(x) = -2i B_{r3}(x, 0^+)^{\dagger}$ belong to $L^1(\mathbf{R}^-)$.

PROOF. We only prove the theorem for $x > 0$, as the proof for $x < 0$ is similar. Put

$$\hat{R}_{\Delta}(\alpha) = \begin{bmatrix} 0 & -\hat{R}(\alpha) \\ -\hat{R}(\alpha)^{\dagger} & 0 \end{bmatrix},$$

and consider the integral equation

$$B_l(x, \alpha) - \int_0^\infty d\beta B_l(x, \beta) \hat{R}_{\Delta}(2x + \alpha + \beta) = \hat{R}_{\Delta}(2x + \alpha), \quad \alpha > 0. \quad (6.20)$$

This integral equation, which follows directly from (4.4) and (6.3)-(6.5), has a unique solution in $L^1(\mathbf{R}^+; \mathbf{C}^{2n \times 2n})$ which coincides with the matrix function $B_l(x, \alpha)$ in (4.3). Moreover, the integral operator with kernel $R_\Delta(2x + \alpha + \beta)$ is selfadjoint and a strict contraction on $L^2(\mathbf{R}^+; \mathbf{C}^{2n \times 2n})$. Iterating the equation obtained by taking the adjoint of the matrices on either side of (6.20) we obtain

$$B_l(x, \alpha)^\dagger = \sum_{j=0}^{\infty} B_l^{(j)}(x, \alpha)^\dagger,$$

where $B_l^{(0)}(x, \alpha) = \hat{R}_\Delta(2x + \alpha)$ and

$$B_l^{(j)}(x, \alpha)^\dagger = \int_0^\infty d\beta \hat{R}_\Delta(2x + \alpha + \beta) B_l^{(j-1)}(x, \beta)^\dagger, \quad j \geq 1. \quad (6.21)$$

Now let $\rho \in [0, 1)$ be the spectral radius of the integral operator appearing in (6.21). Then, by the selfadjointness of this operator on $L^2(\mathbf{R}^+; \mathbf{C}^{2n \times 2n})$ and the Schwarz inequality, we get

$$\begin{aligned} \|B_l^{(j)}(x, \alpha)\| &\leq \left[\int_{2x+\alpha}^\infty d\gamma \|\hat{R}_\Delta(\gamma)\|^2 \right]^{1/2} \left[\int_0^\infty d\beta \|B_l^{(j-1)}(x, \beta)\|^2 \right]^{1/2} \\ &\leq \rho^{j-1} \int_{2x}^\infty d\beta \|\hat{R}_\Delta(\beta)\|^2, \quad j \geq 1, \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^\infty dx \|B_l(x, \alpha)\| &\leq \frac{1}{2} \left[\int_\alpha^\infty d\beta \|\hat{R}_\Delta(\beta)\| + \frac{1}{1-\rho} \int_0^\infty dy \int_{y+\alpha}^\infty d\beta \|\hat{R}_\Delta(\beta)\|^2 \right] \\ &\leq \frac{1}{2} \left[\int_\alpha^\infty d\beta \|\hat{R}_\Delta(\beta)\| + \frac{1}{1-\rho} \int_\alpha^\infty d\beta (\beta - \alpha) \|\hat{R}_\Delta(\beta)\|^2 \right], \end{aligned}$$

which is finite. ■

We note that the finiteness of $\int_0^\infty d\alpha \alpha \|\hat{R}(\alpha)\|^2$, which is implied by (6.18), is equivalent to $\hat{R}(\alpha + \beta)^\dagger$ being Hilbert-Schmidt on $L^2(\mathbf{R}^+; \mathbf{C}^n)$. In fact we have

$$\|\hat{R}\|_{H.S.}^2 = \int_0^\infty \int_0^\infty d\alpha d\beta \operatorname{tr} \{ \hat{R}(\alpha + \beta)^\dagger \hat{R}(\alpha + \beta) \}.$$

Thus, because $\|\hat{R}\|^2 \leq \operatorname{tr} \{ \hat{R}^\dagger \hat{R} \} \leq n \|\hat{R}\|^2$ and

$$\int_0^\infty d\alpha \alpha \|\hat{R}(\alpha)\|^2 = \int_0^\infty d\alpha \int_\alpha^\infty du \|\hat{R}(u)\|^2 = \int_0^\infty \int_0^\infty d\alpha d\beta \|\hat{R}(\alpha + \beta)\|^2,$$

the asserted equivalence follows. In a similar manner, from (6.19) it follows that $\hat{L}(-\alpha - \beta)$ is Hilbert-Schmidt on $L^2(\mathbf{R}^+; \mathbf{C}^n)$.

Note also that the assumptions of Theorem 6.3 are satisfied when $R(\lambda)$ and $L(\lambda)$ are rational matrix functions without real poles that vanish at infinity. This follows easily from the partial fraction decompositions for $R(\lambda)$ and $L(\lambda)$, respectively.

The natural conditions under which one would expect to be able to reconstruct a potential with L^1 -entries for $x > 0$ are $R \in \mathcal{W}^n$ and

$$\sup_{\lambda \in \mathbf{R}} \|R(\lambda)\| < 1, \quad \lim_{\lambda \rightarrow \pm\infty} \|R(\lambda)\| = 0. \quad (6.22)$$

However, evaluating the first iterate of (6.20) as $\alpha \rightarrow 0^+$, we get

$$B_l^{(1)}(x, 0^+) = \int_{2x}^{\infty} d\beta \hat{R}_{\Delta}(\beta)^2 = \begin{bmatrix} \int_{2x}^{\infty} d\beta \hat{R}(\beta) \hat{R}(\beta)^{\dagger} & 0 \\ 0 & \int_{2x}^{\infty} d\beta \hat{R}(\beta)^{\dagger} \hat{R}(\beta) \end{bmatrix},$$

which strongly suggests that condition (6.18) is probably indispensable if the integral $\int_0^{\infty} dx \|B_l^{(1)}(x, 0^+)\|$ is to be finite.

Assuming only the finiteness of $\int_0^{\infty} d\alpha \|\hat{R}(\alpha)\|$ and using the method of the proof of Theorem 2 in [34], one easily obtains the estimate

$$\int_{2x_0}^{\infty} dx \|B_l^{(2j)}(x, 0^+)\| \leq \frac{1}{2} \left(\int_{2x_0}^{\infty} d\beta \|\hat{R}(\beta)\| \right)^{j+1}, \quad j \geq 0,$$

for each fixed $x_0 \in \mathbf{R}$. Unfortunately, this estimate does not extend to the odd iterates of (6.20). However, since \hat{R}_{Δ} belongs to the class \mathbf{K}_{Δ} introduced in [34], it follows from this theorem that the potential $k(x)$ obtained has an L^1 -tail in the sense that

$$\exists x_0 > 0 : \quad \left(\int_{-\infty}^{-x_0} + \int_{x_0}^{\infty} \right) dx \|k(x)\| < +\infty.$$

Moreover, for every $\lambda \in \mathbf{R}$ the Jost solution $F_l(x, \lambda)$ is differentiable with respect to x if $x > x_0$ and $F_r(x, \lambda)$ is differentiable with respect to x for $x < -x_0$. In other words, neither in the work of Melik-Adamyman [34] for the half line nor in the present work for the full line, a complete characterization is given of the scattering data leading to a unique L^1 -potential. One does not obtain such a characterization either if one combines Melik-Adamyman's reduction of the inverse problem on the full line to that on the half line [32] with his solution of the inverse problem on the half line [34].

