TRANSPORT EQUATION ON A FINITE DOMAIN:
I. REFLECTION AND TRANSMISSION OPERATORS AND DIAGONALIZATION

C.V.M. van der Mee ${ }^{1)}$

In this article we study the time-independent linear transport equation in a finite homogeneous non-multiplying medium with anisotropic scattering. For a polynomial phase function the solution is expressed in finitely many auxiliary functions. A diagonalization of an operator associate to the equation is established. Reflection and transmission operators are introduced.

## Introduction

In this article we study the integro-differential
equation

$$
\begin{equation*}
\mu \frac{\partial \psi}{\partial x}(x, \mu)+\psi(x, \mu)= \tag{0.1}
\end{equation*}
$$

$$
=-1 \delta^{+1}\left[\frac{1}{2 \pi} \delta^{2 \pi} \hat{g}\left(\mu \mu^{\prime}+\sqrt{1-\mu^{2}} \sqrt{1-\mu^{\prime}} \cos \alpha\right) \mathrm{d} \alpha\right] \psi\left(\mathrm{x}, \mu^{\prime}\right) \mathrm{d} \mu .
$$

$$
(-1 \leq \mu \leq+1,0<x<\tau<+\infty)
$$

In astrophysics Eq.(0.1) describes the time-independent transfer of unpolarized radiation through a homogeneous planeparallel stellar or planetary atmosphere of finite optical thickness $\tau$. Here $\psi$ denotes the azimuth-averaged intensity of the radiation, $x$ the optical depth and $\hat{g}$ the phase function (with the albedo included as a factor). For physical reasons one

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has $\hat{g}(t) \geq 0(-1 \leq t \leq+1)$ and $c={ }_{-1} \int^{+1} \hat{g}(t) d t \leq 1$. In neutron physics Eq. (0.1) describes the stationary transport of undelayed neutrons with uniform speed through a homogeneous plane-parallel fuel plate of a nuclear reactor. Here $\psi$ denotes the angular density, $x$ a position coordinate, $\tau$ the thickness of the plate (in units of neutron mean free path) and $\hat{g}$ accounts for the scattering properties of the medium. In a non-critical reactor one has $\hat{g}(t) \geq 0(-1 \leq t \leq+1)$ and $c={ }_{-1} s^{+1} \hat{g}(t) d t \leq 1$. In Eq. (0.1)
arccos $\mu$ is the angle describing the direction of propagation. Given a nonnegative phase function $\hat{g} \in L_{1}[-1,+1]$ with $c=-1 \delta^{+1} \hat{g}(t) d t \leq 1$, the problem is to compute the unknown function $\psi$ under suitable boundary conditions. For the physical aspects we refer to various textbooks (see $[2,1,19,3,9]$ ).

In physical applications one usually considers the boundary conditions
(0.2) $\quad \psi(0, \mu)=\phi(\mu)(0 \leq \mu \leq 1), \psi(\tau, \mu)=\phi(\mu)(-1 \leq \mu<0)$, where $\phi$ describes the azimuth-averaged intensity of the radiation or angular density of the neutrons incident to the faces of the medium. In practice mostly polynomial phase functions of the form

$$
\begin{equation*}
\hat{g}(t)=\sum_{n=0}^{N} a_{n}\left(n+\frac{1}{2}\right) P_{n}(t) \quad(-1 \leq t \leq+1) \tag{0.3}
\end{equation*}
$$

are considered, where $P_{n}$ is the usual Legendre polynomial of degree $n$ and $N$ is finite. From the physical constraints $\hat{g}(t) \geq 0$ and $c=-1 f^{+1} \hat{g}(t) d t \leq 1$ it follows that $0 \leq a_{0} \leq 1$ and $-a_{0} \leq a_{n} \leq a_{0}(n=1,2, \ldots, N)$. The cases $0<a_{0}<1$ and $a_{0}=1$ are usually named the non-conservative and conservative case, respectively. Note that $c=a_{0}$.

In astrophysics the solution of the boundary value problem (0.1)-(0.2) (with $\phi(\mu)=0$ for $-1 \leq \mu<0$ ) at $\mathbf{x}=0$ and $x=\tau$ is commonly written as

$$
\begin{array}{r}
\psi(0,-\mu)=\frac{1}{2} \int_{0}^{1} \mu^{-1} S(\mu, \nu) \phi(\nu) d \nu=20 f^{1} \nu \rho(\nu, \mu) \phi(\nu) d \nu ; \\
(0 \leq \mu \leq 1)
\end{array}
$$

$$
\begin{align*}
\psi(\tau, \mu)- & \mathrm{e}^{-\tau / \mu} \phi(\mu)=  \tag{0.4}\\
& =\frac{1}{2} \delta^{1} \mu^{-1} T(\mu, v) \phi(\nu) \mathrm{d} v=20_{0}^{1} \quad v \sigma(\nu, \mu) \phi(\nu) \mathrm{d} v
\end{align*}
$$

Here $S$ and $T$ (resp. $\rho$ and $\sigma$ ) are reflection and transmission functions appearing in the work of Chandrasekhar [2] (resp. Sobolev [19]). Exploiting symmetries in the equation Hovenier [7] expressed these functions in terms of one two-variable exit function. In neutron physics non-rigorous aspects of Case's method of eigenfunction expansion (see [1]) stimulated a mathematically rigorous treatment of the "finite-slab problem" (0.1)-(0.2). On the Hilbert space $L_{2}[-1,+1]$ of square integrable functions on the interval $-1 \leq \mu \leq+1$, the boundary value problem (0.1)-(0.2) may be formulated as the following operator differential equation with boundary conditions:
(0.5a)

$$
\begin{array}{ll}
(0.5 a) & (T \psi)^{\prime}(x)=-(I-B) \psi(x) \quad(0<x<\tau) ; \\
(0.5 b) & {\lim \left\|P_{+} \psi(x)-P_{+} \phi\right\|=0,}_{x \neq 0} \quad \lim _{x \uparrow \tau}\left\|P_{-} \psi(x)-P_{-} \phi\right\|=0 .
\end{array}
$$

Here the vector $\psi(x)$ in $L_{2}[-1,+1]$, the operators $T$ and $B$ and the projections $P_{+}$and $P_{-}$on $L_{2}[-1,+1]$ are defined by (0.6a) $\psi(x)(\mu)=\psi(x, \mu),(T h)(\mu)=\mu h(\mu) ;$

$$
\begin{equation*}
(\mathrm{Bh})(\mu)={ }_{-1} f^{+1}\left[(2 \pi)^{-1} f_{0}^{2 \pi} \hat{g}\left(\mu^{\prime} \mu^{\prime}+\sqrt{1-\mu^{2}} \sqrt{1-\mu^{\prime 2}} \cos \alpha\right) \mathrm{d} \alpha\right] \operatorname{h}\left(\mu^{\prime}\right) \mathrm{d} \mu^{\prime} ; \tag{0.6b}
\end{equation*}
$$

$$
\begin{equation*}
\left(P_{ \pm} h\right)(\mu)=h(\mu)(\mu<0),\left(P_{ \pm} h\right)(\mu)=0(\mu \gg 0) . \tag{0.6c}
\end{equation*}
$$

This statement of the "finite-slab problem" by Hangelbroek [6] stimulated the author to investigate it. For non-multiplying media and $\hat{g} \in L_{r}[-1,+1]$ for some $r>1$, the boundary value problem (0.5) was proved to have a unique solution for every $\phi \in L_{2}^{[-1,+1]}$ (see [12]). Subsequently this result was obtained for the analogous problem in $L_{p}[-1,+1](1 \leq p<+\infty$, including $p=1)$, also for the more general type of boundary conditions
(0.7) $\quad \lim _{x \neq 0}\left\|T P_{+} \psi(x)-P_{+} x\right\|=0, \lim _{x \uparrow \tau}\left\|T P_{-} \psi(x)-P_{-} x\right\|=0$,
where $x \in L_{p}[-1,+1]($ see $[13])$.
Without much emphasis on mathematical rigour astrophysicists have derived analytic expressions for the reflection and transmiṣion functions long before the bulk of mathematical
literature on the subject appeared. Inspired by partial results of Chandrasekhar [2](the cases $N=0,1,2$ ) Mullikin reduced the calculation of the reflection and transmission functions to the computation of finitely many (namely $2 \mathrm{~N}+2$ ) auxiliary functions, for which he found non-linear and linear singular integral equations (see $[15,16]$ ). Transforming to sums and differences of Mullikins functions Hovenier [8] accomplished a decoupling of the linear singular integral equations and wrote the non-linear ones differently.

This article aims at a synthesis of the formulas of astrophysicists and the rigorous approach of mathematicians.

It consists of two parts. In the first part we reduce the solution of the boundary value problem (0.1)-(0.2) for the phase function ( 0.3 ) to the computation of $2 N+2$ auxiliary functions. To rely on the approach of $[12,13]$ it is convenient to introduce the reflection operators $R_{+\tau}$ and $R_{-\tau}$ and the transmission operators $R_{-\tau}$ and $T_{-\tau}$ by

$$
\psi(0)=R_{+\tau} P_{+} \phi+T_{-\tau} P_{-} \phi, \psi(\tau)=T_{+\tau} P_{+} \phi+R_{-\tau} P_{-} \phi,
$$

where $\phi$ is the boundary value function and $P_{+}$and $P_{-}$the projections of ( 0.6 c ). Using the adjoint operators $\mathrm{R}_{+}^{*}$. and $\mathrm{T}_{+}{ }_{\tau}^{*}$ we express $R_{+\tau} \phi, R_{-\tau} \phi, T_{+\tau} \phi$ and $T_{-\tau} \phi$ in the $2 N+2$ auxiliary functions $R_{+}{ }_{\tau}^{*} P_{n}$ and $T_{+}{ }_{\tau}^{*} P_{n}$, where $P_{n}$ is the usual Legendre polynomial of degree $n(n=0,1, \ldots, N)$. It appears that these functions are the $\psi_{n}-$ and $\phi_{n}$ - functions of Chandrasekhar [2]. The main result of Mullikin [16], the non-linear singular equations referred to

## N

above, derived by him for the case ${ }_{n} \sum_{0} a_{n}\left(n+\frac{1}{2}\right) P_{n}(\mu) P_{n}\left(\mu^{\prime}\right) \geq \varepsilon>0$, is obtained as a corollary. The second part of this article is devoted to the analytic continuation of $R_{+}{ }_{\tau}^{*} P_{n}$ and $T+{ }_{\tau}^{*} P_{n}$ and their reduction to Chandrasekhar's $X$ - and $Y$ - functions.

For later use, to derive analytic continuations of and linear singular integral equations for $R_{+}{ }_{\tau}^{*} P_{n}$ and $T+{ }_{\tau}^{*} P_{n}$, we generalize a diagonalization of the operator (I-B) $T^{-1}$ to the conservative case. For the non-conservative case this result originates from Hangelbroek [6] and Lekkerkerker [10,11].

A recent result of Garcia and Siewert [5] on the zeros of the so-called dispersion function plays an important role at the generalization.

Let us describe the contents of the sections. After Section 1 containing necessary concepts and results from [12,13], in Section 2 the reflection and transmission operators are studied in an abstract way. In Section 3 these operators are expressed analytically in $2 N+2$ auxiliary functions. In Section 4 , for later use, we exposefunctions of Transport Theory and some properties of the dispersion function in particular. In Section 5 we diagonalize the operator $(I-B) T^{-1}$ and prove the Hölder continuity of $R_{+}{ }_{\tau}^{*} P_{n}$ and $T+{ }_{\tau}^{*} P_{n}$ as a corollary.

