ELECTRONIC DELIVERY COVER SHEET

Warning Concerning Copyright Restrictions

TRANSPORT THEORY AND STATISTICAL PHYSICS, 11(3), 155-181 (1982-83)

ABSTRACT BOUNDARY VALUE PROBLEMS FROM KINETIC THEORY

by

William Greenberg and Cor van der Mee

Laboratory for Transport Theory and Mathematical Physics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

ABSTRACT

The abstract differential equation $(Tf)' = -Af$ with "partial range" boundary conditions is solved on a Hilbert space. T and A are (possibly unbounded) self-adjoint operators, $A \geq 0$ and semi-Fredholm. Examples from kinetic theory are given.

I. INTRODUCTION

In a recent paper[1] the partial range boundary value problem associated with the abstract kinetic equation

$$\begin{aligned} &\frac{\partial}{\partial x}(Tf) = -Af \\ &(Q_+f)(0) = \phi_0 \\ &\lim_{x\to\infty} \|f(x)\| < \infty \end{aligned} \tag{1}$$

was studied on an abstract Hilbert space, under the assumption that A is positive (possibly unbounded) self-adjoint and Fredholm, and T is bounded self-adjoint and one-one. Here Q_+ is the maximal positive projection associated with T. Such an abstract equation includes as special cases a number of one-dimensional problems in neutron transport, electron transport, and radiative transfer. However, the boundedness restriction on T explicitly excludes those one-dimensional problems in gas kinetics for which explicit representations of solutions are already known.

The present paper removes the boundedness restriction on T, and existence and uniqueness of solutions to (1) will be proved. The approach we utilize, from the theory of strongly continuous linear semigroups, provides a rigorous framework for the method of singular eigenfunctions introduced by K. M. Case[2] to construct solutions of transport equations. This approach was introduced into neutron transport theory by R. Hangelbroek[3] in 1973, and has been extended to a wide range of abstract problems by R. Beals.[11]

The solution of the boundary value problem (1) we give here follows closely the development in Ref. 1, and in some cases, the reader is referred to that source for the proofs of preliminary propositions which are quite similar. The notation in this paper and the previous one is consistent. For completeness, we give the principle result of the earlier paper.

THEOREM 1.

If A is positive, self-adjoint, and Fredholm, and T is bounded self-adjoint and one-one, then for $\Phi_0 \varepsilon Q_+(H_T)$, the boundary value problem (1) has a (differentiable) solution which is unique if and only if Ker A is positive definite with respect to the T-indefinite inner product (defined in Eq. 12).

In Section II, we obtain a reduction of the operator $T^{-1}A$ which enables us to restate the half-space problem. The existence of a suitable signature operator provides $T^{-1}A$ with self-adjoint extensions. In the following section, these extensions are used to construct the Larsen-Habetler albedo operator and obtain half-range expansions. The existence and uniqueness theorems are stated in Section IV. Finally, in the last section, applications from kinetic theory are presented.

II. DEFINITIONS AND DECOMPOSITIONS

Throughout we consider a (possibly unbounded) self-adjoint operator T with trivial kernel Ker T = {0} and a (possibly unbounded) positive self-adjoint operator A with closed range Ran A in an abstract Hilbert space H and with kernel Ker A of finite dimension:

$$\dim \operatorname{Ker} A = n < \infty . \qquad (2)$$

Such a pair (T,A) is called a symmetric pair on H if the following conditions are satisfied:

(i) $D(T) \cap D(A)$ is dense in H. This assumption implies that $T^{-1}A$ is a closable operator. Indeed, AT^{-1} densely defined and extension $(AT^{-1})^* \supseteq T^{-1}A$ imply that $(AT^{-1})^*$ is closed and $T^{-1}A$ is closable.

(ii) Let $K_0 = \overline{T^{-1}A}$ be the closure of $T^{-1}A$ on H. Define the zero root linear manifold $Z_0(K_0)$ of K_0 by

$$Z_0(K_0) = \{f \varepsilon D(K_0) \,|\, f_0 \varepsilon D(K_0^n) \text{ and } K^n f_0 = 0 \text{ for some } n \varepsilon Z_+\},$$

and similarly the zero root linear manifold $Z_0(B)$ for any linear operator B. We assume that $Z_0(K_0) \subseteq D(T)$ and if $x \in \operatorname{Ker} A$ with $x \notin \operatorname{Ran} K_0$, then there exists $y \in \operatorname{Ker} A$ such that $(x,Ty) \neq 0$. This is precisely the requirement that a maximal subspace of eigenvectors which are not the image of generalized eigenvectors be non-degenerate with respect to the indefinite inner product of Eq. (12).

LEMMA 1: $T^{-1}A$ is densely defined.

PROOF The proof depends only upon the closedness of $R = \operatorname{Ran} A$ and its finite co-dimension. Write $D = D(T^{-1})$. If $D \cap R$ is not dense in R, then there exists $q \varepsilon R \backslash \overline{R \cap D}^R$ and a functional $\ell \varepsilon R^*$ such that $\ell(q) = 1$, $\ell(R \cap D) = \{0\}$. But codim $R < \infty$ and $D \subseteq H$ dense imply that $H = R \oplus N$, $N \subseteq D$. Extending ℓ to H by setting $\ell(N) = \{0\}$ gives $\ell \varepsilon H^*$ and $\ell(D) = \{0\}$. Thus $\ell = 0$, which is a contradiction.

LEMMA 2: $\operatorname{Ker} K_0 = \operatorname{Ker} A$ and $Z_0(K_0) = Z_0(T^{-1}A)$.

PROOF Assume $x_n \varepsilon D(T^{-1}A) \cap (\operatorname{Ker} A)^\perp \to x$ and $T^{-1}A\, x_n \equiv h_n \to 0$. Then $h_n \varepsilon D(T), h_n \to 0$ and $A^{-1}Th_n \to x$, where A^{-1} is the bounded operator from Ran A into $(\operatorname{Ker} A)^\perp$ defined by $A^{-1}(Ax) = x$ for $x \varepsilon (\operatorname{Ker} A)^\perp$. But $A^{-1}T$ is closable, because $A^{-1}T \subseteq (TA^{-1})^*$ and TA^{-1} is a densely defined operator from Ran A into H (cf. Lemma 1). Thus $x = 0$.