7. CONSTRUCTION OF THE SCATTERING MATRIX

Throughout this section we assume that $R(\lambda)$ is a rational matrix function satisfying (6.22). We recall that then $R \in \mathcal{W}^n$ by the comments following the proof of Theorem 6.3. From the theory of transfer functions [13], since $R(\lambda) \rightarrow 0$ as $\lambda \rightarrow \pm\infty$, it follows that $R(\lambda)$ can be represented in the form

$$R(\lambda) = iC(\lambda - iA)^{-1}B, \quad \lambda \in \mathbb{C}, \quad (7.1)$$

where A , B , and C are independent of λ and belong to $\mathbb{C}^{p \times p}$, $\mathbb{C}^{p \times n}$, and $\mathbb{C}^{n \times p}$, respectively, for some positive integer p . Here it is assumed that the order p of A is minimal, i.e. the realization (7.1) is minimal and hence unique up to similarity (cf. Theorems 6.1.4 and 6.1.5 in [31]).

Our goal is to construct $S(\lambda)$ in terms of the matrices A , B , and C given in (7.1). Since $R(\lambda)$ is continuous for $\lambda \in \mathbb{R}$, from the minimality of the realization given in (7.1) it follows that A does not have any eigenvalues on the imaginary axis (cf. Theorem 6.2.2 of [31]). Using (7.1) in (5.3) and (5.4), we obtain

$$T_l(\lambda) T_l(\lambda^*)^\dagger = I_n - i[C \ 0](\lambda - iK_l)^{-1} \begin{bmatrix} 0 \\ C^\dagger \end{bmatrix}, \quad (7.2)$$

$$T_r(\lambda^*)^\dagger T_r(\lambda) = I_n + i[0 \ B^\dagger](\lambda - iK_r)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad (7.3)$$

where

$$K_l = \begin{bmatrix} A & -BB^\dagger \\ 0 & -A^\dagger \end{bmatrix}, \quad K_r = \begin{bmatrix} A & 0 \\ C^\dagger C & -A^\dagger \end{bmatrix}. \quad (7.4)$$

Note that the inverses of the right-hand sides in (7.2) and (7.3) can be written as

$$[T_l(\lambda^*)^\dagger]^{-1} T_l(\lambda)^{-1} = I_n + i[C \ 0](\lambda - i\mathcal{E})^{-1} \begin{bmatrix} 0 \\ C^\dagger \end{bmatrix}, \quad (7.5)$$

$$T_r(\lambda)^{-1} [T_r(\lambda^*)^\dagger]^{-1} = I_n - i[0 \ B^\dagger](\lambda - i\mathcal{E})^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad (7.6)$$

where \mathcal{E} is the “state characteristic matrix” given by

$$\mathcal{E} = \begin{bmatrix} A & -BB^\dagger \\ C^\dagger C & -A^\dagger \end{bmatrix}, \quad (7.7)$$

which, apart from some factors $i = \sqrt{-1}$, has been used in [27]. We note that \mathcal{K}_l , \mathcal{K}_r , and \mathcal{E} do not have eigenvalues on the imaginary axis. This follows from the invertibility of $I_n - R(\lambda)R(\lambda)^\dagger$ and Corollary 2.7 in [13]; for \mathcal{K}_l and \mathcal{K}_r this also follows immediately from the special form of the matrices \mathcal{K}_l and \mathcal{K}_r in (7.4) and the fact that \mathcal{A} has no eigenvalues on the imaginary axis. Hence the matrices $(\lambda - i\mathcal{K}_l)^{-1}$, $(\lambda - i\mathcal{K}_r)^{-1}$, and $(\lambda - i\mathcal{E})^{-1}$ in (7.2), (7.3), (7.5), and (7.6) all exist for $\lambda \in \mathbf{R}$.

The contractivity of $R(\lambda)$ and $R(\lambda)^\dagger$ for $\lambda \in \mathbf{R}$ given in (6.17) implies the following result (Theorems 3.2 and 3.4 of [27]):

PROPOSITION 7.1. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be the matrices in the minimal realization given by (7.1) and consider the quadratic matrix equations*

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^\dagger = \mathcal{B}\mathcal{B}^\dagger + \mathcal{X}\mathcal{C}^\dagger\mathcal{C}\mathcal{X}, \quad (7.8)$$

$$\mathcal{A}^\dagger\mathcal{Y} + \mathcal{Y}\mathcal{A} = \mathcal{C}^\dagger\mathcal{C} + \mathcal{Y}\mathcal{B}\mathcal{B}^\dagger\mathcal{Y}. \quad (7.9)$$

Then all hermitian solutions \mathcal{X} of (7.8) are nonsingular, and the number of positive (resp. negative) eigenvalues of \mathcal{X} coincides with the number of poles of $R(\lambda)$ in \mathbf{C}^+ (resp. in \mathbf{C}^-). There is at least one such solution \mathcal{X} . An analogous result holds for hermitian solutions of (7.9).

The nonlinear equations (7.8) and (7.9) are called state characteristic equations in [27] and (continuous algebraic) Riccati equations elsewhere in the literature (e.g. [31]). Since in the literature the term “hermitian” (instead of “selfadjoint”) seems to have some tradition when referring to solutions of Riccati equations, we will use this terminology here. We also remark that in counting the number of poles and eigenvalues, (algebraic) multiplicities have been taken into account. The following result is essential for obtaining explicit expressions for the factors $T_l(\lambda)$ and $T_r(\lambda)$ and their inverses.

PROPOSITION 7.2. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be the matrices in the minimal realization given by (7.1). Then the spectrum of the matrix \mathcal{E} given in (7.7) is symmetric about the imaginary axis. Moreover, the spectral subspace \mathcal{M} of \mathcal{E} corresponding to its eigenvalues in the right half plane is of the form*

$$\mathcal{M} = \left\{ \begin{bmatrix} \mathcal{X} \\ I_p \end{bmatrix} u : u \in \mathbf{C}^p \right\}, \quad (7.10)$$

where \mathcal{X} is a hermitian solution of (7.8), and the spectral subspace \mathcal{L} of \mathcal{E} corresponding to its eigenvalues in the left half plane is of the form

$$\mathcal{L} = \left\{ \begin{bmatrix} I_p \\ \mathcal{Y} \end{bmatrix} u : u \in \mathbf{C}^p \right\}, \quad (7.11)$$

where \mathcal{Y} is a hermitian solution of (7.9). The hermitian matrices \mathcal{X} and \mathcal{Y} are unique.