Finally we make some notational remarks. The inner product of a Hilbert space is denoted bij <.,.> and the orthogonal complement of a subset $M$ by $M^{\perp}$. By $L(H)$ we mean the algebra of bounded linear operators on the complex Banach space $H$; by $I_{H}$ (or $I$ ) we denote its identity element. The spectrum, null space and range of an operator $T$ are written as $\sigma(T)$, Ker $T$ and Im T , respectively.

## 1.STATEMENT OF THE PROBLEM

Radiative transfer and neutron transport may be described by the integro-differential equation (0.1) with boundary conditions (0.2) Here $\hat{g}$ is a real-valued function $\operatorname{inL}_{r}[-1,+1]$ for some $r>1$.

To solve the system of equations (1.1) in the space $L_{p}[-1,+1]$ (with $1 \leq p<+\infty$ ) one introduces the vector $\psi(x)$ in $L_{p}[-1,+1]$, the operators $T$ and $B$ and projections $P_{+}$and $P_{-}$of unit norm on $L_{p}[-1,+1]$ by Eqs. ( $0.6 a$ )-(0.6c).
Now the problem can be written as the operator differential equation ( 0.5 a ) with boundary conditions ( 0.5 b ).

Given $x \in L_{p}[-1,+1]$ one can also consider the more general boundary conditions (0.7).
Given $\phi($ resp. $x)$ in $L_{p}[-1,+1]$, by a solution $\psi$ of the finite-slab problem (0.1)-(0.2) (resp. (0.1)-(0.7)) we mean a vector-
valued function $\psi:(0, \tau) \rightarrow L_{p}[-1,+1]$ such that $T \psi$ is strongly differentiable on ( $0, \tau$ ) and the equalities (0.1) and (0.2) (resp. (0.7) ) are fulfilled.

In [12] the finite-slab problem in $L_{2}[-1,+1]$ has been treated through the abstract notion of a semi-definite admissible pair. By a semi-definite admissible pair on a Hilbert space $H$ we mean a pair of bounded linear operators $T$ and $B$ on $H$ with the following properties:
(C1) $T$ is self-adjoint and has atrivial null space Ker $T$;
(C2) B is compact and $\mathrm{A}=\mathrm{I}-\mathrm{B}$ is positive (i.e.,

$$
\text { <Ah, } h>\geq 0 \text { for every } h \in H) \text {; }
$$

(C3) there exist $0<\alpha<1$ and a bounded operator $D$ on $H$ such that

$$
B=|T|^{\alpha} D .
$$

If $A=I-B$ is strictly positive (i.e., if < Ah, $h \gg 0$ for $0 \neq h \in H$ ), then the pair ( $T, B$ ) is positive definite; otherwise it is called singular.

If $E$ denotes the resolution of the identity of the self-adjoint operator $T$, put $P_{+}=E((0,+\infty))$ and $P_{-}=E((-\infty, 0))$. As Ker $T=\{0\}$, we have $E(\{0\})=0$. By $H_{+}\left(H_{-}\right)$we denote the image of $P_{+}\left(P_{-}\right)$. So
(1.1) $H=H_{+} \oplus H_{-}$.

Now on the Hilbert space $H$ one can formulate the (abstract) finite-slab problem (0.1)-(0.2) (resp.(0.1)-(0.7) ), where $\phi$ (resp. $\chi$ ) is taken in $H$.

Let us consider the pair $(T, B)$ on $L_{2}[-1,+1]$ defined by (0.6a)-(0.6b). If $\hat{g}$ is a real-valued function in $L_{r}[-1,+1]$ for some $r>1$, this pair ( $T, B$ ) satifies (C1) and (C3) and the operator $B$ is compact and self-adjoint (cf.[12], Theorem VI 1.1). As known ([21], Appendix XII.8), one has

$$
(B h)(\mu)=\sum_{n=0}^{+\infty} a_{n}\left(n+\frac{1}{2}\right) P_{n}(\mu)_{-1} f^{+1} h(\nu) P_{n}(\nu) d \nu \quad(-1 \leq \mu \leq+1)
$$

where $P_{n}(\mu)=\left(2^{n} \cdot n!\right)^{-1}\left(\frac{d}{d \mu}\right)^{n}\left(\mu^{2}-1\right)^{n}$ is the usual Legendre polynomial of degree $n$ and $a_{n}=\left(n+\frac{1}{2}\right)^{-\frac{1}{2}}-1 \delta^{+1} \hat{g}(t) P_{n}(t) d t$
$(n=0,1,2, \ldots)$. It appears that $\sigma(B)=\left\{a_{n}: n \geq 0\right\} \cup\{0\}$; so for $a_{n} \leq+1$ (resp. $a_{n}<+1$ ) for every $n=0,1,2, \ldots$, the pair (T,B) is a semi-definite (resp. positive definite) admissible pair on $L_{2}[-1,+1]$. The physical constraints $\hat{g}(t) \geq 0$ and $-1 f^{+1} \hat{g}(t) d t \leq 1$ imply that $0 \leq a_{0} \leq 1$ and $-a_{0} \leq a_{n} \leq a_{0}(n=1,2, \ldots)$, and in this case the pair ( $T, B$ ) is semi-definite; it is positive definite if and only if $0 \leq a_{0}<1$. If $a_{n}=0$ for $n \geq N+1$, the phase function $\hat{g}$ is given by ( 0.3 ).

Let ( $T, B$ ) be a semi-definite admissible pair on a
Hilbert space $H$ and let $A=I-B$. Then there exist natural decompositions of the spectra of the possibly unbounded operators $T^{-1} A$ and $A T^{-1}$ (see Section III. 3 of [12]), which we are about to describe. If $\Gamma$ is a positively oriented contour separating the point $\lambda=0$ from the non-zero part of $\Sigma=\{\lambda \in C: A-\lambda T$ is not invertible\}, put $P_{0}=-(2 \pi i)^{-1} \rho_{\Gamma}(A-\lambda T)^{-1} T \mathrm{~d} \lambda$ and $P_{0}{ }^{+}=-(2 \pi i)^{-1} \rho_{\Gamma} T(A-\lambda T)^{-1} d \lambda$. Then $P_{0}$ is a projection and $P_{0}{ }^{+}=P_{0}{ }^{*}$. We call $H_{0}=\operatorname{Im} P_{0}$ the singular subspace and $H_{1}=\operatorname{Ker} P_{0}$ the regular subspace. We have $\operatorname{dim} \mathrm{H}_{0}<+\infty$ and
$H=H_{0} \oplus H_{1}, T\left[H_{0}\right]=H_{1}^{\perp}, A\left[H_{1}\right]=H_{0}^{\perp}=\bar{T}\left[H_{1}\right]$,
where Ker $A \subset H_{0}$ and $A$ acts as an invertible operator from $H_{1}$ onto $\mathrm{H}_{0}^{\perp}$. On $\mathrm{H}_{1}$ and $\mathrm{H}_{0}^{\perp}=\overline{\mathrm{T}\left[\mathrm{H}_{1}\right]}$ one can define in a unique way the bounded linear operators $S: H_{1} \rightarrow H_{1}$ and $S^{+}: H_{0}^{\perp} \rightarrow H_{0}^{\perp}$ such that ASk $=T k\left(k \in H_{1}\right)$ and $S^{+} A k=T k\left(k \in H_{1}\right)$; then $S$ and $S^{+}$are similar with similarity $A: H_{1} \rightarrow H_{0}^{\perp}$. We call $S$ the associate operator.
It appears that $S$ and $S^{+}$are self-adjoint with respect to the equivalent inner products
$\left\langle k_{1}, k_{2}>A=\left\langle A k_{1}, k_{2}>\left(\right.\right.\right.$ on $\left.H_{1}\right),\left\langle h_{1}, h_{2} A_{A}^{-1}=\left\langle A^{-1} h_{1}, h_{2}>\left(\right.\right.\right.$ on $\left.H_{0}^{\perp}\right)$, respectively.

Continuing our description (cf. Section III. 3 of [12]) we employ the self-adjointness of $S$ and $S^{+}$with respect to the inner products (1.5b) and denote by $F\left(F^{+}\right)$the resolution of the identity of $S\left(S^{+}\right)$. For $h \in H$ let $P_{p} h=F((0,+\infty))\left(I-P_{0}\right) h$, $P_{m_{+}}=F((-\infty, 0))\left(I-P_{0}\right) h, P_{p}^{+} h=F^{+}((0,+\infty))\left(I-P_{0}^{+}\right) h$ and $P_{m}{ }^{+} h=F^{+}((-\infty, 0))\left(I-P_{0}^{+}\right) h$. Then $P_{p}, P_{m}, P_{p}^{+}$and $P_{m}^{+}$are bounded projections on $H$. Their ranges we denote by $H_{p},{\underset{m}{m}}_{H_{m}}, H_{p}^{+}$and $H_{m}^{+}$,
respectively. It appears that

| $(1.4 a)$ | $H_{p}^{+}=\left(H_{m} \oplus H_{0}\right)^{\perp}=A\left[H_{p}\right]=\overline{T\left[H_{p}\right]}, P_{p}^{+}=P_{p}^{*} ;$ |
| :--- | :--- |
| $(1.4 b)$ | $H_{m}^{+}=\left(H_{p} \oplus H_{0}\right)^{\perp}=A\left[H_{m}\right]=\overline{T\left[H_{m}\right]}, P_{m}^{+}=P_{m}^{*} ;$ |
| $(1.4 c)$ | $H=H_{0} \oplus H_{p} \oplus H_{m}$. |

The projections satisfy the intertwining properties
(1.5a) $\quad T P_{0}=P_{0}{ }^{*} T, T P_{p}=P_{p}{ }^{*} T, T P_{m}=P_{m}{ }^{*} T$;
(1.5b) $\quad A P_{0}=P_{0}^{*} A, A P_{p}=P_{p}^{*} A, A P_{m}=P_{m}^{* A}$.

It is easily shown that $\mathrm{T}^{-1} \mathrm{~A}\left(\mathrm{AT}^{-1}\right)$ can be defined as a bounded operator on the finite-dimensional space $H_{0}\left(T\left[H_{0}\right]\right)$ and is nilpotent of order at most 2 (see [12], Proposition III 3.2).

With the operator $\mathrm{T}^{-1} \mathrm{~A}$ one associates three bounded analytic semigroups

$$
\begin{equation*}
\left(e^{-t T^{-1} A_{P^{\prime}}}\right)_{t \geq 0},\left(e^{+t T^{-1} A_{P_{m}}}\right)_{t \geq 0}, e^{-t T^{-1} A_{P_{0}}=\left(I-t T^{-1} A\right) P_{0} ; ~} \tag{1.6}
\end{equation*}
$$

with the adjoint operator $\mathrm{AT}^{-1}$ one connects the bounded analytic semigroups
(1.7) $\quad\left(e^{-t A T}{ }_{p}^{-1} P_{p}^{*}\right)_{t \geq 0},\left(e^{+t A T^{-1}} P_{m}^{*}\right)_{t \geq 0}, e^{-t A T^{-1}} P_{0}^{*}=\left(I-t A T^{-1}\right) P_{0}^{*}$.

For these semigroups the analogues of the intertwining properties (1.5a)-(1.5b) hold true (see Section III. 4 of [12]). In case the pair $(T, B)$ is positive definite, one has $H_{0}=\{0\}$ and $P_{0}=0$.