Similarly, if $T\alpha = Ag$ for some $\alpha \varepsilon \operatorname{Ker} A$, then $x_n \varepsilon D(T^{-1}A) \cap (\operatorname{Ker} A)^\perp \to x$ and $T^{-1}Ax_n \equiv g_n \to \alpha$ imply $Ax_n = Tg_n$, since $\overline{Z_0(T^{-1}A)} \subseteq D(T)$, and thus $g_n \to \alpha, A^{-1}Tg_n \to x$. Therefore, $\alpha \varepsilon D(\overline{A^{-1}T})$ and $(\overline{A^{-1}T})\alpha = A^{-1}T\alpha = x = g$, where we have used the fact that $T\alpha \varepsilon (\operatorname{Ker} A)^\perp$. Note that if $T\alpha = Ag$ has no solution g, then Condition (ii) in the definition of a symmetric pair implies $T\alpha \notin (\operatorname{Ker} A)^\perp$, which is incompatible with $\alpha \varepsilon D(\overline{A^{-1}T})$.

We have defined the zero root linear manifold $Z_0(K_0)$. In a similar way we define the zero root linear manifold $Z_0(K_0^*)$ but with K_0 replaced by K_0^*. The next lemma gives another characterization of $Z_0(K_0)$.

LEMMA 3: If $f_0 \varepsilon Z_0(K_0)$, then there exists $f_1 \varepsilon Z_0(K_0)$ such that

$$K_0 f_0 = f_1, K_0 f_1 = 0 .$$

PROOF We remark that, by virtue of Lemma 2, $Z_0(K_0) = Z_0(T^{-1}A) \subseteq D(A)$. By Condition (ii) in the definition of a symmetric pair, $Z_0(K_0) \subseteq D(T)$. Now the lemma can be proved in the same way as Lemma 1 of Ref. 1.

Lemma 3 implies that the length of a zero Jordan chain of K_0 cannot exceed 2. For special cases similar results were found in Refs. 1 and 3. The next proposition relates $Z_0(K_0)$ and $Z_0(K_0^*)$ and yields two useful decompositions of H.

PROPOSITION 1. One has

$$TZ_0(K_0) = Z_0(K_0^*), \quad A\{Z_0(K_0^*)^\perp \cap D(A)\}$$

$$= \overline{T\{Z_0(K_0^*)^\perp \cap D(T)\}} = Z_0(K_0)^\perp, \tag{3}$$

and the following decompositions hold true:

$$Z_0(K_0) \oplus Z_0(K_0^*)^\perp = H \quad ; \tag{4a}$$

$$Z_0(K_0^*) \oplus Z_0(K_0)^\perp = H \quad . \tag{4b}$$

<u>PROOF</u> Let us first prove the identity

$$TZ_0(K_0) = Z_0(K_0^*) \quad . \tag{5}$$

If $\alpha\varepsilon\text{Ker } A$, then $\alpha\varepsilon D(T), T\alpha\varepsilon D(AT^{-1})$ and $AT^{-1}T\alpha = K_0^*T\alpha = 0$. If $T^{-1}Ag = \alpha\varepsilon\text{Ker } A$ for some $g\varepsilon Z_0(K_0) = Z_0(T^{-1}A)$, then $\alpha\varepsilon D(T)$, $Ag = T\alpha$ and $g = A^{-1}T\alpha + \alpha^1$, where $A^{-1}T\varepsilon(\text{Ker } A)^\perp$ and $\alpha^1\varepsilon\text{Ker } A$. But $g\varepsilon D(T)$, so $Tg = TA^{-1}T\alpha + T\alpha^1$ and $AT^{-1}Tg = K_0^*Tg = T\alpha$. Therefore, $TZ_0(K_0) = Z_0(K_0^*)$.

Next assume $f\varepsilon\text{Ker } K_0^*$. Then $(f,T^{-1}Ag) = 0$ for all $g\varepsilon D(T^{-1}A)$, or $(f,T^{-1}h^\perp) = 0$ for all $h^\perp\varepsilon\text{Ran } A \cap D(T^{-1}) = (\text{Ker } A)^\perp \cap D(T^{-1})$. Then $h^\perp \to (f,T^{-1}h^\perp)$ is bounded on a manifold of finite co-dimension, and thus on all of $D(T^{-1})$. Since T^{-1} is self-adjoint, $f\varepsilon D(T^{-1})$ and $(T^{-1}f,h^\perp) = 0$. Therefore $T^{-1}f = \alpha\varepsilon\text{Ker } A$, or $f = T\alpha$. We have shown that $\text{Ker } K_0^* = T\,\text{Ker } A$.

Assume $K_0^*f = T\alpha, f\varepsilon(\text{Ker } K_0^*)^\perp \cap D(K_0^*)$ and $\alpha\varepsilon\text{Ker } A$. Then $T\alpha\varepsilon\text{Ran } K_0^* \subseteq (\text{Ker } A)^\perp = D(A^{-1})$, so $A^{-1}K_0^*f = A^{-1}T\alpha\varepsilon(\text{Ker } A)^\perp$. We claim $T^{-1}Ag = \alpha$. For if this is not the case,

$T\alpha \notin \text{Ran } A = (\text{Ker } A)^\perp$. Thus $TA^{-1}K_0^*f = Tg$, since $Z_0(K_0) \subseteq D(T)$. But $TA^{-1}K_0^* = I$ on $(\text{Ker } K_0^*)^\perp \cap \{f\varepsilon D(K_0^*)\,|\,K_0^*f\varepsilon D(TA^{-1})\}$. We have shown $K_0^*f\varepsilon D(TA^{-1})$. Therefore, $f = Tg$. Repeating this argument one shows that $\text{Ker}(K_0^*)^n = T\,\text{Ker}(T^{-1}A)^n$ for all $n\varepsilon Z_+$, and thus $\text{Ker}(K_0^*)^2 = Z_0(K_0^*) = TZ_0(K_0)$, which establishes (5).

Next take $x\varepsilon Z_0(K_0) \cap Z_0(K_0^*)^\perp$. Then $x\varepsilon D(T^{-1}A)$, $T^{-1}Ax\varepsilon Z_0(K_0)$ and thus $Ax\varepsilon TZ_0(K_0) \subseteq Z_0(K_0^*)$. But $x\varepsilon Z_0(K_0^*)^\perp$. So $(Ax,x) = 0$, which, by the positivity of A, implies $x\varepsilon\text{Ker } A$. However, we also have $x\varepsilon Z_0(K_0^*)^\perp \subseteq (\text{Ker } K_0^*)^\perp = \overline{\text{Ran } K_0}$. Condition (ii) in the definition of a symmetric pair yields $x\varepsilon\text{Ran } K_0$, and Lemma 2 yields $x = T^{-1}Ay$ for some $y\varepsilon D(T^{-1}A)$. Because $Ty\varepsilon Z_0(K_0^*)$, one gets

$$(Ay,y) = (Tx,y) = (x,Ty) = 0,$$

implying $Tx = Ay = 0$. Thus $x = 0$ and $Z_0(K_0) \cap Z_0(K_0^*)^\perp = \{0\}$.