PROOF. The symmetry of the spectrum of \mathcal{E} about the imaginary axis follows from the similarity $J_{2p}\mathbf{q}_{2p}\mathcal{E}\mathbf{q}_{2p}J_{2p} = -\mathcal{E}^\dagger$, where \mathbf{q}_{2p} is defined by (1.10). The remaining assertions follow from Theorem 7.6.1 in [31] applied to the matrix $J_{2p}\mathcal{E}J_{2p}$ (to comply with the condition $D \geq 0$ there) and the Ξ -neutrality, where $\Xi = iJ_{2p}\mathbf{q}_{2p}$, of the spectral subspaces \mathcal{M} and \mathcal{L} . Note that the spectral subspaces \mathcal{M} and \mathcal{L} both have dimension p , which is the order of \mathcal{A} , because \mathcal{E} has no eigenvalues on the imaginary axis. ■

We remark that the subspace \mathcal{M} can be written in the same form as \mathcal{L} by setting $\mathcal{Y} = \mathcal{X}^{-1}$, since the hermitian solutions of (7.8) are related to those of (7.9) via the substitution $\mathcal{Y} = \mathcal{X}^{-1}$. The subspaces \mathcal{M} and \mathcal{L} are called graph subspaces in the terminology of [31].

The matrices \mathcal{X} and \mathcal{Y} used in Proposition 7.2 allow us to block diagonalize the matrix \mathcal{E} . Since the subspaces \mathcal{L} and \mathcal{M} have dimension p and $\mathcal{M} \cap \mathcal{L} = \{0\}$, the matrix Σ defined by

$$\Sigma = \begin{bmatrix} I_p & \mathcal{X} \\ \mathcal{Y} & I_p \end{bmatrix} \quad (7.12)$$

is nonsingular. Hence, both $I_p - \mathcal{X}\mathcal{Y}$ and $I_p - \mathcal{Y}\mathcal{X}$ are nonsingular, and

$$\Sigma^{-1} = \begin{bmatrix} (I_p - \mathcal{X}\mathcal{Y})^{-1} & -(I_p - \mathcal{X}\mathcal{Y})^{-1}\mathcal{X} \\ -(I_p - \mathcal{Y}\mathcal{X})^{-1}\mathcal{Y} & (I_p - \mathcal{Y}\mathcal{X})^{-1} \end{bmatrix}. \quad (7.13)$$

THEOREM 7.3. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be the matrices in the minimal realization given by (7.1) and let \mathcal{X} and \mathcal{Y} be as in Proposition 7.2. Then

$$\Sigma^{-1}\mathcal{E}\Sigma = \begin{bmatrix} \mathcal{E}_r & 0 \\ 0 & -\mathcal{E}_l^\dagger \end{bmatrix}, \quad (7.14)$$

where

$$\mathcal{E}_r = \mathcal{A} - \mathcal{B}\mathcal{B}^\dagger\mathcal{Y}, \quad \mathcal{E}_l = \mathcal{A} - \mathcal{X}\mathcal{C}^\dagger\mathcal{C}. \quad (7.15)$$

Moreover, the matrices \mathcal{E}_r and \mathcal{E}_l have all their eigenvalues in the left half complex plane and are related via the similarity transformation

$$\mathcal{E}_r = (I_p - \mathcal{X}\mathcal{Y})^{-1}\mathcal{E}_l(I_p - \mathcal{X}\mathcal{Y}). \quad (7.16)$$

PROOF. The relations (7.14)-(7.16) follow by direct computation using (7.7)-(7.9), (7.12), and (7.13). The assertions about the spectra of \mathcal{E}_r and \mathcal{E}_l follow from (7.14) and

Proposition 7.2 which imply that $\mathcal{E} \upharpoonright \mathcal{L}$ is similar to \mathcal{E}_r and $\mathcal{E} \upharpoonright \mathcal{M}$ is similar to $-\mathcal{E}_l^\dagger$. Here the symbol \upharpoonright denotes the restriction of \mathcal{E} to the subspace \mathcal{L} (resp. \mathcal{M}). ■

In the following we also need representations of the form (7.10) and (7.11) for certain invariant subspaces of \mathcal{K}_l and \mathcal{K}_r .

PROPOSITION 7.4. *Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be the matrices in the minimal realization given by (7.1). Then the spectrum of \mathcal{K}_l (\mathcal{K}_r) is symmetric about the imaginary axis and the invariant spectral subspaces corresponding to the left and right half planes both have dimension p . In the case of \mathcal{K}_r both invariant subspaces are of the form*

$$\left\{ \begin{bmatrix} \tilde{\mathcal{X}} \\ I_p \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \quad (7.17)$$

where $\tilde{\mathcal{X}}$ is a solution of the Riccati equation

$$\mathcal{A}\tilde{\mathcal{X}} + \tilde{\mathcal{X}}\mathcal{A}^\dagger = \tilde{\mathcal{X}}\mathcal{C}^\dagger\mathcal{C}\tilde{\mathcal{X}}. \quad (7.18)$$

In the case of \mathcal{K}_l both invariant subspaces are of the form

$$\left\{ \begin{bmatrix} I_p \\ \tilde{\mathcal{Y}} \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \quad (7.19)$$

where $\tilde{\mathcal{Y}}$ is a hermitian solution of the Riccati equation

$$\mathcal{A}^\dagger\tilde{\mathcal{Y}} + \tilde{\mathcal{Y}}\mathcal{A} = \tilde{\mathcal{Y}}\mathcal{B}\mathcal{B}^\dagger\tilde{\mathcal{Y}}. \quad (7.20)$$

PROOF. Apply Theorem 7.2.4 of [31] to $J_{2p}\mathcal{K}_lJ_{2p}$ and $\mathbf{q}_{2p}\mathcal{K}_r\mathbf{q}_{2p}$. The symmetry of the spectrum about the imaginary axis follows as in the proof of Proposition 7.2 for \mathcal{E} . ■

Let us mention that unlike in Proposition 7.1 the matrices $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ in (7.17) and (7.19), respectively, may be singular.

Before we can apply the main factorization result from [13] to (7.2) and (7.3), we need the following proposition ([36], Theorem 6.5.3 of [31]).

PROPOSITION 7.5. *Let \mathcal{L} (resp. \mathcal{M}) be the invariant subspace of the matrix \mathcal{E} given in (7.7) corresponding to the eigenvalues in the left (resp. right) half plane, and let \mathcal{N} (resp. \mathcal{V}) be the invariant subspace of \mathcal{K}_r (resp. \mathcal{K}_l) corresponding to its eigenvalues in the right (resp. left) half plane. Then $\mathcal{L} \oplus \mathcal{N} = \mathbb{C}^{2p}$ and $\mathcal{M} \oplus \mathcal{V} = \mathbb{C}^{2p}$.*

PROOF. First, it follows from Theorem 19.3.1 of [31] (together with the rotation $\lambda \mapsto i\lambda$) that the matrix functions on the right-hand sides of (7.2) and (7.3) admit canonical

factorizations with respect to the real line. Then Theorem 19.1.2 of [31] (or Theorem 6.1 of [23]) implies that the pairs $\{\mathcal{L}, \mathcal{N}\}$ and $\{\mathcal{M}, \mathcal{V}\}$ each form a direct decomposition of \mathbb{C}^{2p} . ■

Now let Π be the projection such that

$$\text{Im } \Pi = \mathcal{L}, \quad \text{Ker } \Pi = \mathcal{N}, \quad (7.21)$$

and let \mathcal{Q} be the projection such that

$$\text{Im } \mathcal{Q} = \mathcal{V}, \quad \text{Ker } \mathcal{Q} = \mathcal{M}. \quad (7.22)$$