THEOREM 1.1 (=Theorem IV2.2 of [12]). Let (T,B) be a semi-definite admissible pair on $H$. Then the boundary value problem (0.1) - (0.2) has a unique solution $\psi$, namely

$$
\begin{align*}
& \psi(x)=\left[e^{\left.-x T^{-1} A_{P_{p}}+e^{(\tau-x)} T^{-1} A_{P_{m}}+\left(I-x T^{-1} A\right) P_{0}\right] V_{\tau}^{-1} \phi . ~ . ~ . ~ . ~}\right.  \tag{1.8a}\\
& \text { ( } 0<x<\tau \text { ) }
\end{align*}
$$

Here $\mathrm{V}_{\tau}$ is the invertible operator given by

$$
\begin{equation*}
V_{\tau}=P_{+}\left[P_{p}+e^{\left.+\tau T^{-1} A_{P_{m}}\right]+P_{-}\left[P_{m}+e^{-\tau T^{-1} A_{P}}\right]+P_{0}-\tau P_{-} T^{-1} A P_{0} . . . . ~}\right. \tag{1.8b}
\end{equation*}
$$

THEOREM 1.2. (cf.Theorem 6.1 of [13]). Let (T,B) be a semi-definite admissible pair on $H$. Then the boundary value
problem (0.1) - (0.7) has a unique solution $\psi$, namely
(1.9a) $\quad \psi(x)=T^{-1}\left[e^{-x A T^{-1}} P_{p}^{*}+e^{(\tau-x) A T^{-1}} P_{m}^{*}+\left(I-x A T^{-1}\right) P_{0}^{*}\right]\left(V_{\tau}^{+}\right)^{-1} x$.

$$
(0<x<\tau)
$$

Here $\mathrm{V}_{\tau}^{+}$is the invertible operator given by
(1.9b) $\quad V_{\tau}^{+}=P_{+}\left[P_{p}^{*}+e^{+\tau A T^{-1}} P_{m}^{*}\right]+P_{-}\left[P_{m}^{*}+e^{-\tau A T^{-1}} P_{p}^{*}\right]+P_{0}^{*}-P_{-} A T^{-1} P_{0}^{*}$.

From the intertwining properties (1.5a) (and their semigroup analogues) it is immediate that
(1.10) $\quad \mathrm{TV}_{\tau}=\mathrm{V}_{\tau}^{+} \mathrm{T}$.

The compactness of the operator $I-V_{\tau}^{+}$can be proved in the same way as the compactness of $I-V_{\tau}$ (see [12], proof of Theorem IV 2.1), and thus $\mathrm{V}_{\tau}^{+}$has a closed range. $A s \mathrm{~V}_{\tau}$ is invertible, (1.10) implies the invertibility of $V_{\tau}^{+}$. Now Theorem 1.2 can be derived by employing the proof of Theorem IV 2.1 of [12] up to formula (2.9) and by substituting the boundary conditions (0.7). Although not within the context of semi-definite pairs, an analogue of Theorem 1.2 appears in Section 6 of [13].

By specifying the theory of semi-definite admissible pairs for the specific pair ( $T, B$ ) in ( $0.6 a$ ) - ( 0.6 b ) on $L_{2}[-1,+1]$ one obtains statements on the unique solvability of the finite-slab problem in non-multiplying media (i.e., when $\hat{g}(t) \geq 0$ and $\left.{ }_{-1} \delta^{+1} \hat{\mathrm{~g}}(\mathrm{t}) \mathrm{dt} \leq 1\right)$. In [13] it is shown that basically the same results on unique solvability hold on $L_{p}[-1,+1](1 \leq p<+\infty)$ provided $\hat{g} \in L_{r}[-1,+1]$ for some $r>1$.

A few remarks of a historical nature are worthwhile. The abstract statement (0.1) - (0.2) of the finite-slab problem goes back to Hangelbroek [6] and triggered the author's research on this problem. Independent of and parallel to the investigation leading to [12] Hangelbroek proved the invertibility of $V_{\tau}$ for a case when the pair ( $T, B$ ) is positive definite. The author's invertibility proof was subsequently generalized for semi-definite pairs [12] and $L_{p}^{[-1,+1] ~(c f .[13]) . ~}$

## 2. REFLECTION AND TRANSMISSION OPERATORS

In this section we define and study reflection and transmission operators. Let (T, B) be a semi-definite admissible pair on the Hilbert space $H$ and introduce the short-hand notations

$$
\begin{equation*}
U_{p}^{\tau}=P_{p}+e^{+\tau T T^{-1} A_{P_{m}}+P_{0}, \quad U_{m}^{\tau}=P_{m}+e^{-\tau T^{-1}} A_{P_{p}}+\left(I-\tau T^{-1} A\right) P_{0} . . . . ~} \tag{2.1}
\end{equation*}
$$

Using these abbreviations we define the reflection operators $R_{+\tau}, R_{-\tau}, R_{+\tau}^{+}$and $R_{-\tau}^{+}$, and the transmission operators $T_{+\tau}, T_{-\tau}, T_{+\tau}^{+}$and $T_{-\tau}^{+}$as follows:
(2.2a) $\quad R_{+\tau}=U_{p}{ }^{\tau} V_{\tau}^{-1} P_{+}, R_{-\tau}=U_{m}{ }^{\tau} V_{\tau}^{-1} P_{-}$;
(2.2b) $T_{+\tau}=U_{m}{ }^{\tau} V_{\tau}^{-1} P_{+}, T_{-\tau}=U_{p}{ }^{\tau} V_{\tau}^{-1} P_{-}$;
(2.2c) $\quad R_{+\tau}^{+}=\left(U_{p}^{\tau}\right)^{*}\left(V_{\tau}^{+}\right)^{-1} P_{+}, R_{-\tau}^{+}=\left(U_{m}^{\tau}\right)^{*}\left(V_{\tau}^{+}\right)^{-1} P_{-} ;$

$$
\begin{equation*}
T_{+\tau}^{+}=\left(U_{m}^{\tau}\right) *\left(V_{\tau}^{+}\right)^{-1} P_{+}, T_{-\tau}^{+}=\left(U_{p}^{\tau}\right)^{*}\left(V_{\tau}^{+}\right)^{-1} P_{-}, \tag{2.2d}
\end{equation*}
$$

where the invertible operators $V_{\tau}$ and $V_{\tau}^{+}$are given by (1.8b) and (1.9b). Theorems 1.1 and 1.2 justify the existence of the above operators.

Let us explain the terms "reflection operator" and "transmission operator". From Theorem 1.1 it is clear that the unique solution $\psi_{\phi}$ of the boundary value problem (0.1)-(0.2) is continuous on $[0, \tau]$ and satifies

$$
\begin{equation*}
\psi_{\phi}(0)=R_{+\tau} P_{+} \phi+T_{-\tau} P_{-} \phi, \psi_{\phi}(\tau)=T_{+\tau} P_{+} \phi+R_{-\tau} P_{-} \phi . \tag{2.3a}
\end{equation*}
$$

The unique solution $\hat{\psi}_{X}$ of the boundary value problem (0.1)-(0.7) has the property that $T \hat{\psi}_{X}$ is continuous on $[0, \tau]$ and satifies

$$
\begin{equation*}
\left(T \hat{\psi}_{\chi}\right)(0)=R_{+\tau}^{+} P_{+} \chi+T+{ }_{-\tau}^{+} P_{-} \chi,\left(T \hat{\psi}_{\chi}\right)(\tau)=T{ }_{+\tau}^{+} P_{+} \chi+R_{-\tau}^{+} P_{-} \chi \tag{2.3b}
\end{equation*}
$$

Thinking of the specific example of radiative transfer, the operators $R_{+\tau}$ and $R_{-\tau}$ map the intensity of the radiation incident to the faces $x=0$ and $x=\tau$ into the sum of the intensities of incident and reflected radiation; the operators $T_{+\tau}$ and $T_{-\tau}$ map the former intensity into the sum of the intensities of incident
radiation and radiation transmitted from the opposite faces of the medium. In this way our choice of terminology is justified.

With the help of (1.5a) and (1.10) one gets the intertwining properties

$$
\begin{equation*}
\mathrm{TR}_{ \pm \tau}=\mathrm{R}_{ \pm}+{ }^{+} \mathrm{T}, \quad \mathrm{TT} \mathrm{~T}_{ \pm \tau}=\mathrm{T}_{ \pm \tau}^{+} \mathrm{T} \tag{2.4}
\end{equation*}
$$

Using the definitions of $V_{\tau}$ and $\mathrm{V}_{\tau}^{+}$it is clear that
(2.5a) $\quad P_{+} R_{+\tau}=P_{+}, P_{-} R_{-\tau}=P_{-}, P_{-} T_{+\tau}=0, P_{+} T_{-\tau}=0$;
(2.5b) $\quad P_{+} R_{+\tau}^{+}=P_{+}, P_{-} R_{-\tau}^{+}=P_{-}, P_{-} T_{+\tau}^{+}=0, P_{+} T_{-\tau}^{+}=0$.

Thus $R_{+\tau}, R_{-\tau}, R_{+\tau}^{+}$and $R_{-\tau}^{+}$are projections.
LEMMA 2.1. One has

$$
\begin{equation*}
R_{+\tau}^{+}=I-R_{-\tau}^{*}, R_{-\tau}^{+}=I-R_{+\tau}^{*} ; \tag{2.6a}
\end{equation*}
$$

(2.6b) $T_{+\tau}^{+}=T_{+\tau}^{*} \cdot T_{-\tau}^{+}=T_{-\tau}^{*}$.

In particular, one has the intertwining properties
(2.7a) $T R_{+\tau}=\left(I-R_{-}^{*}\right) T, T R_{-\tau}=\left(I-R_{+}^{*}\right) T$;
(2.7b) $\quad \mathrm{TT}_{+\tau}=\mathrm{T}_{+\tau}^{*} \mathrm{~T}^{\prime}, \mathrm{TT}_{-\tau}=\mathrm{T}_{-\tau}^{*} \mathrm{~T}^{( }$.

PROOF. First we prove that $\left(R_{-\tau}^{+}\right)^{*} R_{+\dot{\tau}}=0$. To see this, note that

$$
\left(R_{-\tau}^{+}\right)^{*}=P_{-}\left[\left(V_{\tau}^{+}\right)^{*}\right]^{-1} U_{m}^{\tau},\left(T_{-\tau}^{+}\right)^{*}=P_{-}\left[\left(V_{\tau}^{+}\right)^{*}\right]^{-1} U_{p}^{\tau}
$$

With the help of (2.2a) and (2.2b) one obtains

$$
\begin{aligned}
& \left(R_{-\tau}^{+}\right) * R_{+\tau}=P_{-}\left[\left(V_{\tau}^{+}\right) *\right]^{-1}\left[e^{\left.-\tau T^{-1} A_{P_{p}}+e^{+\tau T^{-1}} A_{P_{m}}+\left(I-\tau T^{-1} A\right) P_{0}\right] V_{\tau}^{-1} P_{+}}\right. \\
& \left(T_{-\tau}^{+}\right) * T_{+\tau}=P_{-}\left[\left(V_{\tau}^{+}\right) *\right]^{-1}\left[e^{\left.-\tau T^{-1} A_{P_{p}}+e^{+\tau T^{-1}} A_{P_{m}}+\left(I-\tau T^{-1} A\right) P_{0}\right] V_{\tau}^{-1} P_{+},}\right.
\end{aligned}
$$

and thus $\left(R_{-\tau}^{+}\right) * R_{+\tau}=\left(T_{-\tau}^{+}\right) * T_{+}$. But for every $h, k \in H$ we have

$$
\begin{aligned}
\left\langle\left(T_{-\tau}^{+}\right){ }^{*} T_{+\tau} h, k\right\rangle & =\left\langle T_{+\tau} h, T_{-\tau}^{+} k\right\rangle=\left\langle\left(I-P_{-}\right) T_{+\tau} h, T_{-\tau}^{+} k\right\rangle= \\
& =\left\langle T_{+\tau} h, P_{+} T_{-\tau}^{+} k\right\rangle=0,
\end{aligned}
$$

where we used that $P_{-} T_{+\tau}=P_{+} T_{-\tau}^{+}=0(\operatorname{see}(2.5))$. Therefore,

$$
\begin{equation*}
\left(R_{-\tau}^{+}\right) * R_{+\tau}=\left(T_{-\tau}^{+}\right) * T_{+\tau}=0 \tag{2.8}
\end{equation*}
$$