Take $y\varepsilon Z_0(K_0^*) \cap Z_0(K_0)^\perp$ and $z\varepsilon Z_0(K_0^*)$. Then (5) implies $y = Tx$ with $x\varepsilon Z_0(K_0)$ and $z = Tu$ with $u\varepsilon Z_0(K_0)$. So $(x,z) = (x,Tu) = (Tx,u) = (y,u) = 0$. Thus $x\varepsilon Z_0(K_0) \cap Z_0(K_0^*) = \{0\}$ and $y = Tx = 0$. Hence,

$$Z_0(K_0) \cap Z_0(K_0^*)^\perp = Z_0(K_0^*) \cap Z_0(K_0)^\perp = \{0\}. \tag{6}$$

The remaining part of the proof is the same as the corresponding part of Ref.1. The decompositions (4a)-(4b) follows from (5), (6) and a simple dimension argument.

Now, modifying an idea originally introduced by Hangelbroek,[4] let $H_A = D(A^{\frac{1}{2}}) \cap Z_0(K_0^*)^\perp$ be the Hilbert space with inner product

$$(x,y)_A = (A^{\frac{1}{2}}x,A^{\frac{1}{2}}y) \quad . \tag{7}$$

Note that $(x,y)_A = (Ax,y)$ for $x,y \varepsilon D(A) \cap Z_0(K_0^*)^\perp \subseteq H_A$, and that H_A is continuously and densely embedded in $Z_0(K_0^*)^\perp$. We define

$$K_1 = \overline{T^{-1}A}^A \oplus T^{-1}A|_{Z_0(K_0)}$$

to be the direct sum of two operators: (i) the restriction of $T^{-1}A$ to $Z_0(K_0)$, which is bounded; (ii) the closure in $(\ , \ )_A$-topology of the restriction of $T^{-1}A$ to $Z_0(K_0^*)^\perp$. Note that

$$T^{-1}A \subseteq K_1 \subseteq K_0 \quad ,$$

with $Z_0(K_0)$ in the domain of all three operators. Then Lemma 2 and 3 and Proposition 1 are also valid for K_1. Henceforth we write T and A also for their restrictions on H_A.

Obviously the operator $T^{-1}A|_{Z_0(K_0^*)^\perp}$ is symmetric with respect to the inner product (7) and K_1 is its second adjoint with respect to (7). In general, it is quite difficult to find out whether $K_1|_{Z_0(K^*)^\perp}$ is $(\ , \ )_A$--self-adjoint or even if it has a self-adjoint extension. However, if either T or A is bounded, then $K_1 = T^{-1}A$ is closed and its restriction to $Z_0(K_0^*)^\perp$ is $(\ , \ )_A$-self-adjoint. The following lemma gives the existence of self-adjoint extensions for the most interesting kinetic models.

<u>LEMMA</u> 4: Let F be a signature operator on H (i.e., $F = F^*$, $F^2 = I$) such that $FD(T) \subseteq D(T)$, $FD(A) \subseteq D(A)$ and

$$FTx = -TFx, \ FAy = AFy; \ x\varepsilon D(T), \ y \varepsilon D(A) \quad .$$

Then $K_1|_{Z_0(K_0^*)^\perp}$ has a $(\ , \ )_A$-self-adjoint extension K which satisfies

$$FD(K) \subseteq D(K); \ FKz = -KFz, \ z\varepsilon D(K) \subseteq H_A \quad . \tag{8}$$

<u>PROOF</u> Certainly $FH_A \subseteq H_A$ and F induces a $(\ , \ )_A$-signature operator on H_A, which has the following properties:

$$FD(T^{-1}A) \subseteq D(T^{-1}A); \ FT^{-1}Az = -T^{-1}AFz, \ z\varepsilon D(T^{-1}A) \ .$$

Hence, a similar property holds for the second $(\ , \ )_A$-adjoint $K_1|_{Z_0(K_0^*)^\perp}$ of $T^{-1}A|_{Z_0(K_0^*)^\perp}$. Let us denote the $(\ , \ )_A$-adjoint of B by $B^\dagger$. Then the relations

$$FD(K_1^\dagger) \subseteq D(K_1^\dagger); \ FK_1^\dagger z = -K_1^\dagger Fz, \ z\varepsilon D(K_1^\dagger),$$

imply

$$F \operatorname{Ker}(K_1^\dagger - i) = \operatorname{Ker}(K_1^\dagger + i),$$

and thus K_1 has equal deficiency indices. Using the procedure of Theorem X.2 of Ref. 6, one defines

$$D(K) = \{x + x_+ + Fx_+ | x\varepsilon D(K_1), \ x_+ \varepsilon \operatorname{Ker}(K_1^\dagger - i)\}; \tag{9a}$$

$$K(x + x_+ + Fx_+) = K_1x + i(x_+ - Fx_+). \tag{9b}$$

Then $K|_{Z_0(K_0^*)^\perp}$ is a self-adjoint extension of $K_1|_{Z_0(K_0^*)^\perp}$.

A signature operator F on H that leaves invariant D(T) and D(A), and anticommutes with T and commutes with A, will be called an inversion symmetry for the symmetric pair (T,A). It is easy to prove that in this case $\dim Z_0(K) = 2m$ = even and the spectra of T and K are real and symmetric with respect to $\lambda = 0$ (cf. Refs. 3 and 7; the dimensional statement will be proved at the end of this section.).

As in Ref. 1 define a matrix operator β on $Z_0(K_0)$ and a positive operator A_β to reduce the half-space problem to one where Ker A = {0}.

PROPOSITION 2. Let (T,A) be a symmetric pair and let P denote the projection of H onto $Z_0(K_0^*)^\perp$ along $Z_0(K_0)$. For some invertible operator β on the finite-dimensional space $Z_0(K_0)$ put

$$A_\beta = A\,P + T\beta^{-1}(I - P) \quad . \tag{10}$$

Then the operator A_β is densily defined with bounded inverse and

$$A_\beta^{-1}\,T = \beta \oplus \left(T^{-1}A|_{Z_0(K_0^*)^\perp}\right)^{-1}. \tag{11}$$

One may choose β in such a way that $(T\beta^{-1}x,x) \geq 0$ for all $x \in Z_0(K_0)$, in which case A_β will be a positive operator.