Applying Theorem 1.5 of [13] we can express the transmission coefficients in terms of the matrices appearing in (7.1) and the projections Π and \mathcal{Q} as follows:

$$T_r(\lambda^*)^\dagger = I_n + i \begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix} (\lambda - i\mathcal{K}_r)^{-1} (I_{2p} - \Pi) \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \quad (7.23)$$

$$T_r(\lambda) = I_n + i \begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix} \Pi (\lambda - i\mathcal{K}_r)^{-1} \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \quad (7.24)$$

$$T_r(\lambda)^{-1} = I_n - i \begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix} (\lambda - i\mathcal{E})^{-1} \Pi \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \quad (7.25)$$

$$[T_r(\lambda^*)^\dagger]^{-1} = I_n - i \begin{bmatrix} 0 & \mathcal{B}^\dagger \end{bmatrix} (I_{2p} - \Pi) (\lambda - i\mathcal{E})^{-1} \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \quad (7.26)$$

$$T_l(\lambda) = I_n - i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (\lambda - i\mathcal{K}_l)^{-1} \mathcal{Q} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \quad (7.27)$$

$$T_l(\lambda^*)^\dagger = I_n - i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (I_{2p} - \mathcal{Q}) (\lambda - i\mathcal{K}_l)^{-1} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \quad (7.28)$$

$$[T_l(\lambda^*)^\dagger]^{-1} = I_n + i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} (\lambda - i\mathcal{E})^{-1} (I_{2p} - \mathcal{Q}) \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \quad (7.29)$$

$$T_l(\lambda)^{-1} = I_n + i \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} \mathcal{Q} (\lambda - i\mathcal{E})^{-1} \begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}. \quad (7.30)$$

With the expressions (7.23)-(7.30) we have accomplished the desired canonical factorizations of the matrix functions on the right-hand sides of (7.2), (7.3), (7.5), and (7.6). Our

next goal is to find more explicit representations for the projections Π and \mathcal{Q} and for the invariant subspaces \mathcal{N} and \mathcal{V} .

PROPOSITION 7.6. *Let \mathcal{X} and \mathcal{Y} be as in Proposition 7.2 and let $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_+$ and $\tilde{\mathcal{Y}} = \tilde{\mathcal{Y}}_-$ be as in Proposition 7.4, where the subscript $+$ ($-$) indicates that the spectral subspaces given in (7.17) and (7.19) are those associated with the right (left) half plane. Then the invariant subspaces \mathcal{N} and \mathcal{V} and the projections Π and \mathcal{Q} can be expressed as*

$$\mathcal{N} = \left\{ \begin{bmatrix} \tilde{\mathcal{X}}_+ \\ I_p \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \quad \mathcal{V} = \left\{ \begin{bmatrix} I_p \\ \tilde{\mathcal{Y}}_- \end{bmatrix} u : u \in \mathbb{C}^p \right\}, \quad (7.31)$$

$$\Pi = \begin{bmatrix} (I_p - \tilde{\mathcal{X}}_+ \mathcal{Y})^{-1} & -(I_p - \tilde{\mathcal{X}}_+ \mathcal{Y})^{-1} \tilde{\mathcal{X}}_+ \\ \mathcal{Y}(I_p - \tilde{\mathcal{X}}_+ \mathcal{Y})^{-1} & -\mathcal{Y}(I_p - \tilde{\mathcal{X}}_+ \mathcal{Y})^{-1} \tilde{\mathcal{X}}_+ \end{bmatrix}, \quad (7.32)$$

$$\mathcal{Q} = \begin{bmatrix} (I_p - \mathcal{X} \tilde{\mathcal{Y}}_-)^{-1} & -(I_p - \mathcal{X} \tilde{\mathcal{Y}}_-)^{-1} \mathcal{X} \\ \tilde{\mathcal{Y}}_-(I_p - \mathcal{X} \tilde{\mathcal{Y}}_-)^{-1} & -\tilde{\mathcal{Y}}_-(I_p - \mathcal{X} \tilde{\mathcal{Y}}_-)^{-1} \mathcal{X} \end{bmatrix}. \quad (7.33)$$

Furthermore, if \mathcal{A} has all its eigenvalues in the left half plane, then $\tilde{\mathcal{X}}_+ = \tilde{\mathcal{Y}}_- = 0$ and

$$\mathcal{N} = \{0\} \oplus \mathbb{C}^p, \quad \mathcal{V} = \mathbb{C}^p \oplus \{0\}, \quad (7.34)$$

$$\Pi = \begin{bmatrix} I_p & 0 \\ \mathcal{Y} & 0 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} I_p & -\mathcal{X} \\ 0 & 0 \end{bmatrix}. \quad (7.35)$$

PROOF. First, (7.31) is an immediate consequence of (7.17), (7.19), (7.21), and (7.22). Then (7.32) and (7.33) follow from (7.21), (7.22), and Proposition 7.2. If \mathcal{A} has all its eigenvalues in the right half plane, then $\tilde{\mathcal{X}}_+ = \tilde{\mathcal{Y}}_- = 0$, by the particular form of \mathcal{K}_l and \mathcal{K}_r in (7.4), and so (7.34) and (7.35) follow from (7.31)-(7.33). ■

In order to find more explicit expressions for Π and \mathcal{Q} when \mathcal{A} has at least one eigenvalue in the right half plane, we employ suitable similarity transformations which bring the images of \mathcal{K}_l and \mathcal{K}_r in a form amenable to the same treatment as if \mathcal{A} had only eigenvalues in the left half plane. To set up these similarity transformations it is convenient to choose a basis such that \mathcal{A} , \mathcal{B} , and \mathcal{C} are partitioned as

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_- & 0 \\ 0 & \mathcal{A}_+ \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_- \\ \mathcal{B}_+ \end{bmatrix}, \quad \mathcal{C} = [\mathcal{C}_- \quad \mathcal{C}_+]. \quad (7.36)$$

Here \mathcal{A}_- (\mathcal{A}_+) has all its eigenvalues in the left (right) half plane and we denote its order by p_- (p_+), so that $p_- + p_+ = p$. Moreover, \mathcal{B}_\pm and \mathcal{C}_\pm are $p_\pm \times n$ and $n \times p_\pm$ matrices,

respectively. Now put

$$\Phi_l = \begin{bmatrix} I_{p-} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{p+} \\ 0 & 0 & I_{p-} & 0 \\ 0 & -I_{p+} & 0 & P_1 \end{bmatrix}, \quad \Phi_r = \begin{bmatrix} I_{p-} & 0 & 0 & 0 \\ 0 & -P_2 & 0 & I_{p+} \\ 0 & 0 & I_{p-} & 0 \\ 0 & -I_{p+} & 0 & 0 \end{bmatrix}, \quad (7.37)$$

where P_1 and P_2 are the unique solutions of the equations (cf. Theorem I4.1 of [23], Theorem VII.2.4 of [17])

$$\mathcal{A}_+ P_1 + P_1 \mathcal{A}_+^\dagger = \mathcal{B}_+ \mathcal{B}_+^\dagger, \quad (7.38)$$

$$P_2 \mathcal{A}_+ + \mathcal{A}_+^\dagger P_2 = \mathcal{C}_+^\dagger \mathcal{C}_+. \quad (7.39)$$