Next we derive the intertwining formulas

$$
\begin{equation*}
\left(I-R_{-\tau}^{*}\right) T=T R_{+\tau}, \quad\left(I-R_{+\tau}^{*}\right) T=T R_{-\tau} . \tag{2.9}
\end{equation*}
$$

For every $h, k \in H$ one has

$$
\begin{aligned}
& \left.<\left(I-R_{-\tau}^{*}\right) T h, k\right\rangle=\left\langle T h,\left(I-R_{-\tau}\right) k\right\rangle= \\
& =\left\langle T\left(I-R_{+\tau}\right) h,\left(I-R_{-\tau}\right) k\right\rangle+\left\langle T R_{+\tau} h, k\right\rangle-\left\langle T R_{+\tau} h, R_{-\tau} k\right\rangle .
\end{aligned}
$$

Since $\left(I-R_{+\tau}\right) h \in H_{-}$and $\left(I-R_{-\tau}\right) k \in H_{+}$(of. (2.5a)), the first
term at the right-hand side disappears. As $T R_{+\tau}=R_{+\tau}^{+}{ }_{T}(c f .(2.4)$ ) and (2.8) the third term vanishes and the first identity of (2.9)
follows. The second identity of (2.9) follows by taking adjoints.
Eqs (2.6a) are a corollary of (2.4) and (2.9).
To derive (2.6b), notice that

$$
\left(R_{+\tau}^{+}\right)^{*}=P_{+}\left[\left(V_{\tau}^{+}\right)^{*}\right]^{-1} U_{p}^{\tau},\left(T_{+\tau}^{+}\right) *=P_{+}\left[\left(V_{\tau}^{+}\right)^{*}\right]^{-1} U_{m}^{\tau} .
$$

Using these expressions one easily checks that

$$
\begin{equation*}
\left(T_{+\tau}^{+}\right) R_{+\tau}=\left(R_{+\tau}^{+}\right){ }^{*} T_{+\tau} \tag{2.10}
\end{equation*}
$$

We compute that

$$
\begin{aligned}
& \left\langle T_{+\tau}^{+} \mathrm{Th}, \mathrm{k}\right\rangle=\left\langle\mathrm{Th},\left(\mathrm{~T}_{+\tau}^{+}\right)^{*}\left(\mathrm{I}-\mathrm{R}_{+\tau}\right) \mathrm{k}\right\rangle+\left\langle\mathrm{Th},\left(\mathrm{R}_{+\tau}^{+}\right){ }^{*} \mathrm{~T}_{+\tau} \mathrm{k}\right\rangle= \\
& =\left\langle\mathrm{T}_{+\tau}^{+} \mathrm{Th},\left(\mathrm{I}-\mathrm{R}_{+\tau}\right) \mathrm{k}\right\rangle+\left\langle\mathrm{R}_{+\tau}^{+} \mathrm{Th}, \mathrm{~T}_{+\tau} \mathrm{k}\right\rangle= \\
& =\left\langle T+\tau T h, P_{-}\left(I-R_{+\tau}\right) k\right\rangle+\left\langle T R_{+\tau} h, P_{+} T+\tau\right\rangle= \\
& =\left\langle\mathrm{P}_{-} \mathrm{T}_{+}{ }_{+}^{+} \mathrm{Th},\left(\mathrm{I}-\mathrm{R}_{+\tau}\right) \mathrm{k}\right\rangle+\left\langle\mathrm{TP}+\mathrm{R}_{+\tau} \mathrm{h}, \mathrm{~T}_{+\tau} \mathrm{k}\right\rangle=\left\langle\mathrm{T}_{+\tau}^{*} \mathrm{Th}, \mathrm{k}\right\rangle,
\end{aligned}
$$

where we employed (2.4), (2.5a)-(2.5b) and (2.10). Hence, $\mathrm{T}_{+}^{+} \mathrm{T}=\mathrm{T}_{+}{ }^{*} \mathrm{~T}$ and the first identity of (2.6b) is clear. The second identity is proved likewise. Eqs (2.7b) follow by incorporating (2.4).

THEOREM 2.2. The following commutator relations hold
true:
(2.11a) $\dot{R}_{+\dot{\tau}}{ }^{T-T R}+\tau=R_{-\tau}^{*} B R_{+\tau}^{T-T}{ }_{-\tau}^{* B T}+\tau$;
(2.11b) $T_{+\tau} T-T T_{+\tau}=R_{+\tau}^{*} B T+\tau T_{+\tau}^{*} B R_{+\tau} T$.

PROOF. To establish (2.11a) we first note that
$\left(I-R_{+\tau}\right) h \in H_{-},\left(I-R_{-\tau}\right) k \in H_{+}$and thus $\left\langle\left(I-R_{+\tau}\right) h,\left(I-R_{-\tau}\right) k\right\rangle=0$.
So $0=\left(I-R_{-\tau}^{*}\right)\left(I-R_{+\tau}\right)=\left(I-R_{-\tau}^{*}\right)-R_{+\tau}+R_{-\tau}^{*} R_{+\tau}$. However, for every $h, k \in H$ one has

$$
\left\langle T_{-}^{*} T_{+} h, k\right\rangle=\left\langle T \tau_{+} h, T_{-\tau} k\right\rangle=\left\langle P_{+} T_{+\tau} h, T_{-\tau} k\right\rangle=\left\langle T+\tau, P_{+} T_{-\tau} k\right\rangle=0
$$

(cf.(2.5a)), and thus

$$
\begin{equation*}
R_{+\tau}-\left(I-R_{-\tau}^{*}\right)=R_{-\tau}^{*} R_{+\tau}=R_{-\tau}^{*} R_{+\tau}-T-\tau T_{+\tau}^{*} \tag{2.12}
\end{equation*}
$$

Next we calculate $R_{-\tau}^{*} A R_{+\tau}$ and $T_{-\tau}^{* A T}{ }_{+\tau}$. We have

$$
R_{-}{ }_{-}^{*} A R_{+\tau}=P_{-}\left(V_{\tau}^{*}\right)^{-1}\left(U_{m}^{\tau}\right)^{*} \cdot A U_{p}^{\tau} V_{\tau}^{-1} P_{+} .
$$

Using (1.5b) and its semigroup analogues we obtain

$$
R_{-}^{*} A R_{+\tau}=P_{-}\left(V_{\tau}^{*}\right)^{-1} A\left[e^{\left.+\tau T^{-1} A_{P_{m}}+e^{-\tau T^{-1}} A_{P_{p}}+\left(I-\tau T^{-1} A\right) P_{0}\right] V_{\tau}^{-1} P_{+} . . . . . . . . .}\right.
$$

In a similar way one computes $\mathrm{T}_{-}{ }_{-}^{*} \mathrm{AT}_{+\tau}$ and discovers that
(2.13)

$$
R_{-\tau}^{*} A R_{+\tau}=T_{-\tau}^{* A T}+\tau .
$$

From (2.12), (2.13) and $I-A=B$ one gets

$$
R_{+\tau}-\left(I-R_{-\tau}^{*}\right)=R_{-\tau}^{*} B R_{+\tau}-T-{ }_{-\tau}^{* B T}+\tau,
$$

and the commutator relation (2.11a) follows with the help of (2.7a).
To establish (2.11b) we first compute that
(2.14)

$$
\left.+\left(I-\tau T^{-1} A\right) P_{0}\right] V_{\tau}^{-1} P_{+}=T+{ }_{\tau}^{* A R}+\tau
$$

With the help of (2.5a) it is clear that

$$
\begin{equation*}
R_{+\tau}^{* T} T_{+\tau}=T_{+\tau}, T_{+\tau}^{*} R_{+\tau}=T_{+\tau}^{*} . \tag{2.15}
\end{equation*}
$$

Hence, employing (2.14) and (2.15) we obtain

$$
\begin{aligned}
& R_{+\tau}^{*} B T_{+\tau}^{-T}+{ }_{\tau}^{* B R_{+\tau}=\left\{R_{+\tau}^{*} T_{+\tau}-T_{+\tau}^{*} R_{+\tau}\right\}-} \\
& -\left\{R_{+\tau}^{* A T}+\tau{ }^{-T}+\tau A R_{+\tau}^{*}\right\}={ }^{*}+\tau-T_{+\tau}^{*}
\end{aligned}
$$

and formula (2.11b) follows with the help of (2.7b).
Obviously, (-T, B) is a semi-definite admissible pair on H. But now for this pair the roles of $P_{+}, P_{-}, P_{p}, P_{m}, P_{0}, A, T^{-1} A$ are played by the respective entities $P_{-}, P_{+}, P_{m}, P_{p}, P_{0}, A,-T^{-1} A$. Writing down (2.11a) and (2.11b) for the pair ( $-T, B$ ) we get
(2.16a) $R_{-\tau} T-P R_{-\tau}=R_{+\tau}^{* B R}-\tau{ }^{T}-T{ }_{+\tau}^{*} B T-\tau T$;

Substituting (2.7a)-(2.7b) at the left-hand side of (2.16a) and (2.11b) we get
(2.17a) $\quad \mathrm{R}_{+\tau}{ }^{*} \mathrm{~T}-\mathrm{TR}+{ }^{*}=\mathrm{R}_{+}{ }^{*} \mathrm{BR}-\tau, \mathrm{T}-\mathrm{T}+{ }^{*} \mathrm{BT}_{-\tau} \mathrm{T}$;

The same operations may be applied to the left-hand sides of (2.11a) and (2.16b).

PROPOSITION 2.3. Let $(T, B)$ be a semi-definite admissible pair on $H$. Then the following operators are compact:

$$
\begin{equation*}
R_{ \pm \tau}-P_{ \pm}, R_{ \pm \tau}^{*}-P_{ \pm}, T_{ \pm \tau}^{-e^{-\tau} \tau T^{-1}} P_{ \pm}, T_{ \pm \tau}^{*}-e^{-T_{T}^{-1}} P_{ \pm} \tag{2.18}
\end{equation*}
$$

In particular, if for some $0<\alpha<1$ we have $B=|T|^{\alpha} D$ and $D$ belongs to the $p$-th Von Neumann-Schatten class $C_{p}(1 \leq p<+\infty)$, then the operators (2.18) belong to the class $C_{p}$ too.

PROOF. Let us simplify the above problem first. Let $\mathcal{C}_{p}$ $(1 \leq p<+\infty)$ denote the $p-t h$ Von Neumann-Schatten class in $L(H)$ (of.[17] for the definition, examples and main properties of such a class).