The proof of this proposition is the same as the one of Proposition 2 of Ref. 1. Whenever $(T\beta^{-1}x,x) \geq 0$ for all $x \in Z_0(K_0)$, we define a Hilbert space H_{A_β} and its inner product $(.,.)_{A_\beta} = (A_\beta .,.)$ in analogy with H_A and $(.,.)_A$. Because (T,A_β) is a symmetric pair on H with Ker $A_\beta = \{0\}$ (which one easily checks), the operator $T^{-1}A_\beta$ is closable in H_{A_β}. However,

$$D(A_\beta) = D(A) = \{D(A) \cap Z_0(K_0^*)^\perp\} \oplus Z_0(K_0)$$

and $\dim Z_0(K_0) < \infty$. Thus H_{A_β} does not depend on the particular choice of β and so we suppress β and write H_A. (Note that the new H_A equals $H_A \oplus Z_0(K_0)$). The minimal closure of $A_\beta^{-1}T$ in H_A is $K_1^{-1} \oplus \beta$. If on $Z_0(K_0^*)^\perp$ the operator $T^{-1}A$ has a $(\ ,\ )_A$-self-adjoint extension K, then $K^{-1} \oplus \beta$ will be a self-adjoint extension of $A_\beta^{-1}T$.

LEMMA 4: The subspace $Z_0(K_0)$ is a Pontryagin space[8] with respect to the indefinite inner product

$$[u,v] = (Tu,v). \tag{12}$$

If M is a complement of $K_0\{Z_0(K_0)\}$ in $Z_0(K_0)$ and $N_\pm$ is a maximal positive/negative subspace of M with respect to (12), then $K_0\{Z_0(K_0)\} \oplus N_\pm$ is a maximal positive/negative subspace of $Z_0(K_0)$ and there exists a maximal negative/positive subspace $M_\mp$ of $Z_0(K_0)$ orthogonal to $K_0\{Z_0(K_0)\} \oplus N_\pm$ such that

$$K_0\{Z_0(K_0)\} \oplus N_\pm \oplus M_\mp = Z_0(K_0) \quad .$$

This lemma can be proved in the same way as Lemma 2 of Ref. 1. As in Ref. 1 we may derive the following: in order that there exists a unique maximal positive subspace M_+ such that $K_0\{Z_0(K_0)\}$

$\subseteq M_+ \subseteq \mathrm{Ker}\ K_0 = \mathrm{Ker}\ A$, it is necessary and sufficient that Ker A is definite (i.e., either positive or negative) with respect to (12). If all zero eigenvectors of K_0(or $T^{-1}A$; see Lemma 2) are the image under K_0 (or $T^{-1}A$) of a generalized eigenvector, then $\mathrm{Ker}\ K_0 = \mathrm{Ker}\ A$ is neutral (i.e., consists of zero norm vectors only) and M_+ is uniquely specified. In fact, in this case $M_+ = \mathrm{Ker}\ K_0 = K_0\{Z_0(K_0)\}$.

In case the symmetric pair (T,A) has an inversion symmetry F, one obviously has

$$F\ \mathrm{Ker}\ A = \mathrm{Ker}\ A,\ FZ_0(K_0) = Z_0(K_0)\ . \tag{13}$$

Then

$$(u,v)_F = (Fu,v) \tag{14}$$

is another indefinite inner product on $Z_0(K_0)$ in which $iT^{-1}A|_{Z_0(K_0)}$ is self-adjoint. (Note that $FT^{-1}Ax = -T^{-1}AFx$ for $x \varepsilon Z_0(K_0)$). The subspace $Z_0^{\pm} = \{x \varepsilon Z_0(K_0) | Fx = \pm x\}$ of even/odd vectors in $Z_0(K_0)$ is strictly positive/negative with respect to (14) and for $x_\pm \varepsilon Z_0^{\pm}$ one has

$$(x_+,x_-)_F = (Fx_+,x_-) = (x_+,Fx_-)\ .$$

As $Fx_\pm = \pm x_\pm$, one gets $(x_+,x_-)_F = 0$, and thus $Z_0^{\pm}$ are orthogonal in (14) with $Z_0^+ \oplus Z_0^- = Z_0(K_0)$. (For the latter we note that $\frac{1}{2}(I + F) + \frac{1}{2}(I - F) = I$). Let us make the connection of (12) and (14). As T and F anticommute, the subspaces $Z_0^{\pm}$ satisfy

$$[x,x] = 0\ ,\ x \varepsilon Z_0^{\pm}\ .$$

Hence, the dimension $m_\pm$ of a maximal [,]-positive/negative subspace of $Z_0(K_0)$ equals or exceeds $\max\{\dim Z_0^+, \dim Z_0^-\}$. But

$$m_+ + m_- = \dim Z_0^+ + \dim Z_0^- = \dim Z_0(K_0) < \infty.$$

Hence, $m = m_+ = m_- = \dim Z_0^+ = \dim Z_0^-$ and $\dim Z_0(K_0) = 2m$ even. For symmetric pairs (T,A) for which T is bounded and I-A is compact this has been observed before in Ref. 3. The presence of the inversion symmetry F implies that Ker A is definite in (12) if and only if every zero eigenvector of $T^{-1}A$ is the image under $T^{-1}A$ of a generalized eigenvector.

III. HALF-RANGE EXPANSIONS

Throughout Sections III to V we assume that $T^{-1}A|_{Z_0(K_0^*)^\perp}$ has a$(\ , \ )_A$- self-adjoint extension K which we extend linearly to H_A by putting $Kx = T^{-1}Ax$ for $x \varepsilon Z_0(K_0)$. This assumption is satisfied if (T,A) has an inversion symmetry F (cf. Lemma 4) or if either T or A is bounded. In the latter case one simply takes $K = T^{-1}A$. We define $Q_\pm$ to be the H-orthogonal projections of H onto the maximal T-invariant subspace on which T is positive/negative. We already defined the $(\ , \ )_{A_\beta}$-inner product on H_{A_β} and took the decision to suppress β in H_{A_β} (because this space does not depend on β); then $H_A \subseteq H$ densely. In analogy with $Q_\pm$, we define $P_\pm$ to be the $(\ , \ )_{A_\beta}$-orthogonal projections of H_A onto the maximal $\hat{K}_\beta$-invariant subspace on which

$$\hat{K}_\beta = \beta \oplus (K|_{Z_0(K_0^*)^\perp})^{-1} \tag{15}$$

is $(\ ,\ )_{A_\beta}$-selfadjoint.