In fact, we have

$$P_1 = \int_0^\infty dt e^{-t\mathcal{A}_+} \mathcal{B}_+ \mathcal{B}_+^\dagger e^{-t\mathcal{A}_+^\dagger}, \quad P_2 = \int_0^\infty dt e^{-t\mathcal{A}_+^\dagger} \mathcal{C}_+^\dagger \mathcal{C}_+ e^{-t\mathcal{A}_+}, \quad (7.40)$$

so that P_1 and P_2 are positive selfadjoint. Then, we easily compute

$$\Omega_1 = \Phi_l \mathcal{K}_l \Phi_l^{-1} = \begin{bmatrix} \Omega_3 & -\Omega_5 \Omega_5^\dagger \\ 0 & -\Omega_3^\dagger \end{bmatrix}, \quad \Omega_2 = \Phi_r \mathcal{K}_r \Phi_r^{-1} = \begin{bmatrix} \Omega_4 & 0 \\ \Omega_6^\dagger \Omega_6 & -\Omega_4^\dagger \end{bmatrix}, \quad (7.41)$$

where

$$\Omega_3 = \begin{bmatrix} \mathcal{A}_- & -\mathcal{B}_- \mathcal{B}_+^\dagger \\ 0 & -\mathcal{A}_+^\dagger \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} \mathcal{A}_- & 0 \\ \mathcal{C}_+^\dagger \mathcal{C}_- & -\mathcal{A}_+^\dagger \end{bmatrix}, \quad \Omega_5 = \begin{bmatrix} \mathcal{B}_- \\ 0 \end{bmatrix}, \quad \Omega_6 = [\mathcal{C}_- \quad 0].$$

Note that all the eigenvalues of Ω_1 and Ω_2 lie in the open left half plane. Therefore, in analogy to (7.34), (7.35), and Proposition 7.2, the projection operators \mathcal{Q} and Π are such that

$$\text{Im } \mathcal{Q} = \Phi_l^{-1} [\mathbf{C}^p \oplus \{0\}], \quad \text{Ker } \mathcal{Q} = \text{Im } \begin{bmatrix} \mathcal{X} \\ I_p \end{bmatrix}, \quad (7.42)$$

$$\text{Ker } \Pi = \Phi_r^{-1} [\{0\} \oplus \mathbf{C}^p], \quad \text{Im } \Pi = \text{Im } \begin{bmatrix} I_p \\ \mathcal{Y} \end{bmatrix}. \quad (7.43)$$

Let us partition the inverses of Φ_l and Φ_r defined in (7.37) into $p \times p$ blocks as

$$\Phi_l^{-1} = \begin{bmatrix} \Lambda_{l1} & \Lambda_{l2} \\ \Lambda_{l3} & \Lambda_{l4} \end{bmatrix}, \quad \Phi_r^{-1} = \begin{bmatrix} \Lambda_{r1} & \Lambda_{r2} \\ \Lambda_{r3} & \Lambda_{r4} \end{bmatrix}. \quad (7.44)$$

Note that

$$\Phi_l^{-1} = \mathbf{q}_{2p} \Phi_l \mathbf{q}_{2p}, \quad \Phi_r^{-1} = \mathbf{q}_{2p} \Phi_r \mathbf{q}_{2p}, \quad (7.45)$$

so that the entries of Φ_l^{-1} and Φ_r^{-1} are readily available from (7.37).

PROPOSITION 7.7. *Suppose a basis is chosen such that the matrices \mathcal{A} , \mathcal{B} , and \mathcal{C} in the realization (7.1) have the form indicated in (7.36). Then the matrices \mathcal{X}_+ , \mathcal{Y}_- , Π , and \mathcal{Q} in Proposition 7.6 can be expressed as*

$$\tilde{\mathcal{Y}}_- = \Lambda_{l3} \Lambda_{l1}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & P_1^{-1} \end{bmatrix}, \quad \tilde{\mathcal{X}}_+ = \Lambda_{r2} \Lambda_{r4}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & P_2^{-1} \end{bmatrix}, \quad (7.46)$$

$$\mathcal{Q} = \begin{bmatrix} \Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1} & -\Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \\ \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1} & -\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \end{bmatrix}, \quad (7.47)$$

$$I_{2p} - \Pi = \begin{bmatrix} -\Lambda_{r2}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y} & \Lambda_{r2}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1} \\ -\Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y} & \Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1} \end{bmatrix}. \quad (7.48)$$

PROOF. It follows from (7.37), (7.42), (7.44), and (7.45) that

$$\mathcal{V} = \text{Im } \mathcal{Q} = \left\{ \begin{bmatrix} \Lambda_{l1} \\ \Lambda_{l3} \end{bmatrix} u : u \in \mathbb{C}^p \right\}.$$

Now (7.31), (7.37), (7.44), and (7.45) imply (7.46) for $\tilde{\mathcal{Y}}_-$. Similarly, by (7.31), (7.37), and (7.43)-(7.45), we have

$$\mathcal{N} = \text{Ker } \Pi = \left\{ \begin{bmatrix} \Lambda_{r2} \\ \Lambda_{r4} \end{bmatrix} u : u \in \mathbb{C}^p \right\},$$

and so, by comparison with (7.31), we obtain (7.46) for $\tilde{\mathcal{X}}_+$. Then (7.47) and (7.48) follow on using (7.46) in (7.32) and (7.33). In the derivation of (7.48) we have also used the identity $(I_p - \tilde{\mathcal{X}}_+\mathcal{Y})^{-1}\tilde{\mathcal{X}}_+ = \tilde{\mathcal{X}}_+(I_p - \mathcal{Y}\tilde{\mathcal{X}}_+)^{-1}$. ■

Note that in (7.48) we have stated the result for $I_{2p} - \Pi$ rather than Π because we will only need the former. By using (7.36) and (7.37) one easily verifies that $\tilde{\mathcal{X}}_+$ and $\tilde{\mathcal{Y}}_-$ given in (7.46) satisfy (7.18) and (7.20), respectively.