Using (2.2a)-(2.2b) and the fact that $V_{\tau}-I \in C_{p}$ (see the proof of Th.IV 2.1 together with Lemma III 5.3 of [12]), we see
that in order to show that $R_{+\tau}-P_{+} \in C_{p}$ and $T_{+\tau}-e^{-\tau T^{-1}} P_{+} \in C_{p}$ it suffices to prove that $U_{p}^{\tau} P_{+}-P_{+} \in C_{p}$ and $U_{m}^{\tau} P_{+}-e^{-\tau T^{-1}} P_{+} \in C_{p}$.
 $e^{+\tau T^{-1}} A_{P_{m}}-e^{+\tau T T^{-1}} P_{-} \in C_{p}$. Then $U_{p}^{\tau} P_{+}-P_{+}=\left[P_{p}-P_{+}\right] P_{+}+\left[e^{+\tau T^{-1}} A_{P_{m}}-e^{+\tau T^{-1}} P_{-}\right] P_{+}+P_{0} P_{+} \epsilon C_{p}$ and $U_{m}^{\tau} P_{+}-e^{-\tau T^{-1}} P_{+}=\left[P_{m}-P_{-}\right] P_{+}+\left[e^{-\tau T^{-1}} A_{P} P^{-\tau T^{-1}} P_{+}\right]+\left(I-\tau T^{-1} A\right) P_{0} P_{+} \epsilon C_{p}$
(cf. Lemma III 5.3 of [12]). Therefore, $R_{+\tau}-P_{+} \in C_{p}$ and $T_{+\tau}-e^{-\tau T T^{-1}} P_{+} \epsilon C_{p}$, and in the same way we get $R_{-\tau}-P_{-} \in C_{p}$ and $T_{-\tau}-e^{+\tau T^{-1}} P_{-} \in C_{p}$.

It is sufficient to prove that for any semi-definite pair ( $T, B$ ) with $B=|T|^{\alpha} D$ for some $0<\alpha<1$ and $D \in C_{p}$ the operator $e^{-\tau T^{-1}} A_{P_{p}}-e^{-\tau T^{-1}} P_{+}$belongs to the class $C_{p}$. (By applying this
 For this we only have to prove that $e^{-\tau T^{-1}} A_{P_{p}}-e^{-\tau T} T^{-1} P_{+} A \in C_{p}$. (Note that $B=|T|^{\alpha} D \in C_{p}$ ). By Proposition III 1.4 of [12] we have

$$
\begin{equation*}
e^{-\tau T^{-1} A_{P}} P_{p} e^{-\tau T^{-1}} P_{+} A=-(2 \pi i)^{-1}{ }_{\Gamma} \int e^{-\tau / \lambda}\left[(T-\lambda A)^{-1}-(T-\lambda)^{-1}\right] A d \lambda \tag{2.19}
\end{equation*}
$$

where $\Gamma$ is the positively oriented pentagon with vertices 0 , (1-i) $\frac{1}{2} \sqrt{2}, M-i, M+i$ and (1+i) $\frac{1}{2} \sqrt{ } 2$ for some $M>\max (\|T\|,\|S\|)$; here $S$ is the associate operator of Section 2. Repeating the proof of Lemma III 5.3 of [12] we eventually conclude that the operator of (2.19) belongs to $C_{p}$. Hence, $R_{ \pm \tau}-P_{ \pm} \in C_{p}$ and $T_{ \pm \tau}-e^{\mp \tau T^{-1}}{ }_{P_{ \pm}} \in C_{p}$.

Finally, one easily proves that their adjoints belong to $C_{p}$ (cf. [17]). []

Let us return to the pair (T,B) in (1.2a)-(1.2b), where the expansion coefficients satisfy $a_{n} \leq 1(n=0,1,2, \ldots)$. For a polynomial phase function $\hat{g}(\mu)={ }_{n} \sum_{0}^{N} a_{n}\left(n+\frac{1}{2}\right) P_{n}(\mu)$ the conclusion of Proposition 2.3 obviously holds for $C_{1}$ being the ideal of trace class operators; the same conclusion is drawn if ${ }_{n} \sum_{0}^{+\infty}\left|a_{n}\right|<+\infty$ (cf. [12], Proposition VI 1.3). More generally, if the phase function $\hat{E} \in L_{2}[-1,+1]$ ( or equivalently, if $\left.{ }_{n} \sum_{0}^{+\infty}\left|a_{n}\right|^{2}<+\infty\right)$, then the same Proposition VI 1.3 of [12] implies that $R_{ \pm} \tau_{ \pm} P$ and $T_{ \pm \tau}-e^{\mp \tau T^{-1}} P_{ \pm}$ are Hilbert-Schmidt operators. So for $\hat{g} \in L_{2}[-1,+1]$ there exist functions $S(\mu, \nu)($ resp. $\rho(\nu, \mu))$ and $T(\mu, \nu)($ resp. $\sigma(\nu, \mu)$ ) such that

$$
\left(R_{+\tau} \phi\right)(-\mu) \quad=\frac{1}{2} \delta^{1} \mu^{-1} S(\mu, \nu) \phi(\nu) d \nu=2_{0} \delta^{1} \nu \rho(\nu, \mu) \phi(\nu) d v ;
$$

$$
\begin{equation*}
(0 \leq \mu \leq 1) \tag{2.20}
\end{equation*}
$$

$$
\left(T_{+\tau} \phi\right)(\mu)-e^{-\tau / \mu} \phi(\mu)=\frac{1}{2} \delta^{1} \delta^{-1} T(\mu, v) \phi(v) d v=2_{0} s^{1} v \sigma(v, \mu) \phi(v) d v,
$$

where the four integral operators have square integrable kernels. The functions $S(\mu, \nu)(r e s p . ~ \rho(\nu, \mu)$ ) and $T(\mu, \nu)(r e s p . \sigma(\nu, \mu))$ are known as the reflection and transmission functions of Chandrasekhar [2](resp. Sobolev [19]). As we shall see in the next section, for polynomial phase functions these functions are continuous for $0 \leq \mu, \nu \leq 1$.

## 3. REDUCTION TO AUXILIARY FUNCTIONS

In this section we deal with the concrete semi-definite admissible pair ( $T, B$ ) of ( 0.1 ) - (0.2) only. For the polynomial phase function

$$
\hat{g}(t)={ }_{n} \sum_{0}^{N} a_{n}\left(n+\frac{1}{2}\right) P_{n}(t) ; 0 \leq a_{0} \leq 1,-a_{0} \leq a_{n} \leq a_{0}(n=1,2, \ldots, N),
$$

we express the reflection operators $R_{+\tau}$ and $R_{-\tau}$, the transmission operators $T_{+\tau}$ and $T_{-\tau}$ and their adjoints in the $2 N+2$ auxiliary functions ${ }^{R}+{ }_{+}^{*} P_{n}$ and $T{ }_{+T}^{*} P_{n}(n=0,1, \ldots, N)$. On $L_{2}[-1,+1]$ (but also on $L_{p}[-1,+1]$ for other $\left.1 \leq p<+\infty\right)$ we define the "inversion symmetry" $J$ by $(J h)(\mu)=h(-\mu)(-1 \leq \mu \leq+1)$; this operator is an isometry with $J^{2}=I$ and will be employed throughout this paper. It has the following properties:
(3.1) $T J=-J T, J B=B J$.

In the context of semi-definite admissible pairs it was systematically studied in [12](Sections III.6-III.7) and it was shown that

$$
\begin{aligned}
& J P_{+}=P_{-} J, J P_{p}=P_{m}^{J}, J P_{p}^{*}=P_{m}^{* J}, J P_{0}=P_{0} J \\
& J e^{-t T^{-1} A_{P_{p}}=e^{+t T^{-1}} A_{P_{m}} J, J e^{-t A T^{-1}} P_{p}^{*}=e^{+t A T^{-1}} P_{m}^{* J} .}
\end{aligned}
$$

Hence, it necessarily satisfies the identities
(3.2a) $\quad J V_{\tau}=V_{\tau} J, J V_{\tau}^{+}=V_{\tau}^{+} J$;
(3.2b) $\quad J R_{+\tau}=R_{-\tau} J, J T_{+\tau}=T_{-\tau} J, J R_{+\tau}^{+}=R_{-\tau}^{+}{ }_{J}, J T_{+\tau}^{+}=T_{-\tau}^{+} J$.

So it suffices to derive expressions for $R_{+\tau}, T_{+\tau}$ and $R_{+\tau}^{+}, T_{+\tau}^{+}$.
THEOREM 3.1. For every $\phi, x \in L_{1}[-1,+1]$ one has

$$
\begin{aligned}
& \left(R_{+\tau} \phi\right)(\mu)=\left\{\begin{array}{ll}
\phi(\mu) & , 0 \leq \mu \leq 1 ; \\
{ }_{2} f^{1} \nu \nu(\nu,-\mu) \phi(\nu) d \nu ; \\
& (-1 \leq \mu<0)
\end{array}\left(T_{+\tau} \phi\right)(\mu)= \begin{cases}0 \quad,-1 \leq \mu<0 ; \\
{ }^{0} f^{1} f_{\nu \sigma}(\nu, \mu) \phi(\nu) d \nu+ \\
+\mathrm{e}^{-\tau / \mu} \phi(\mu), 0 \leq \mu \leq 1 ;\end{cases} \right. \\
& \left(R_{+\tau}^{+} \chi\right)(\mu)=\left\{\begin{array}{ll}
\chi(\mu) & , 0 \leq \mu \leq 1 ; \\
{ }^{2} f^{1}{ }^{1} \mu \rho(\nu,-\mu) \chi(\nu) d \nu ; \\
(-1 \leq \mu<0)
\end{array} \quad\left(T_{+\tau}^{+} \chi\right)(\mu)= \begin{cases}0 \quad,-1 \leq \mu<0 ; \\
{ }^{0} \delta^{1} \mu \sigma(\nu, \mu) \chi(\nu) \mathrm{d} \nu+ \\
+\mathrm{e}^{-\tau / \mu} \chi(\mu), 0 \leq \mu \leq 1 .\end{cases} \right.
\end{aligned}
$$

Here the reflection function $\rho(\nu, \lambda)$ and the transmission function $\sigma(\nu, \lambda)$ are given by
(3.3a) $\rho(\nu, \lambda)=\frac{E(\nu, \lambda)+E(\lambda, \nu)}{2(\lambda+\nu)}, \sigma(\nu, \lambda)=\frac{E(\nu, \lambda)-E(\lambda, \nu)}{2(\lambda-\nu)}$, where
(3.3b) $E(v, \lambda)=\frac{1}{2}{ }_{n} \sum_{0}^{N} a_{n}\left(n+\frac{1}{2}\right)(-1)^{n} \rho_{n}(v) \sigma_{n}(\lambda)$;
(3.3c) $\rho_{n}(v)=\left(R_{+T}^{*} P_{n}\right)(v)-(-1)^{n}\left(T_{+\tau}^{*} P_{n}\right)(v)$;
(3.3d) $\sigma_{n}(\lambda)=\left(R_{+\tau}^{*} P_{n}\right)(\lambda)+(-1)^{n}\left(T_{+T}^{*} P_{n}\right)(\lambda)$.