Let us introduce two additional inner products; namely

$$(x,y)_T = (|T|x,y), \quad (x,y\in D(T)) \tag{16}$$

with the completion of $D(T)$ denoted H_T, and

$$(x,y)_{K_\beta} = (|\hat{K}_\beta|x,y)_{A_\beta}, \ (x,y\in D(\hat{K}_\beta)) \tag{17}$$

with the completion of $D(\hat{K}_\beta)$ denoted H_{K_β}. We remark that

$$D(\hat{K}_\beta) = Z_0(K_0) \oplus \{D(K) \cap Z_0(K_0^*)^\perp\} \tag{18}$$

(cf. (15)), and thus $D(\hat{K}_\beta)$ does not depend on β. Also $(x,y)_{A_\beta} = (Ax,y)$ for $x,y\in D(A) \cap Z_0(K_0^*)^\perp$. Therefore, all norms (17) are equivalent on the set (18) and henceforth we shall suppress β in H_{K_β} and write H_K.

One of the main differences with the case when T is bounded is that H is not "naturally" embedded in H_T. However, the set $D(|T|^{1/2})$ is complete with respect to the graph inner product

$$(x,y)_{GT} = (x,y) + (|T|^{1/2}x,|T|^{1/2}y), \quad (x,y\in D(|T|^{1/2}))$$

(note that $|T|^{1/2}$ is closed) and densely embedded in both H and H_T. The domains $D(T)$ and $D(|T|^{1/2})$ are invariant under $Q_\pm$ and $Q_\pm$ is orthogonal with respect to $(\ ,\ )_{GT}$. In a straightforward way one shows that $Q_\pm$ extends to an orthogonal projection with respect to (16) also[9]. In an analogous way one shows that the set $D(|K_\beta|^{1/2})$, which is a complete Hilbert space with respect to the graph inner product

$$(x,y)_{GK_\beta} = (x,y)_{A_\beta} + (|\hat{K}_\beta|^{1/2}x,|\hat{K}_\beta|^{1/2}y)_{A_\beta}, \ x,y\in D(|\hat{K}_\beta|^{1/2}) ,$$

is densely embedded in both H_A and H_K, while $P_\pm$ extends to a $(\ ,\ )_{K_\beta}$-orthogonal projection on H_A. Finally, the identity $\operatorname{Ker} P = Z_0(K_0) \subseteq D(A) \cap D(T)$ allows us to extend continuously the projection P of H onto $Z_0(K_0^*)^\perp$ along $Z_0(K_0)$ to bounded projections on the spaces H_A and H_K. In both cases, $\operatorname{Ker} P = Z_0(K_0)$.

Let us introduce the Larsen-Habetler albedo operator E.[10] This operator is defined by the conditions that, for all $f\in D(T)$,

$$\begin{aligned} &\text{(i)} && Q_\pm E Q_\pm f = Q_\pm E f; \\ &\text{(ii)} && P_\mp E Q_\pm f = 0 . \end{aligned} \tag{19}$$

In transport theory terminology, these conditions imply that if $f\in\operatorname{Ran} Q_+$ is an incoming flux for a right half-space problem, then Ef will be the corresponding total (incoming plus reflected) flux, and if $f\in\operatorname{Ran} Q_-$ is an incoming flux for a left half-space problem, then Ef will be the corresponding total flux. In this way E will depend on the particular self-adjoint extension K, contrary to the case when T is bounded or A is bounded, because in these cases $K = T^{-1}A$ is uniquely specified by T and A.

To derive an explicit representation for $E:H_T \to H_K$, we establish first the intertwining relation

$$P_\pm E = EQ_\pm$$

on H_T. We have

$$P_\pm E = P_\pm E(Q_+ + Q_-) = P_\pm EQ_\pm = EQ_\pm ,$$

where we have used Eqs. (19). Now by (19) again,

$$Q_\pm P_\pm E Q_\pm \quad = \quad Q_\pm$$

whence, by adding the $\pm$ equations,

$$Q_+P_+EQ_+ + Q_-P_-EQ_- \;=\; (Q_+P_+ + Q_-P_-)E \;=\; I. \tag{20}$$

<u>PROPOSITION 3</u>. There exists a unique albedo operator $E: H_T \to H_K$ that is bounded, injective and satisfies the conditions (19). Further, E acts as a bounded operator from H_T into H_T.

<u>PROOF</u>. On H_A we define the Hangelbroek operators[5]

$$V \;=\; Q_+P_+ + Q_-P_- : H_A \to H, \qquad W \;=\; Q_+P_- + Q_-P_+ : H_A \to H. \tag{21}$$

Following an argument of Beals[11] we compute that, for $f \in H_A$, if the terms on the left hand side are both finite, then

$$\begin{aligned} \|Vf\|_T^2 - \|Wf\|_T^2 \;&=\; \{(TP_+f, Q_+P_+f) - (TP_-f, Q_-P_-f)\} \\ &\quad - \{(TP_-f, Q_+P_-f) - (TP_+f, Q_-P_+f)\} \qquad (22) \\ &= (TP_+f, P_+f) - (TP_-f, P_-f) \;=\; (|K_\beta^{-1}|f, f)_A = \|f\|_{K_\beta}^2 \end{aligned}$$

Let us prove that E extends to a bounded operator from H_T into H_T. A straightforward calculation shows that, for $f \in H_A$,[12]

$$(Q_+ - Q_-)(P_+ - P_-)f \;=\; Vf - Wf \;=\; (2V - I)f \;, \tag{23}$$

and therefore, for $f \in H_A \cap D(T)$,

$$\begin{aligned} ((2V - I)f, f)_T &= (|T|(Q_+ - Q_-)(P_+ - P_-)f, f) \;= \\ &= (K_\beta^{-1}(P_+ - P_-)f, f)_{A_\beta} \;=\; (|K_\beta^{-1}|f, f)_{A_\beta} \;= \\ &= \|f\|_{K_\beta}^2 . \end{aligned} \tag{24}$$

This implies the following identity:

$$2(Vf, f)_T \;=\; \|f\|_T^2 + \|f\|_{K_\beta}^2, \qquad f \in H_A \cap D(T). \tag{25}$$

Introduce the semi-bounded quadratic form

$$q(f,g) \;=\; 2(Vf, g)_T \;, \qquad f, g \in H_A \cap D(T) \tag{26}$$

on the Hilbert space H_T. Note that q can be extended to a closed form with domain $D(q) = H_T \cap H_K$, and $H_A \cap D(T)$ is a form core for q. Now q is the quadratic form of a unique self-adjoint operator whose domain D satisfies[6]

$$H_A \cap D(T) \subseteq D \subseteq H_T \cap H_K \subseteq H_T \;.$$

Hence V extends to a unique self-adjoint operator on H_T (with domain D), and moreover,

$$2(Vf, f)_T \;\geq\; \|f\|_T^2 \;, \qquad f \in D. \tag{27}$$

From this we find V to have trivial kernel and dense range in H_T. Putting E on the manifold $D_0(E)$ as

$$D_0(E) \;=\; VD \subseteq H_T \;, \qquad E(Vf) \;=\; f \in D \;,$$

E extends to a bounded operator on H_T.