In order to use the results of Proposition 7.7 in (7.23)-(7.30) we need some additional notation. We decompose the solution \mathcal{X} of (7.8) as

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_3 & \mathcal{X}_4 \end{bmatrix},$$

so that \mathcal{X}_1 and \mathcal{X}_4 are selfadjoint and have orders p_- and p_+ , respectively, and $\mathcal{X}_2^\dagger = \mathcal{X}_3$. We denote by P_3 the unique solution of the equation

$$\Omega_4 P_3 + P_3 \mathcal{E}_l^\dagger = -\Lambda_{r4} \mathcal{B} \mathcal{B}^\dagger, \quad (7.49)$$

which is given by

$$P_3 = \int_0^\infty dt e^{i\Omega_4 t} \Lambda_{r4} \mathcal{B} \mathcal{B}^\dagger e^{i\mathcal{E}_l^\dagger t}, \quad (7.50)$$

and we define P_4 to be the unique (and generally nonsquare matrix) solution of the equation

$$\Omega_4 P_4 + P_4 \mathcal{A}_-^\dagger = \Lambda_{r4} \mathcal{B} \mathcal{B}_-^\dagger,$$

given by

$$P_4 = - \int_0^\infty dt e^{i\Omega_4 t} \Lambda_{r4} \mathcal{B} \mathcal{B}_-^\dagger e^{i\mathcal{A}_-^\dagger t}. \quad (7.51)$$

Note that in contrast to the solutions P_1 and P_2 of (7.38) and (7.39), respectively, the matrices P_3 and P_4 are in general not selfadjoint. Furthermore, we let

$$\mathcal{J}_l = (\Lambda_{l1} - \Lambda_{l3} \mathcal{X})^{-1} \Lambda_{l1}, \quad \mathcal{J}_r = \mathcal{Y} (\Lambda_{r4} - \Lambda_{r2} \mathcal{Y})^{-1}, \quad (7.52)$$

and introduce the matrices

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{\mathcal{A}}_+ & 0 \\ 0 & \tilde{\mathcal{A}}_- \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} \tilde{\mathcal{B}}_+ \\ \tilde{\mathcal{B}}_- \end{bmatrix}, \quad \tilde{\mathcal{C}} = [\tilde{\mathcal{C}}_+ \quad \tilde{\mathcal{C}}_-], \quad (7.53)$$

where

$$\tilde{\mathcal{A}}_+ = \begin{bmatrix} -\mathcal{E}_l^\dagger & 0 \\ 0 & -\mathcal{A}_-^\dagger \end{bmatrix}, \quad \tilde{\mathcal{A}}_- = \begin{bmatrix} -\mathcal{A}_+^\dagger & 0 \\ \Lambda_{r4} \mathcal{B} \mathcal{B}_+^\dagger & \Omega_4 \end{bmatrix}, \quad (7.54)$$

$$\tilde{\mathcal{B}}_+ = \begin{bmatrix} \mathcal{J}_l \mathcal{C}^\dagger \\ 0 \end{bmatrix}, \quad \tilde{\mathcal{B}}_- = \begin{bmatrix} (P_1 - \mathcal{X}_4)^{-1} (\mathcal{X}_3 \mathcal{C}_-^\dagger + \mathcal{X}_4 \mathcal{C}_+^\dagger) \\ P_3 \mathcal{J}_l \mathcal{C}^\dagger \end{bmatrix}, \quad (7.55)$$

$$\tilde{\mathcal{C}}_+ = [\mathcal{B}^\dagger (I_p + \mathcal{J}_r P_3) \quad \mathcal{B}^\dagger \mathcal{J}_r P_4 - \mathcal{B}_-^\dagger], \quad \tilde{\mathcal{C}}_- = [-\mathcal{B}_+^\dagger \quad -\mathcal{B}^\dagger \mathcal{J}_r]. \quad (7.56)$$

We mention that $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, and $\tilde{\mathcal{C}}$ are $3p \times 3p$, $3p \times n$, and $n \times 3p$ matrices, respectively. Moreover, $\tilde{\mathcal{A}}_\pm$, $\tilde{\mathcal{B}}_\pm$, and $\tilde{\mathcal{C}}_\pm$ are $(p_+ + p) \times (p_+ + p)$, $(p_+ + p) \times n$, and $n \times (p_+ + p)$ matrices, respectively.

Next we present the main result of this section, expressing the scattering matrix in terms of the quantities defined above in connection with the similarity transformations induced by Φ_l and Φ_r .

THEOREM 7.8. *Let $R(\lambda)$ be a rational reflection coefficient satisfying (6.22). Then the remaining entries of the scattering matrix (3.11) are given by*

$$T_l(\lambda) = I_n + iC\Lambda_{l1}(\lambda - i\Omega_3)^{-1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}C^\dagger, \quad (7.57)$$

$$T_r(\lambda) = I_n + iB^\dagger\mathcal{J}_r(\lambda - i\Omega_4)^{-1}\Lambda_{r4}\mathcal{B}, \quad (7.58)$$

$$L(\lambda) = i\tilde{C}(\lambda - i\tilde{A})^{-1}\tilde{B}. \quad (7.59)$$

In the special case when \mathcal{A} has all its eigenvalues in the left half plane, these expressions simplify to

$$T_l(\lambda) = I_n + iC(\lambda - i\mathcal{A})^{-1}\mathcal{X}C^\dagger, \quad (7.60)$$

$$T_r(\lambda) = I_n + iB^\dagger\mathcal{Y}(\lambda - i\mathcal{A})^{-1}\mathcal{B}, \quad (7.61)$$

$$L(\lambda) = iB^\dagger\mathcal{Y}(\lambda - i\mathcal{A})^{-1}\mathcal{X}C^\dagger + iB^\dagger(I_p - \mathcal{Y}\mathcal{X})(\lambda + i\mathcal{E}_l^\dagger)^{-1}C^\dagger. \quad (7.62)$$

PROOF. Using (7.27), (7.37), (7.41), (7.44), (7.45), (7.47), and the equality

$$\Phi_l \mathcal{Q} \begin{bmatrix} 0 \\ C^\dagger \end{bmatrix} = - \begin{bmatrix} I_p \\ 0 \end{bmatrix} (\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}C^\dagger,$$

we obtain (7.57). From (7.23), (7.37), (7.41), (7.44), (7.45), (7.48), as well as the identity

$$\Phi_r(I_{2p} - \Pi) \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} = - \begin{bmatrix} 0 \\ I_p \end{bmatrix} (\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y}\mathcal{B},$$

it follows that

$$T_r(\lambda^*)^\dagger = I_n - iB^\dagger\Lambda_{r4}(\lambda + i\Omega_4^\dagger)^{-1}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y}\mathcal{B}.$$

Now (7.58) follows by taking the adjoint and using (7.52). Note that Λ_{r2} , Λ_{r4} , and \mathcal{Y} are hermitian.

With the help of (7.14), (7.26), (7.47), and (7.52), we derive

$$[T_l(\lambda^*)^\dagger]^{-1} = I_n + iC\mathcal{X}(\lambda + i\mathcal{E}_l^\dagger)^{-1}\mathcal{J}_lC^\dagger. \quad (7.63)$$

From (7.1), (7.63), and using

$$I_p - \mathcal{J}_l = -\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X},$$

$$iC^\dagger C\mathcal{X} = (\lambda + i\mathcal{A}^\dagger) - (\lambda + i\mathcal{E}_l^\dagger),$$

we get

$$-R(\lambda^*)^\dagger [T_l(\lambda^*)^\dagger]^{-1} = -i\mathcal{B}^\dagger(\lambda + i\mathcal{A}^\dagger)^{-1}\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}\mathcal{C}^\dagger + i\mathcal{B}^\dagger(\lambda + i\mathcal{E}_l^\dagger)^{-1}\mathcal{J}_l\mathcal{C}^\dagger. \quad (7.64)$$