PROOF. Because of the remarks made in the second last paragraph of Section 2 we may take $\phi, \chi \in L_{2}[-1,+1]$. Since we have the identities $\quad T R_{+\tau}=R_{+\tau}^{+} T$ and $T T_{+\tau}=T_{+\tau}^{+} T(c f .(2.4)$ ), it suffices to deduce the formulas for $R_{+\tau}$ and $T_{+\tau}$. First the derivation of the one for $R_{+\tau}$ is given. Repeated application of the commutator relation (2.11a) yields

$$
R_{+\tau} T^{k}-T^{k} R_{+\tau}={ }_{j}^{k-1}{ }_{=0}^{\Sigma} T^{k-1-j}\left\{R_{-\tau}^{*} B R_{+\tau}-T{ }_{-\tau}^{*} B T_{+\tau}\right\} T^{j+1}, k \in \mathbb{N} .
$$

For $k \in \mathbb{N}$ and $h \in L_{2}[-1,+1]$ we rewrite this identity as

$$
\begin{aligned}
& \left(R_{+\tau^{T}} T^{k}\right)(\mu)=\mu^{k}\left(R_{+\tau} h\right)(\mu)+{ }_{-1} f^{+1}{ }_{j}^{k-1} \sum_{0} \mu^{k-1-j} v^{j+1} h(v){ }_{n} \sum_{0}^{N} a_{n}\left(n+\frac{1}{2}\right) \\
& \left\{\left(R_{+\tau}^{*} P_{n}\right)(\nu)\left(R_{-\tau}^{*} P_{n}\right)(\mu)-\left(T{ }_{+\tau}^{*} P_{n}\right)(\nu)\left(T{ }_{-\tau}^{*} P_{n}\right)(\mu)\right\} d v .
\end{aligned}
$$

As $R_{ \pm}{ }^{*} \mathrm{P}_{\mathrm{n}}$ and $\mathrm{T}_{ \pm \mathrm{T}}{ }^{*} \mathrm{P}_{\mathrm{n}}$ belong to $\mathrm{H}_{+}=L_{2}[0,1]$, this formula remains correct when the integration is performed over [0,1] instead of $[-1,+1]$. Writing $j_{j=0}^{k-1} \mu^{k-1-j} v^{j}=\left(\nu^{k}-\mu^{k}\right) /(\nu-\mu)$ and $\phi(\mu)=\mu^{k}$ one gets

$$
\begin{equation*}
\left(R_{+\tau} \phi h\right)(\mu)=\phi(\mu)\left(R_{+\tau} h\right)(\mu)+ \tag{3.4}
\end{equation*}
$$

$$
+{ }_{0} \delta^{1} v \frac{\phi(v)-\phi(\mu)}{v-\mu} h(v)_{n} \sum_{=0}^{N} a_{n}\left(n+\frac{1}{2}\right)
$$

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As this identity is correct for $\phi(\mu) \equiv 1$, by linearity it is correct if $\phi$ is a polynomial and $h \in L_{2}[-1,+1]$. As for $h \in L_{\infty}[-1,+1]$ the integral operator in (3.4) is bounded, Eq. (3.4) is correct for all pairs $\phi \in L_{2}[-1,+1]$ and $h \in L_{\infty}[-1,+1]$. Define $K$ by

$$
\begin{equation*}
(K f)(\mu)=\left(R_{+\tau} f\right)(\mu)- \tag{3.5}
\end{equation*}
$$

$-0^{1} v f(v)_{n} \sum_{0}^{N} d_{n}\left(n+\frac{1}{2}\right) \frac{\left(R_{+T}^{*} P_{n}\right)(v)\left(R_{-t}^{*} P_{n}\right)(\mu)-\left(T_{+T}^{*} P_{n}\right)(\nu)\left(T_{-T}^{*} P_{n}\right)(\mu)}{v-\mu} d v$.
Then $K-R_{+\tau}$ is a compact operator from $L_{2}[0,1]$; from (3.4) (with $\phi(\mu)=\mu)$ it follows that

$$
T K h=K T h, \quad h \in L_{\infty}[-1,+1],
$$

and thus TK=KT. According to a classical result of Stone ([20], Theorem 8.1) there exists a function $\chi \in L_{\infty}[-1,+1]$ such that $K=\chi(T)$ (i.e., $K$ is the multiplication by $X$ ). Using that $R_{+\tau}-P_{+}$ is compact (cf. Proposition 2.3), one sees that $K-P_{+}$is a multiplyer by an $L_{\infty}$-function; such an operator is trivial (see the last paragraph of the proof of Theorem VI 3.1 of [12]); thus $K=P_{+}$. Substituting the identities (3.2b) and noting that $\left(J P_{n}\right)(\mu)=P_{n}(-\mu)=(-1)^{n} P_{n}(\mu)$ one obtains from (3.4) and (3.5) a formula for $R_{+\tau} f$, in which

$$
\rho(\nu, \lambda)=\frac{1}{2} n \sum_{0}^{N}{ }_{0}^{*}{ }_{n}\left(n+\frac{1}{2}\right)(-1-)^{n} \stackrel{\left(R_{+}^{*}+{ }_{\tau} P_{n}\right)(\nu)\left(R_{+}^{*} P_{n}\right)(\lambda)-\left(T+{ }_{\tau}^{*} P_{n}\right)(\nu)\left(T_{+}{ }_{\tau}^{*} P_{n}\right)(\lambda)}{\lambda+\nu} ;
$$

Defining $\rho_{n}(v)$ and $\sigma_{n}(\lambda)$ by (3.3c)-(3.3d) one easily reduces (3.6a) to the first part of (3.3a) where $E(\nu, \lambda)$ is given by (3.3b). The expression for $T_{+\tau}$ is derived analogously by first applying (2.11b) repeatedly. Now one exploits that
$T_{+\tau}-e^{-\tau T^{-1}} P_{+}$is a compact operator, but formulas (3.2b) do not play a role. The result obtained reads

$$
\sigma(\nu, \lambda)=\frac{{ }^{\frac{1}{2}} n}{N} \underline{\Sigma}_{=}^{N} a_{n}\left(n+\frac{1}{2}\right) \frac{\left(R_{+}^{*} P_{n}\right)(\lambda)\left(T{ }_{+\tau}^{*} P_{n}\right)(\nu)-\left(T{ }_{+\tau}^{*} P_{n}\right)(\lambda)\left(R_{+}^{*} P_{n}\right)(\nu)}{\lambda-\nu} .
$$

Usine (3.3c)-(3.3d) one finds from (3.6b) the second part of (3.3a).

In the notations of Sobolev [18] we have
$\phi_{n}=R_{+}{ }^{*} P_{n}$ and $\psi_{n}=T_{+}{ }^{*} P_{n}$; formulas (3.6) correspond to Eqs (30)-(31) of [18]. Mullikin [16] writes $\psi_{n}=R_{+\tau}^{*} P_{n}$ and $\phi_{n}=T_{+\tau}^{*} P_{n}$; formulas (3.6) correspond to Eqs (2.42) of [16]. Substituting $x=P_{k}(k=0,1, \ldots, N)$ into the formulas for $R_{+\tau}^{+} X$ and $T_{+\tau}^{+} X$ and employing that $\left(R_{+}{ }^{*} P_{k}\right)(\mu)=P_{k}(\mu)-(-1)^{k}\left(R_{+\tau}^{+} p_{k}\right)(-\mu)($ see $(2.6 a),(3.2 b))$ and $\left(T_{+}{ }_{\tau}^{*} P_{k}\right)(\mu)=\left(T_{+}{ }^{+} P_{k}\right)(\mu)(c f .(2.6 b))$ we obtain a coupled system of $2 N+2$ non-linear singular equations for the $2 N+2$ unknown $R_{+}{ }^{*} P_{n}$ and $T_{+}^{*} P_{n}(n=0,1, \ldots, N)$ corresponding to Eqs(2.43)-(2.44) of [16].

Hovenier defines one two-variable exit function $E(\nu, \lambda)$ as follows:

$$
E(\nu, \lambda)=(\lambda+\nu) \rho(\nu, \lambda)+(\lambda-v) \sigma(\nu, \lambda)
$$

(cf.[8], (2)). As $\rho(\nu, \lambda)=\rho(\lambda, v)$ and $\sigma(\nu, \lambda)=\sigma(\lambda, v)(c f .(3.6))$, it is clear that $E(\nu, \lambda)$ is given by Eqs (3.3a)-(3.3d) (cf. [8], (3)-(4); also (10)-(11)). In this way Hovenier obtained a simplification of formulas presented by Mullikin and Sobolev.

## 4. SPECIAL FUNCTIONS OF TRANSPORT THEORY

Fon later use and to enhance the readibility of section 5 we expose functions known to transport theorists and present the structure of the singular and regular subspaces $H_{0}$ and $H_{1}$ in the case of the semi-definite pair ( $T, B$ ) in ( 0.1 )-(0.2). Consider the polynomials $\left(H_{n}\right)_{n=0}^{+\infty}$ satisfying the recurrence relation (4.1a) $(2 n+1)\left(1-a_{n}\right) \mu H_{n}(\mu)=(n+1) H_{n+1}(\mu)+n H_{n-1}(\mu)$;
(4.1b) $\quad H_{0}(\mu) \equiv 1, H_{1}(\mu)=\left(1-a_{0}\right) \mu$.

For $a_{0}=a_{1}=\ldots=0$ these polynomials are just the usual Legendre polynomials. Further, substituting $\mu=0$ one sees that $H_{n}(0)=P_{n}(0)(n \geq 0)$.

In terms of these and the Legendre polynomials one defines the characteristic binomial $\psi(\nu, \mu)$ and the characteristic function $\psi(\mu)$ by

$$
\begin{equation*}
\psi(\nu, \mu)={ }_{n=0}^{N} a_{n}\left(n+\frac{1}{2}\right) H_{n}(\nu) P_{n}(\mu), \psi(\mu)=\psi(\mu, \mu) . \tag{4.2}
\end{equation*}
$$

The dispersion function $\wedge$ is defined by either one of the expression
(4.3a)

$$
\Lambda(\lambda)=1+\lambda-1 f^{+1}(\mu-\lambda)^{-1} \psi(\mu) d \mu=1-a_{0}+_{-1} \delta^{+1} \mu(\mu-\lambda)^{-1} \psi(\lambda ; \mu) d \mu ;
$$

taking Cauchy principal values one gets
(4.3b) $\quad \lambda(\nu)=\lim _{\varepsilon+0} \frac{1}{2}\{\Lambda(\nu+i \varepsilon)+\Lambda(\nu-i \varepsilon)\}=1-a_{0}{ }^{+}-1^{q^{+1}} \mu(\mu-\nu)^{-1} \psi(\nu, \mu) d \mu$.

Due to symmetries in the polynomials in (4.1a) we have
(4.4a) $\psi(-v,-\mu)=\psi(\nu, \mu), \psi(-\nu, \mu)=\psi(\nu,-\mu), \psi(-\mu)=\psi(\mu)$;
(4.4b) $\quad \lambda(-\lambda)=\lambda(\lambda) \quad, \lambda(-\nu)=\lambda(\nu)$.

The dispersion function $\Lambda$ is analytic on $\psi_{\infty}$ cut along
$[-1,+1]$ and continuous on the imaginary axis. Because in terms of a determinant
(4.5) $\left.\quad \wedge(\lambda)=\operatorname{det}(T-\lambda A)(T-\lambda)^{-1}(\lambda \in \mathbb{C} \backslash-1,+1]\right)$
for a polynomial phase function (cf.[6], Lemma 3.1;[11], Proposition 1 ; these results continuously extend to the conservative case), outside $[-1,+1]$ the zeros of $\Lambda(\lambda)$ correspond to the spectrum of the operator polynomial $T-\lambda A$. In fact, for the associate operator we have

$$
\left.\sigma(S)=\sigma\left(S^{+}\right)=[-1,+1] \cup\{\lambda \in \mathbb{C} \backslash-1,+1]: \wedge(\lambda)=0\right\} ;
$$

at infinity $\Lambda(\lambda)$ has a zero of multiplicity $s=\operatorname{dim} H_{0}$, where $H_{0}$ is the singular subspace. The finite zeros of $\Lambda(\lambda)$ are real and simple (see the second paragraph at page 238 of [4]; later proofs appear for $0 \leq c<1$ in $[6,11]$ ). Note that

$$
\text { (4.6) } \quad \lim _{\varepsilon \neq 0} \wedge(\nu \pm i \varepsilon)=\lambda(\nu) \pm i \pi \nu \psi(\nu) \quad(-1<\nu<+1) \text {. }
$$

The next result has recently been proved in a clear way by Garcia and Siewert [5].