Since $V(H_A \cap D(T))$ is dense in H_T, we may consider E as a densely defined operator on $D_1(E)$:

$$D_1(E) = \{Vf \,|\, f \in H_A \cap D(T)\}, \quad E(Vf) = f \in H_A \cap D(T) .$$

From Eq. (22) it follows that

$$\|Eg\|^2_{K_\beta} \leq \|g\|^2_T , \qquad g \in D(E) \subseteq H_T, \tag{28}$$

which establishes the existence of E as a bounded operator from H_T into H_K.

For arbitrary T and bounded injective A the invertibility of the Hangelbroek operator V from H_K into H_T and the equivalence of the $(\, , \,)_{K_\beta}$ and $(\, , \,)_T$ inner products on $D(A) \cap D(T)$ were proved[11] by Beals, after which he could simply put $E = V^{-1}$. Subsequently these results were generalized to the case when T is bounded and A may be unbounded with non-trivial kernel (see Refs. 1 and 13), but as in the present article the proof of the boundedness of V (rather than $E = V^{-1}$) can not be obtained. For a discussion of the implications of the boundedness of V we refer to Lemma 3 in Ref. 1.

Earlier, Hangelbroek[4] proved the invertibility of V as an operator from H into H for neutron transport with isotropic (and later also anisotropic) scattering kernels. In that work I-A was assumed compact. Under the conditions that I-A is compact and Ran (I-A) $\subseteq$ Ran $|T|^\alpha$ for $0 \leq \alpha \leq 1$, van der Mee[3] proved the invertibility of V and of TVT^{-1} on H, which in that case implies Beals' result on H_T.

IV. EXISTENCE AND UNIQUENESS THEORY FOR HALF SPACE PROBLEMS

To solve the half-space problem, one seeks a solution of Eq. (1), $f:[0,\infty) \to H_K$, subject to

$$Q_+ f(0) = f_+ \in \text{Ran } Q_+ \tag{29a}$$

$$\limsup_{x \to \infty} \|f(x)\| \text{ finite} . \tag{29b}$$

Because the albedo operator E acts from H_T into H_K, a statement of this type is required. Below, we give a more precise statement of the problem.

The decomposition of H into reducing subspaces of K, Proposition 1, decouples the half-space problem, into a half-space problem on PH (with a different f_+) and a finite-dimensional first order system on (I - P)H. However, the use of a suitable operator A_β makes it possible to extend the half-space problem on PH to one on H of a simpler structure than the original problem, the simplicity stemming from the injectivity of A_β. The main difficulty of the newly obtained half-space problem is that the albedo operator E acts from H_T into H_K and might not act from H into H. For this reason we state the following weakened version of the half-space problem:

Given $f_+ \in \text{Ran } Q_+$, construct a continuous function $\phi:[0,\infty) \to H_K$, with both $KP\phi$ and $(I-P)\phi$ differentiable on $(0,\infty)$, such that

$$\frac{d}{dx} KP\phi = -P\phi \qquad (\text{on } P\, H_K) \tag{30a}$$

$$\frac{d}{dx} (I - P)\phi = -T^{-1}A\phi \qquad (\text{on } Z_0(K_0)) \tag{30b}$$

$$\phi(0)\varepsilon H_T \text{ and } Q_+\phi(0) = f_+ \tag{30c}$$

$$||P\phi(x)||_K = O(1), \quad ||(I-P)\phi(x)|| = O(1)(x\to\infty). \tag{30d}$$

We did not make use of β in this statement of the half-space problem. In (30d) it is immaterial which β one applies in the K_β-norm.

The decompositions of H_K into reducing subspaces of K, Proposition 1 extended to H_K, decouples the weak half-space problem (30) into an infinite dimensional evolution equation on PH_K (namely, (30a) with initial value PEf_+) and a finite-dimensional first order system on $(I - P)H_K = Z_0(K_0)$. On PH_K, the weak half-space problem is equivalent to the semigroup problem

$$\frac{\partial}{\partial x} T\phi = -A\phi \ ;$$

$$\phi(0) = PEf_+ \ ;$$

$$||\phi(x)||_K = O(1)(x\to\infty) \ ,$$

which has a unique solution once $\phi(0) = PEf_+$ is specified uniquely. The albedo operator E satisfies conditions (19). On $(I - P)H_K = Z_0(K_0)$, boundedness at infinity requires that $(I - P)Ef_+ \varepsilon \operatorname{Ker} A$, after which the solution on $Z_0(K_0)$ can be written as a constant; more precisely,

$$(I - P)\phi(x) = e^{-xT^{-1}A}(I - P)Ef_+ \equiv (I - P)Ef_+ \ .$$

<u>THEOREM 2</u>. For every $f_+\varepsilon Q_+(H_T)$, the half-space problem has a unique (differentiable) solution if and only if Ker A is

positive definite with respect to the indefinite inner product (12). This will be the case if each $\lambda = 0$ eigenvector of K has a corresponding generalized eigenvector. If Ker A is not positive definite; there exist non-trivial solutions with incoming flux $f_+ = 0$ (non-uniqueness). On PH_K, $\lim_{x\to\infty} ||P\ (x)||_K = 0$.

The theorem follows immediately from standard semigroup theory, assuming the construction of E (which dependes on β) gives a unique albedo operator E. We observe that $PE:H_T \to PH_K$ is independent of the choice of β. The reasoning involving the uniqueness or non-uniqueness of the construction of E is precisely the same as in Ref. 1. As in Ref. 1 we may derive the following measure of non-uniqueness:

$$\delta = \dim\ [\operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-] \cap \operatorname{Ker} A \ ,$$

which is the dimension of the maximal strictly negative subspace of Ker A with respect to the indefinite inner product (12).