Using (5.5), (7.57), (7.64), and some standard results on realizations (Chapter 1 of [13]), we obtain

$$L(\lambda) = i\Omega_8(\lambda - i\Omega_7)^{-1}\Omega_9, \quad (7.65)$$

where

$$\Omega_7 = \begin{bmatrix} \Omega_4 & -\Lambda_{r4}\mathcal{B}\mathcal{B}^\dagger & \Lambda_{r4}\mathcal{B}\mathcal{B}^\dagger \\ 0 & -\mathcal{A}^\dagger & 0 \\ 0 & 0 & -\mathcal{E}_l^\dagger \end{bmatrix},$$

$$\Omega_8 = \mathcal{B}^\dagger \begin{bmatrix} \mathcal{J}_r & -I_p & I_p \end{bmatrix}, \quad \Omega_9 = \begin{bmatrix} 0 & & \\ \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} & & \\ & \mathcal{J}_l & \end{bmatrix} \mathcal{C}^\dagger.$$

In order to bring $L(\lambda)$ into the form (7.59) we use a similarity transformation. Let

$$\Psi = \begin{bmatrix} 0 & 0 & 0 & I_p \\ 0 & I_{p-} & 0 & 0 \\ 0 & 0 & I_{p+} & 0 \\ -I_p & P_4 & 0 & P_3 \end{bmatrix}, \quad \Psi^{-1} = \begin{bmatrix} P_3 & P_4 & 0 & -I_p \\ 0 & I_{p-} & 0 & 0 \\ 0 & 0 & I_{p+} & 0 \\ I_p & 0 & 0 & 0 \end{bmatrix},$$

where P_3 and P_4 have been defined in (7.50) and (7.51). Then it is straightforward to verify that

$$\tilde{\mathcal{A}} = \Psi\Omega_7\Psi^{-1}, \quad \tilde{\mathcal{C}} = \Omega_8\Psi^{-1}, \quad \tilde{\mathcal{B}} = \Psi\Omega_9, \quad (7.66)$$

where $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, and $\tilde{\mathcal{C}}$ are the matrices defined in (7.53). Using (7.66) in (7.65), together with the fact that for one of the blocks of Ω_9 we can write

$$\begin{aligned} \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} &= \begin{bmatrix} 0 & 0 \\ 0 & I_{p+} \end{bmatrix} \begin{bmatrix} I_{p-} & -\mathcal{X}_2 \\ 0 & P_1 - \mathcal{X}_4 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_3 & \mathcal{X}_4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ (P_1 - \mathcal{X}_4)^{-1}\mathcal{X}_3 & (P_1 - \mathcal{X}_4)^{-1}\mathcal{X}_4 \end{bmatrix}, \end{aligned}$$

we obtain (7.59) with the matrices (7.54)-(7.56). The expressions (7.60)-(7.62) can be obtained from (5.5) and (7.23)-(7.30) by using the special forms (7.35) for Π and \mathcal{Q} , or by obvious reductions from (7.57)-(7.59). The details are omitted. ■

8. INVERSE PROBLEM WITH RATIONAL SCATTERING MATRICES

Let $R(\lambda)$ have the form (7.1) for certain matrices \mathcal{A} , \mathcal{B} , and \mathcal{C} , where \mathcal{A} has minimal order and hence does not have zero or purely imaginary eigenvalues. Then

$$R(\lambda) = i\mathcal{C}(\lambda - i\mathcal{A})^{-1}\mathcal{B} = - \int_{-\infty}^{\infty} dt e^{-i\lambda t} \mathcal{C} E(t; -\mathcal{A}) \mathcal{B}, \quad (8.1)$$

where

$$E(t; -\mathcal{A}) = \begin{cases} e^{-t\mathcal{A}} P_{\mathcal{A}}^{(+)} = \frac{1}{2\pi i} \int_{\Gamma_+} dz e^{-tz} (z - \mathcal{A})^{-1}, & t > 0, \\ -e^{-t\mathcal{A}} P_{\mathcal{A}}^{(-)} = -\frac{1}{2\pi i} \int_{\Gamma_-} dz e^{-tz} (z - \mathcal{A})^{-1}, & t < 0, \end{cases} \quad (8.2)$$

is the bisemigroup generated by \mathcal{A} (cf. [14,15]). Here Γ_+ and Γ_- are the positively oriented simple Jordan contours in the right and left half planes enclosing all of the eigenvalues of \mathcal{A} in the open right and left half planes, respectively, and $P_{\mathcal{A}}^{(+)}$ and $P_{\mathcal{A}}^{(-)}$ are the spectral projections of \mathcal{A} corresponding to its eigenvalues in the right and left half planes, respectively.

Our strategy for reconstructing $k(x)$ from $R(\lambda)$ is as follows. When $x > 0$ we will solve the Marchenko equation (6.10) by using $R(\lambda)$ as the input, and when $x < 0$ we will solve the Marchenko equation (6.11) by using $L(\lambda)$ as given in (7.59). Then we use (4.19) to determine $k(x)$. First consider (6.10) with

$$\hat{R}(t) = -\mathcal{C} E(t; -\mathcal{A}) \mathcal{B}, \quad \hat{R}(t)^\dagger = -\mathcal{B}^\dagger E(t; -\mathcal{A}^\dagger) \mathcal{C}^\dagger,$$

which are obtained from (6.1) and (8.1). Introducing the positive selfadjoint $p \times p$ matrices

$$\mathcal{D}_1 = \int_0^\infty dt E(t; -\mathcal{A}) \mathcal{B} \mathcal{B}^\dagger E(t; -\mathcal{A}^\dagger), \quad \mathcal{D}_2 = \int_0^\infty dt E(t; -\mathcal{A}^\dagger) \mathcal{C}^\dagger \mathcal{C} E(t; -\mathcal{A}),$$

and assuming $x > 0$, we obtain for the hermitian integral kernel in (6.10)

$$\int_0^\infty d\beta \hat{R}(\gamma + \beta + 2x) \hat{R}(\alpha + \beta + 2x)^\dagger = \mathcal{C} E(\gamma + 2x; -\mathcal{A}) \mathcal{D}_1 E(\alpha + 2x; -\mathcal{A}^\dagger) \mathcal{C}^\dagger.$$

The unique solution of the separable integral equation (6.10) is then given by

$$B_{I3}(x, \alpha) = \mathcal{B}^\dagger [I_p - E(2x; -\mathcal{A}^\dagger) \mathcal{D}_2 E(2x; -\mathcal{A}) \mathcal{D}_1]^{-1} E(\alpha + 2x; -\mathcal{A}^\dagger) \mathcal{C}^\dagger, \quad (8.3)$$

where the inverse exists because of the unique solvability of (6.10). For later use we note that, by (7.36) and (7.40), we have

$$\mathcal{D}_1 = \begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix}, \quad \mathcal{D}_2 = \begin{bmatrix} 0 & 0 \\ 0 & P_2 \end{bmatrix}. \quad (8.4)$$

When $x < 0$ we start from $L(\lambda)$ as given in (7.59), so that

$$\hat{L}(t) = -\tilde{C} E(t; -\tilde{A}) \tilde{B}, \quad \hat{L}(t)^\dagger = -\tilde{B}^\dagger E(t; -\tilde{A}^\dagger) \tilde{C}^\dagger.$$