PROPOSITION 4.1. For $v \notin[-1,+1]$ thefunctions $\wedge(\nu)$
and $\psi(\nu)$ do not have common zeros. For $v \in(-1,+1)$ there are no common zeros of $\lambda(\nu)$ and $\psi(\nu)$. If $\psi( \pm 1)=0$, then the limits $\lim _{\nu \downarrow 1} \wedge(\nu)$ and $\lim _{\nu \uparrow-1} \wedge(\nu)$ exist and are non-zero.

In the non-conservative case $0 \leq c<1$ the results of this proposition are claimed by Hangelbroek [6] and Lekkerkerker [11]. At page 313 of [11]a proof of the third statement is given. In [6] the second statement is proved. Garcia and Siewert [5] prove all three statements in a completely different way and their
proof also applies to the conservative case $c=1$ and critical $c>$ media.

As exposed in Section 1, to every semi-definite admissible pair ( $T, B$ ) on a Hilbert space $H$ there exist a singular subspace $H_{0}$ and a regular subspace $H_{1}$ such that $H_{0} \oplus H_{1}=H$ and (4.7) $\quad T\left[H_{0}\right]=H_{1}^{\perp}, \quad \overline{T\left[H_{1}\right]}=A\left[H_{1}\right]=H_{0}^{\perp}$
(cf. (1.2)). On $H_{1}\left(\right.$ resp. $H_{0}^{\perp}$ ) the bounded operator $S\left(r e s p . S^{+}\right.$) is uniquely defined by $A S k=T k=S^{+} A k$ for every $k \in H_{1}$.

PROPOSITION 4.2. (=special case of Th.VI 4.1 of [12]). Let $a_{0}=a_{1}=\ldots=a_{m-1}=1$ and $-1 \leq a_{n}<+1(n=m, m+1, \ldots, N)$. Let $s=m$ for even $m$ and $s=m+1$ for odd $m$; then $s$ is the dimension of the singular subspace $H_{0}$. In particular,
(4.8a) $\quad H_{0}=\operatorname{span}\left\{P_{0}, P_{1}, \ldots, P_{S-1}\right\}, H_{1}=\overline{\operatorname{span}\left\{P_{n}: n \geq s+1\right\} \oplus} \operatorname{span}\left\{T^{-1} P_{\text {: }}\right.$
(4.8b) $T\left[H_{0}\right]=\operatorname{span}\left\{T P_{0}, T P_{1}, \ldots, T P_{s-1}\right\}, \overline{T\left[H_{1}\right]}=\overline{\operatorname{span}\left\{P_{n}: n \geq s\right\}}$.

The formula for $H_{0}$ is immediate from Theorem VI 4.1 of [12]. The subspaces $\overline{\mathrm{T}\left[\mathrm{H}_{1}\right]}=\mathrm{H}_{0}^{1}$ and $\mathrm{T}\left[\mathrm{H}_{0}\right]$ follow directly. Finally, $H_{1}=\left(T\left[H_{0}\right]\right)^{\perp}$ is computed. For the case needed the elementary formul (4.8a)-(4.8b) do not appear in literature.
5. DIAGONALIZATION OF THE ASSOCIATE OPERATOR

In this section we diagonalize the associate operators $S \in L\left(H_{1}\right)$ and $S^{+} \in L\left(H_{0}\right)$ and apply these results to prove the Hollder continuity of the auxiliary functions $R_{+}{ }_{\tau}^{*} P_{n}$ and $T+{ }_{\tau}^{*} P_{n}$ ( $n=0,1, \ldots, N$ ). For the non-conservative case $0<a_{0}<1$ and the conservative isotropic case $\left(N=0, a_{0}=1\right)$ these diagonalizations are due to Hangelbroek [6] and Lekkerkerker [10,11]. Here we present them in a form inspired by Eq. (2.10) of [14], which is more suita to our purposes, and generalize them to the conservative case $a_{0}=1$. In this way a non-routine extension of $[6,10,11]$ is derived.

THEOREM 5.1. Leet $a_{0}=a_{1}=\ldots=a_{m-1}=1$ and $-1 \leq a_{n}<+1$
$(n=m, m+1, \ldots, N)$, and put $N=\sigma\left(S^{+}\right)$. Let $s=m$ for even $m$ and $s=m+1$ for odd m . Then there exists a finite Borel measure $\sigma$ on $N=\sigma\left(\mathrm{S}^{+}\right)$
and an operator $\mathrm{F}^{+}$from $L_{2}[-1,+1]$ onto the Hilbert space $L_{2}(N)$ of $\sigma$-square integrable functions on N with the following properties:
(i) $\operatorname{Ker} \mathrm{F}^{+}=T\left[\mathrm{H}_{0}\right]=\operatorname{span}\left\{T \mathrm{P}_{\mathrm{n}}: 0 \leq \mathrm{n} \leq \mathrm{s}-1\right\}, \operatorname{Im~} \mathrm{F}^{+}=L_{2}(\mathrm{~N})_{\sigma}$;

$$
\begin{equation*}
\delta_{N}\left(F^{+} h_{1}\right)(v) \overline{\left(F^{+} h_{2}\right)(v)} d \sigma(v)=\left\langle A^{-1} h_{1}, h_{2}\right\rangle\left(h_{1}, h_{2} \in \overline{T\left[H_{1}\right]}\right) ; \tag{ii}
\end{equation*}
$$

(iii) $\left(F^{+} P_{n}\right)(v)=H_{n}(v) \quad(v \in N ; n=0,1,2, \ldots)$.

In terms of a Cauchy principal value one has
(5.1) $\quad\left(\mathrm{F}^{+} \mathrm{h}\right)(\nu)=\lambda(\nu) h(\nu){ }_{-1} £^{+1} \nu(\mu-\nu)^{-1} \psi(\nu, \mu) h(\mu) d \mu$. $\left(\nu \in N, h \in L_{2}[-1,+1]\right)$

The unique vector $h_{S} \in \overline{T\left[H_{1}\right]}$ such that $\left(F^{+} h_{s}\right)(v) \equiv 1$ is given by $h_{s}=P_{s} / P_{s}(0)$.

As A acts as an invertible operator from $H_{1}$ onto $\overline{\mathrm{T}\left[\mathrm{H}_{1}\right]}$,
the operator $F F^{+} A: L_{2}[-1,+1] \rightarrow L_{2}(N)_{\sigma}$ has the property $\left(\mathrm{F}^{+} \mathrm{Th}\right)(v)=v\left(\mathrm{~F}^{+} \mathrm{Ah}\right)(v)=v(\mathrm{Fh})(v)\left(v \in N, h \in L_{2}[-1,+1]\right)$; so it is given by

$$
\text { (5.2) } \quad(F h)(v)=\lambda(v) h(\nu)-1^{£^{+1}} \mu(\mu-v)^{-1} \psi(\nu, \mu) h(\mu) d \mu
$$

The unique vector $k_{s} \in H_{1}$ such that $\left(\mathrm{Fk}_{\mathrm{s}}\right)(\nu) \equiv 1$ is given by $k_{S}=\left\{(2 s+1)\left(1-a_{s}\right) P_{S}(0)\right\}^{-1}(s+1) T^{-1} P_{S+1}$. The properties of $F$ are proved in a straight:orward way once Theorem 5.1 is established.

PROOF OF THEOREM 5.1. Let us denote the right-hand side of (5.1) by (Gh) $(v)$. Then
$\left(\operatorname{GP}_{0}\right)(v)=\lambda(v){ }_{-1} \mathbb{f}^{+1} v(\mu-v)^{-1} \psi(v, \mu) d \mu=1-a_{0}{ }^{+}{ }_{-1} \Phi^{+1}\{(\mu-v)+v\}(\mu-v)^{-1} \psi(\nu, \mu)$,
${ }_{-1} f^{+1} \nu(\mu-v)^{-1}{ }_{\psi}(\nu, \mu) \mathrm{d} \mu=1-\mathrm{a}_{0}{ }^{+}{ }_{-1} \delta^{+1} \psi(\nu, \mu) \mathrm{d} \mu=1=\mathrm{H}_{0}(\nu), \quad v \in \mathrm{~N} \quad(c \mathrm{f} .(4.3 \mathrm{~b})$,
(4.2)). Assume that $\left(\operatorname{GP}_{n}\right)(v)=H_{n}(v)$ for $n=0,1, \ldots, k$, and let us compute $(k+1)\left(G P_{k+1}\right)(v)$. Inserting the recurrence relation $(k+1) P_{k+1}=(2 k+1) T P_{k}-k P_{k-1}$ for the Legendre polynomials (cf.(4.1a) with $\left.a_{0}=a_{1}=\ldots=0\right)$, one gets $(k+1)\left(G P_{k+1}\right)(v)=(k+1) \lambda(v) P_{k+1}(v)$
${ }_{-1} £^{+1} \nu(\mu-\nu)^{-1} \psi(\nu, \mu)\left[(2 k+1)\{(\mu-\nu)+\nu\} P_{k}(\mu)-k P_{k-1}(\mu)\right] d \mu=$ $(k+1) \lambda(\nu) P_{k+1}(\nu)-(2 k+1) \nu_{-1} \rho^{+1} \psi(\nu, \mu) P_{k}(\mu) d \mu-(2 k+1) \nu_{-1}{ }^{+1} \nu(\mu$ $-v)^{-1} \psi(\nu, \mu) P_{k}(\mu) d \mu+k_{-1} f^{+1} \nu(\mu-\nu)^{-1} \psi(\nu, \mu) P_{k-1}(\mu) d \mu$. Now we employ
the induction hypothesis and the recurrence relation for the Legendre polynomials to simplify this expression and get

$$
\begin{aligned}
& (k+1)\left(G P_{k+1}\right)(\nu)=-(2 k+1) \nu \nu_{-1} \rho^{+1} \psi(\nu, \mu) P_{k}(\mu) d \mu+(2 k+1) \nu H_{k}(\nu) \\
& -k H_{k-1}(\nu)=(2 k+1)\left(1-a_{k}\right) \nu H_{k}(\nu)-k H_{k-1}(\nu)=(k+1) H_{k+1}(\nu) ;
\end{aligned}
$$

By induction we may conclude that $\left(G P_{n}\right)(v)=H_{n}(v)$ for $n=0,1,2, \ldots$.
From (4.1a)-(4.1b) it follows that
(5.3a) $\quad H_{1}=H_{3}=\ldots=H_{s-1}=0$, deg $H_{0}=\operatorname{deg} H_{2}=\ldots .=\operatorname{deg} H_{s}=0$;
(5.3b) $\quad \operatorname{deg} H_{s+k}=k(k \geq 0), \lim _{n \rightarrow+\infty}\left\{n-\operatorname{deg} H_{n}\right\}=s$.

So for a polynomial $h$ we have ( $G h(v)=0(\nu \in N)$ if and only if $h \in \operatorname{span}\left\{T P_{n}: 0 \leq n \leq s-1\right\}=T\left[H_{0}\right]$; further, $G$ maps the polynomials onto polynomials. If $P$ is a polynomial, then $\{P: G P$ is constant\} $=\{P: \operatorname{deg} P \leq s\}$, while $\operatorname{deg} P-\operatorname{deg} G P=s$ whenever $\operatorname{deg} P \geq s+1$.