V. APPLICATIONS

This section contains several physical models leading to an equation of the form (1), which were not contained in Ref. 1. However, all models in Ref. 1 could be added here as applications. All of them (the present ones and those in Ref. 1) involve a time-independent one-dimensional transport problem in a semi-infinite medium with spatial variable $x\varepsilon(0,\infty)$. For all these models we shall specify the Hilbert space H, the operators T and A, whether or not (T,A) is a symmetric pair on H, and the structure of the zero root linear manifold $Z_0(K_0)$. All models will involve an unbounded operator T.

1. SCALAR BGK EQUATION[14,15,16,17]

$$\mu \frac{\partial f}{\partial x}(x,\mu) = -f(x,\mu) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x,\nu)e^{-\nu^2}d\nu, \quad -\infty < \mu < \infty.$$

In this case we take $H = L_2(-\infty,\infty)_\delta$ with δ the measure on $(-\infty,\infty)$ with Radon-Nikodym derivative $d\delta/d\mu = \pi^{-\frac{1}{2}}e^{-\mu^2}$. We define A and T by

$$(Af)(\mu) = f(\mu) - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\nu)e^{-\nu^2}d\nu, \quad (Tf)(\mu) = \mu f(\mu) .$$

Then T is unbounded self-adjoint, A bounded positive and I - A compact. Further,

$$\operatorname{Ker} A = \operatorname{span}\{1\}, \quad Z_0(K_0) = \operatorname{span}\{1,\mu\} \subseteq D(T) \tag{31}$$

The map $(Ff)(\mu) = f(-\mu)$ is an inversion symmetry of the symmetric pair (T,A). As the assumption (2) is fulfilled, the half-space problem can be solved and has a unique solution (cf. (31)).

2. BGK EQUATION FOR HEAT TRANSFER[15,18,19]

$$\mu \frac{\partial}{\partial x}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = -\begin{bmatrix} f_1(x,\mu) \\ f_2(x,\mu) \end{bmatrix} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \begin{bmatrix} 1+\frac{2}{3}(\mu^2-\frac{1}{2})(\nu^2-\frac{1}{2}) & \frac{2}{3}(\mu^2-\frac{1}{2}) \\ \frac{2}{3}(\nu^2-\frac{1}{2}) & \frac{2}{3} \end{bmatrix} \begin{bmatrix} f_1(x,\nu) \\ f_2(x,\nu) \end{bmatrix} e^{-\nu^2}d\nu.$$

$$(-\infty < \mu < \infty)$$

We take $H = L_2(-\infty,\infty)_\delta \oplus L_2(-\infty,\infty)_\delta$ with $d\delta/d\mu = \pi^{-\frac{1}{2}}e^{-\mu^2}$ and define A and T by

$$\begin{bmatrix} (Af)_1(\mu) \\ (Af)_2(\mu) \end{bmatrix} = \begin{bmatrix} f_1(\mu) \\ f_2(\mu) \end{bmatrix} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \begin{bmatrix} 1+\frac{2}{3}(\mu^2-\frac{1}{2})(\nu^2-\frac{1}{2}) & \frac{2}{3}(\mu^2-\frac{1}{2}) \\ \frac{2}{3}(\nu^2-\frac{1}{2}) & \frac{2}{3} \end{bmatrix} \begin{bmatrix} f_1(\nu) \\ f_2(\nu) \end{bmatrix} e^{-\nu^2}d\nu ;$$

$$(Tf)_i(\mu) = \mu f_i(\mu) \quad (i = 1, 2) ,$$

where f is the column vector with entries f_1 and f_2. Then T is unbounded, self-adjoint, A bounded positive, I - A compact and

$$\operatorname{Ker} A = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \mu^2 \\ 1 \end{bmatrix}\right\}$$

$$Z_0(K_0) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \mu \\ 0 \end{bmatrix}, \begin{bmatrix} \mu^2 \\ 1 \end{bmatrix}, \begin{bmatrix} \mu^3 \\ \mu \end{bmatrix}\right\}. \tag{32}$$

The map $(Ff)_i(\mu) = f_i(-\mu)$ $(i = 1,2; f = (f_1,f_2))$ is an inversion symmetry of the inversion symmetric pair (T,A). As (2) is fulfilled, the half-space problem can be solved and has a unique solution (cf. (32)).

3. NEUTRON TRANSPORT WITH ANGULARLY DEPENDENT CROSS-SECTIONS[1,20,21]

$$\mu \frac{\partial f}{\partial x}(x,\mu) = -\Sigma(\mu)f(x,\mu) + \frac{1}{2}\int_{-1}^{+1} \Sigma_s(\mu')f(x,\mu')d\mu', \quad -1 \le \mu \le 1 . \tag{33}$$

We assume that Σ and Σ_s are measurable and $\Sigma \ge \Sigma_s \ge \varepsilon > 0$. Now premultiply Eq. (33) by $\Sigma_s(\mu)$ and consider the new equation on $H = L_2[-1,+1]$. Assume that $\int_{-1}^{1} \Sigma_s(\mu)^2 d\mu < \infty$. Put

$$(Af)(\mu) = \Sigma_s(\mu) \quad \{\Sigma(\mu)f(\mu) - \frac{1}{2}\int_{-1}^{+1}\Sigma_s(\mu')f(\mu')d\mu'\} ,$$

$$(Tf)(\mu) = \mu\Sigma_s(\mu)f(\mu) .$$

Then T is self-adjoint, A is self-adjoint with closed range.

Schwarz's inequality implies that

$$\frac{1}{2}\left|\int_{-1}^{+1}\Sigma_s f d\mu\right|^2 \le \frac{1}{2}\int_{-1}^{+1}\Sigma_s^2|f|^2 d\mu \cdot \int_{-1}^{+1} d\mu \le \int_{-1}^{+1}\Sigma_s\Sigma|f|^2 d\mu$$

and therefore A is positive. Note that T (resp. A) is bounded if and only if Σ_s (resp. Σ) is bounded. As $\Sigma_s \epsilon L_2[-1,1]$, it is clear that

$$D(A) = \{f|\Sigma_s\Sigma f \epsilon L_2[-1,1]\}, \quad D(T) = \{f|\mu\Sigma_s f \epsilon L_2[-1,1]\}.$$

Thus $D(A) \cap D(T)$ is dense in $L_2[-1,1]$.