Proceeding as in the derivation of (8.3), we obtain

$$B_{r2}(x, \alpha) = \tilde{B}^\dagger [I_{3p} - E(-2x; -\tilde{A}^\dagger) \mathcal{D}_4 E(-2x; -\tilde{A}) \mathcal{D}_3]^{-1} E(\alpha - 2x; -\tilde{A}^\dagger) \tilde{C}^\dagger, \quad (8.5)$$

where the inverse exists because of the unique solvability of (6.11). Here \mathcal{D}_3 and \mathcal{D}_4 are the positive selfadjoint matrices given by

$$\mathcal{D}_3 = \int_0^\infty dt E(t; -\tilde{A}) \tilde{B} \tilde{B}^\dagger E(t; -\tilde{A}^\dagger), \quad \mathcal{D}_4 = \int_0^\infty dt E(t; -\tilde{A}^\dagger) \tilde{C}^\dagger \tilde{C} E(t; -\tilde{A}),$$

which, by means of (7.53)-(7.56), can be written as

$$\mathcal{D}_3 = \begin{bmatrix} P_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{D}_4 = \begin{bmatrix} P_6 & P_7 & 0 & 0 \\ P_8 & P_9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$P_5 = \int_0^\infty dt e^{t\mathcal{E}_l^\dagger} \mathcal{J}_l C^\dagger C \mathcal{J}_l^\dagger e^{t\mathcal{E}_l}, \quad P_6 = \int_0^\infty dt e^{t\mathcal{E}_l} (I_p + P_3^\dagger \mathcal{J}_r^\dagger) B B^\dagger (I_p + \mathcal{J}_r P_3) e^{t\mathcal{E}_l^\dagger}, \quad (8.6)$$

and P_7 , P_8 , and P_9 are irrelevant because they will not contribute to $k(x)$, as we will see.

Now we are ready to prove the main result of this section. Again we first state the general result and then specialize it to the particular case when \mathcal{A} has all its eigenvalues in the left half plane or, equivalently, when $R(\lambda)$ is analytic in \mathbf{C}^+ .

THEOREM 8.1. *Suppose that $R(\lambda)$ satisfies (6.22) and is given by the minimal representation (7.1) in a basis where (7.36) holds. Then the matrix potential $k(x)$ in (1.2) is given by*

$$k(x) = \begin{cases} 2i C_+ e^{-2xA_+} [I_{p_+} - P_1 e^{-2xA_+^\dagger} P_2 e^{-2xA_+}]^{-1} B_+, & x > 0, \\ -2i C \mathcal{J}_l^\dagger [I_p - e^{-2x\mathcal{E}_l} P_6 e^{-2x\mathcal{E}_l^\dagger} P_5]^{-1} e^{-2x\mathcal{E}_l} (I_p + P_3^\dagger \mathcal{J}_r^\dagger) B, & x < 0. \end{cases} \quad (8.7)$$

If $R(\lambda)$ is analytic in \mathbf{C}^+ , then

$$k(x) = \begin{cases} 0, & x > 0, \\ -2i C [I_p - e^{-2x\mathcal{E}_l} \mathcal{X} e^{-2x\mathcal{E}_l^\dagger} \mathcal{Y}]^{-1} e^{-2x\mathcal{E}_l} (I_p - \mathcal{X} \mathcal{Y}) B, & x < 0, \end{cases} \quad (8.8)$$

where \mathcal{X} and \mathcal{Y} are the unique solutions of (7.8) and (7.9), respectively. Moreover, if $R(\lambda)$ is analytic in \mathbb{C}^+ , then the jump in the potential at $x = 0$ is given by

$$k(0^+) - k(0^-) = -k(0^-) = 2iCB. \quad (8.9)$$

PROOF. The representation (8.7) for $k(x)$ is a direct consequence of (4.19) and (8.3)-(8.6). Thus we need only establish the simplifications that occur when $R(\lambda)$ is analytic in \mathbb{C}^+ . In this case \mathcal{A} has all its eigenvalues in the left half plane so that from (8.2) we get $E(t; -\mathcal{A}) = 0$ for $t > 0$. Thus $k(x) = 0$ for $x > 0$. For $x < 0$, starting with (7.36), we can simplify the expression in (8.7) by deleting the blocks associated with the spectrum of \mathcal{A} in the right half plane. This reduction is implemented by the following substitutions: $\Lambda_{l1} \mapsto I_p$, $\Lambda_{l3} \mapsto 0$, and hence

$$\mathcal{J}_l^\dagger \mapsto I_p, \quad (8.10)$$

by (7.52). Similarly, $\Lambda_{r2} \mapsto 0$, $\Lambda_{r4} \mapsto I_p$, and hence $\mathcal{J}_r^\dagger \mapsto \mathcal{Y}$. Since $\Omega_4 \mapsto \mathcal{A}$, the solution to (7.49) becomes $P_3 = -\mathcal{X}$ and thus

$$I_p + P_3^\dagger \mathcal{J}_r^\dagger \mapsto I_p - \mathcal{X}\mathcal{Y}. \quad (8.11)$$

Furthermore we can compute P_5 and P_6 in (8.6). We observe that P_5 and P_6 are solutions to the following Riccati equations:

$$P_5 \mathcal{E}_l + \mathcal{E}_l^\dagger P_5 = -\mathcal{C}^\dagger \mathcal{C}, \quad (8.12)$$

$$P_6 \mathcal{E}_l^\dagger + \mathcal{E}_l P_6 = -(I_p - \mathcal{X}\mathcal{Y}) \mathcal{B} \mathcal{B}^\dagger (I_p - \mathcal{Y}\mathcal{X}). \quad (8.13)$$

First note the identity

$$\mathcal{E}_l^\dagger \mathcal{Y} + \mathcal{Y} \mathcal{E}_r = \mathcal{C}^\dagger \mathcal{C} (I_p - \mathcal{X}\mathcal{Y}), \quad (8.14)$$

which follows from (7.9) and (7.15). On multiplying (8.14) from the right by $(I_p - \mathcal{X}\mathcal{Y})^{-1}$, using (7.16), and comparing the result with (8.12), we find that

$$P_5 = -\mathcal{Y} (I_p - \mathcal{X}\mathcal{Y})^{-1} = -(I_p - \mathcal{Y}\mathcal{X})^{-1} \mathcal{Y}. \quad (8.15)$$

Similarly, on multiplying the identity

$$\mathcal{E}_r \mathcal{X} + \mathcal{X} \mathcal{E}_l^\dagger = \mathcal{B} \mathcal{B}^\dagger (I_p - \mathcal{Y}\mathcal{X}),$$

which follows from (7.8) and (7.15), from the left by $I_p - \mathcal{X}\mathcal{Y}$, using (7.16), and comparing the result with (8.13), we obtain

$$P_6 = -\mathcal{X} (I_p - \mathcal{Y}\mathcal{X}). \quad (8.16)$$

Since, by (7.16) and its adjoint,

$$(I_p - \mathcal{Y}\mathcal{X})e^{-2x\mathcal{E}_l^\dagger} = e^{-2x\mathcal{E}_r^\dagger}(I_p - \mathcal{Y}\mathcal{X}),$$

the result (8.8) for $x < 0$ follows from inserting (8.10), (8.11), (8.15), (8.16) in (8.7). Finally, letting $x \rightarrow 0$ from below gives $k(0^-) = -2i\mathcal{CB}$, and hence (8.9) follows. ■

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