In the same way as in Ref. 1 (Ch. VI, Appl. 6) we compute the zero root linear manifold. We find that Ker A $\neq \{0\}$ if and only if $\Sigma(\mu) = \Sigma_s(\mu)$ almost everywhere, in which case

$$Z_0(K_0) = \begin{cases} \operatorname{span}\{\Sigma^{-1}, \ \mu\Sigma(\mu)^{-2}\} \text{ if } \int_{-1}^{+1} \mu\Sigma(\mu)^{-1} d\mu = 0 \ ; \\ \operatorname{span}\{\Sigma^{-1}\} = \operatorname{Ker} A \text{ if } \int_{-1}^{+1} \mu\Sigma(\mu)^{-1} d\mu \neq 0 \ . \end{cases}$$

We observe that $Z_0(K_0) \subseteq D(T)$ and Eq. (2) is satisfied. Note that $[\Sigma^{-1},\Sigma^{-1}] = \int_{-1}^{1} \mu\Sigma(\mu)^{-1} d\mu$. Thus Ker A is positive definite if the integral is non-negative, and strictly negative otherwise.

Now let us calculate the deficiency subspaces of $T^{-1}A$. If $0 \neq f\epsilon \operatorname{Ker}(T^{-1}A \mp i)$, then

$$(\Sigma(\mu) \mp i\mu) f(\mu) = \frac{1}{2}\int_{1}^{+1} \Sigma_s(\mu')f(\mu')d\mu' \ , \qquad (34)$$

and thus $\int_{-1}^{+1}\Sigma_s(\mu')f(\mu')d\mu' \neq 0$. Thus $f(\mu) = a(\Sigma(\mu) \mp i\mu)^{-1}$, $a \neq 0$,

which certainly is an L_2-function. By substitution into (34) one gets

$$0 = 1 - \frac{1}{2}\int_{-1}^{+1}\frac{\Sigma_s(\mu)}{\Sigma(\mu)\pm i\mu} d\mu = \frac{1}{2}\int_{-1}^{+1}\frac{\Sigma(\Sigma-\Sigma_s)+\mu^2}{\Sigma(\mu)^2+\mu^2} d\mu$$

$$\pm i\mu \int_{-1}^{+1}\frac{\mu\Sigma_s(\mu)}{\Sigma(\mu)^2+\mu^2} d\mu,$$

which cannot possibly hold true. (Note that $\Sigma(\Sigma-\Sigma_s) + \mu^2 \ge 0$). Thus $\operatorname{Ker}(T^{-1}A \mp i) = \{0\}$. An easy calculation gives the invertiblity of $T^{-1}A \mp i$. In fact,

$$((T^{-1}A \mp i)^{-1}g)(\mu) = \frac{\mu g(\mu) + \phi(g)}{\Sigma(\mu) \mp i\mu} \ ,$$

where

$$\phi(g) = \frac{1}{2}\int_{-1}^{+1}\frac{\Sigma_s(\mu')\mu' g(\mu')}{\Sigma(\mu') \mp i\mu'} d\mu' \left\{1 - \frac{1}{2}\int_{-1}^{+1}\frac{\Sigma_s(\mu')}{\Sigma(\mu')\mp i\mu'} d\mu'\right\}^{-1}.$$

(Note that ϕ is a bounded functional, because $\int_{-1}^{+1}(\Sigma^2+\mu^2)^{-1}\mu^2\Sigma_s^2 d\mu < \infty$.) Hence, $T^{-1}A$ is essentially self-adjoint in $(\ ,\ )_A$.

All information considered, we may conclude that (T,A) is a symmetric pair on $L_2[-1,1]$ and the half-space problem has a unique solution if and only if $\int_{-1}^{1}\mu\Sigma(\mu)^{-1}d\mu \ge 0$. Otherwise the measure of non-uniqueness $\delta = 1$.

ACKNOWLEDGEMENT

This work was supported in part by the Department of Energy under grant number DE-AS05-80ER10711. The article was written while the second author was visiting Blacksburg. The authors are indebted to Prof. P. F. Zweifel for helpful discussions and to Prof. R. J. Hangelbroek for a useful communication.

Permanent Address:

Cor van der Mee
Dept. of Physics & Astronomy
Vrÿe Universiteit
De Boelelaan 1081
1081 HV Amsterdam
THE NETHERLANDS

REFERENCES

1. W. Greenberg, C.V.M. van der Mee, P. F. Zweifel, "Abstract Kinetic Equations", submitted to Int. Eqs. Op. Theory.

2. K. M. Case, Ann. Phys. (N.Y.) *9*, 1 (1960).

3. C. V. M. van der Mee, "Semigroup and Factorization Methods in Transport Theory", Math. Centre Tract no. 146, Amsterdam (1981).

4. R. J. Hangelbroek, TTSP *5*, 1 (1976). Also R. J. Hangelbroek, "A functional analytic approach to the linear transport equation". Thesis, Rijks Universiteit Groningen (1973).

5. R. J. Hangelbroek and C. G. Lekkerkerker, SIAM J. Math. Anal. *8*, 458 (1977).

6. M. Reed, B. Simon, "Methods of Modern Mathematical Physics. II. Fourier analysis, self-adjointness," Academic Press, New York (1972). Section X.1.

7. M. V. Maslennikov, "The Milne Problem with Anisotropic Scattering," Proc. of the Steklov institute of mathematics, vol. *97*, A.M.S., Providence, RI (1968).

8. J. Bognár, "Indefinite Inner Product Spaces," Springer, Heidelberg (1974).

9. I. C. Gohberg, N. Krupnik, "Einführung in die Theorie der eindimensionalen singulären Integraloperatoren," Birkhäuser, Basel (1979). Theorem 3.1.

10. E. W. Larsen and G. J. Habetler, Comm. on Pure Appl. Math. *26*, 525 (1973).

11. R. Beals, J. Funct. Anal. *34*, 1 (1979).

12. R. J. Hangelbroek, Private Communication.

13. R. Beals, J. Math. Phys. 22, 954 (1981).

14. C. Cercignani, Ann. Phys. *20*, 219 (1962).

15. C. Cercignani, "Mathematical Methods in Kinetic Theory," Plenum Press, New York (1969).

16. R. L. Bowden and L. Freede Garbanati, TTSP *7*, 1 (1978).

17. H. G. Kaper, SIAM J. Math. Anal. *10*, 161 (1979).

18. R. L. Bowden and W. L. Cameron, TTSP *8*, 45 (1979).

19. C. Cercignani and C. E. Siewert, Zeitschr. Angew. Math. Phys. *33*, 297 (1982).

20. Th. F. Nonnenmacher and P. F. Zweifel, Phys. stat. sol. (b) *96*, 653 (1979).

21. M. M. R. Williams, Proc. Cambridge Phil. Soc. *84*, 549 (1978).