Note that $h_{S}=P_{s} / P_{S}(0) \in \overline{T\left[H_{1}\right]}$ (see Proposition 4.2) and $\left(F^{+} h_{s}\right)(v)=H_{s}(v) / P_{S}(0)=H_{S}(v) / H_{S}(0) \equiv 1$. As one easily checks from (4.1a), for a polynomial $k \in \bar{T}\left[\mathrm{H}_{1}\right]=A\left[H_{1}\right]$ we have
(5.4) $\quad\left(\operatorname{GS}^{+} h\right)(v)=v(G h)(v), v \in N=\sigma\left(S^{+}\right)$.

So G maps span $\left\{\left(S^{+}\right)^{n_{h}}{ }_{s}: n \geq 0\right\}$ onto the set of polynomials (on $H$ ). But on $\left.\overline{\mathrm{TH}_{1}}\right]$ the operator $\mathrm{S}^{+}$is self-adjoint with respect to a suitable inner product (cf. (1.3)). Since deg ( $\left.S^{+}\right)^{n_{n}} h_{s}=s+\operatorname{deg} G\left(S^{+}\right)^{n_{1}}$ $=s+n+\operatorname{deg} G h_{s}=s+n(n=0,1,2, \ldots)$, it is clear that span $\left\{\left(\mathrm{S}^{+}\right)^{\mathrm{n}_{\mathrm{h}}}: \mathrm{n} \geq 0\right\}$ is a dense linear subspace of $\overline{\mathrm{T}\left[\mathrm{H}_{1}\right]}$. By Von Neumann's spectral theorem for self-adjoint operators with a simplt spectrum ([20], Theorem 7.10) there exists a finite Borel measure $\sigma$ on $N=\sigma\left(S^{+}\right)$and an invertible operator $\mathrm{F}^{+}: \overline{\mathrm{T}\left[\mathrm{H}_{1}\right]} \rightarrow L_{2}(N)$, which has the property $\left(F^{+} S^{+} h\right)(\nu)=\nu\left(F^{+} h\right)(\nu)\left(\nu \in N ; h \in T\left[H_{1}\right]\right)$ and the property (ii) of the theorem. It is clear that $\mathrm{F}^{+}$extends to a bounded operator $\mathrm{F}^{+}: L_{2}[-1,+1] \rightarrow L_{2}(N)$ by setting $F^{+} h=0$ for $h \in T\left[H_{0}\right]$. By (5.4) we have $\mathrm{F}^{+} \mathrm{h}=\mathrm{Gh}$ on the polynomials $h$. Hence, formula (5.1) and properties (i) and (iii) are clear.

A complete orthogonal system of $L_{2}(N)_{\sigma}$ is the sequence $\left(H_{s+n}\right)_{n=0}^{+\infty}$; it has the property

$$
\begin{equation*}
N^{\int H_{s+n}}(v) H_{s+k}(v) d \sigma(v)=\left\langle A^{-1} P_{s+n}, P_{s+k}\right\rangle=\frac{\delta_{n+k}}{\left(s+n+\frac{1}{2}\right)\left(1-a_{s+n}\right)} . \tag{5.5}
\end{equation*}
$$

THEOREM 5.2. Let $a_{0}=a_{1}=\ldots a_{m-1}=1$ and $-1 \leq a_{n}<+1$ $(n=m, m+1, \ldots, N)$, and put $N=\sigma\left(S^{+}\right)$. Let $s=m$ for even $m$ and $s=m+1$ for odd m . If $\sigma$ is the measure and $\overline{F^{+}}: L_{2}\left[\overline{-1,+1]} \rightarrow L_{2}(\bar{N}) \sigma\right.$ the surjective operator of Theorem 5.1 , then on the polynomials $p$ the right inverse of $\mathrm{F}^{+}$with range $\overline{\mathrm{T}\left[\mathrm{H}_{1}\right]}=\operatorname{span}\left\{\mathrm{P}_{\mathrm{n}}: \mathrm{n} \geq \mathrm{s}\right\}$ has the form
(5.6a) $\quad\left(\left(F^{+}\right)^{-1} p\right)(\mu)=p(\mu)+_{N} f \mu \frac{p(\mu)-p(\nu)}{\mu-\nu} \psi_{S}(\nu, \mu) d \sigma(\nu)(-1 \leq \mu \leq+1)$, where $\psi_{S}(\nu, \mu)$ is defined by

$$
\begin{equation*}
\psi_{S}(\nu, \mu)=\stackrel{N-s}{\sum_{k}} 0 a_{s+k}\left(s+k+\frac{1}{2}\right) H_{s+k}(\nu) P_{s+k}(\mu) \tag{5.6b}
\end{equation*}
$$

Moreover, $\mathrm{F}^{+}$maps functions on $[-1,+1]$ that are Hölder continuous of exponent $0<\alpha<1$ except for a possible jump discontinuity at $\mu=0$, onto functions on $N$ of the same type.

PROOF. Recall that $\left(\mathrm{H}_{\mathrm{s}+\mathrm{k}}\right)_{\mathrm{k}=0}^{+\infty}$ is a complete orthogonal system of $L_{2}(N)_{\sigma}$ satisfying the recurrence relation
(5.7) $\quad(2 s+2 k+1)\left(1-a_{s+k}\right) T_{N} H_{s+k}=(s+k+1) H_{s+k+1}+(s+k) H_{s+k-1}$, where $\left(T_{N} p\right)(\nu)=\nu p(\nu)\left(\nu \in N, p \in L_{2}(N)_{\sigma}\right)$. Let us denote by (Gp)( $\mu$ ) the right-hand side of (5.6a). To prove (5.6a) it suffices to check it for $p=H_{s+k}$ by induction on $k$.

As for $h_{s}=P_{S} / P_{S}(0) \in \overline{T\left[H_{1}\right]}$ one has $\left(F^{+} h_{s}\right)(v) \equiv 1=H_{s}(v) / H_{s}(0)=$
$H_{S}(\nu) / P_{S}(0)$, formula (5.6a) is correct for $p=H_{s}$. Suppose that (5.6a) is correct for $p \in\left\{H_{S}, H_{s+1}, \ldots, H_{s+k}\right\}$. Using (5.7) one gets

$$
\begin{aligned}
& (s+k+1)\left(\mathrm{GH}_{s+k+1}\right)(\mu)=(s+k+1) H_{S+k+1}(\mu)+ \\
& +(2 s+2 k+1)\left(1-a_{s+k}\right)\left\{\mu_{N} \int \mu \frac{H_{s+k}(\mu)-H_{s+k}(\nu)}{\mu-\nu} \psi_{S}(\nu, \mu) d \sigma(\nu)+\right. \\
& \left.+\mu_{N} \int H_{s+k}(\nu) \psi_{S}(\nu, \mu) d \sigma(\nu)\right\}- \\
& -(s+k)_{N} \delta \mu \frac{H_{s+k-1}(\mu)-H_{s+k-1}(\nu)}{\mu-v} \psi_{S}(\nu, \mu) d \sigma(\nu) .
\end{aligned}
$$

Using the induction hypothesis and the recurrence relation for the Legendre polynomials it is clear that

$$
\begin{aligned}
& (s+k+1)\left(\mathrm{GH}_{s+k+1}\right)(\mu)=(s+k+1) P_{s+k+1}(\mu)+ \\
& +(2 s+2 k+1) \mu\left\{-a_{s+k} P_{s+k}(\mu)+{ }_{N} \int H_{s+k}(\nu) \psi_{s}(\nu, \mu) d \sigma(v)\right\}
\end{aligned}
$$

Inserting (5.6b) and employing (5.5) one obtains $\mathrm{GH}_{\mathrm{S}+\mathrm{k}+1}=\mathrm{P}_{\mathrm{s}+\mathrm{k}+1}$,
and formula (5.6a) is clear for $p=H_{s+k+1}$.
Given the phase function $\hat{g}(t)={ }_{n} \sum_{0}^{N} a_{n}\left(n+\frac{1}{2}\right) P_{n}(t)$ with $0<a_{0} \leq+1$ and $-a_{0} \leq a_{n} \leq a_{0}(n=1,2, \ldots, N)$, we consider the nonconservative phase functions $c \hat{g}$ for $c \uparrow 1$. If $S$ is the operator on the regular subspace $H_{1}$ satisfying $A S h=T h$ for $h \in H_{1}$, choose $M>\|S\|$ and let $\Gamma$ be the positively oriented circle with centre 0 and radius $M$. Then for $1>c \geqslant c_{0}$ the number of zeros of the dispersion function $\Lambda_{C}(\lambda)$ of the phase function $c \hat{g}$ outside $\Gamma$ equals the multiplicity $s=d i m H_{0}$ of the zero of $\wedge(\lambda)$ at infinity; this follows from Rouché's theorem provided max $\left\{\left|\wedge(\lambda)^{-1}\left[\Lambda(\lambda)-\Lambda_{c}(\lambda)\right]\right|: \lambda \in \Gamma\right\}<1$ for $c_{0} \leq c<1$. Let $\left(A_{c} h\right)(\mu)=h(\mu)-c_{n} \sum_{0}^{N} a_{n}\left(n+\frac{1}{2}\right) P_{n}(\mu){ }_{-1} f^{+1} h\left(\mu^{\prime}\right) P_{n}\left(\mu^{\prime}\right) d \mu^{\prime}$ for $-1 \leq \mu \leq+1$. Define the projections $P_{1, c}$ and $P_{1, c}+$ by

$$
P_{1, c}=-(2 \pi i)^{-1} \delta\left(A_{c}-\lambda T\right)^{-1} T d \lambda, P_{1,} \stackrel{+}{c}=-(2 \pi i)^{-1} \int T\left(A_{c}-\lambda T\right)^{-1} d \lambda
$$

Then $P_{1, c}\left(\right.$ resp. $\left.P_{1},{ }_{c}^{+}\right)$tends to $I-P_{0}\left(r e s p . I-P_{0}^{*}\right)$ in the norm as $c \uparrow 1$, while

Put $\mathrm{H}_{1, c}=\operatorname{ImP}_{1, c}, \mathrm{H}_{1, \mathrm{c}}{ }^{+}=\operatorname{ImP}_{1, \mathrm{c}^{+}}$; then $\overline{\mathrm{T} \mathrm{H}_{1, c}}=\mathrm{H}_{1,}{ }^{+}$and $\mathrm{A} \mathrm{H}_{1, c}=\mathrm{H}_{1,}{ }^{+}$.
In the non-conservative case the Borel measure $\sigma$ on $N$ is absolutely continuous on $[-1,+1]$ with Radon-Nikodym derivative (5.8) $\quad(d \sigma / d \nu)=\left[\lambda(\nu)^{2}+\pi^{2} \nu^{2} \psi(\nu)^{2}\right]^{-1}, \quad-1<\nu<+1$
(cf. $[6,11]$ ). Let us extend this result to the conservative case with the help of a stability argument. Let $N=\sigma(S)=\sigma\left(S^{+}\right)$and $N_{c}=\left\{\nu \in \sigma\left(A_{c}^{-1} T\right):|\nu|<M\right\}$, where $c_{0} \leq c<1$. If $p$ is a polynomial and $c_{0} \leq c<1$ we have

$$
\delta_{N_{c}} p(\nu) d \sigma_{c}(v)=\left\langle A_{c}^{-1} P_{1},{ }_{c}^{*} p\left(S_{c}^{+}\right) h_{0}, h_{0}\right\rangle=\left\langle A_{c}^{-1} p\left(S_{c}^{+}\right) P_{1, c}^{*} h_{0}, P_{1}, c_{c}^{*} h_{0}\right\rangle,
$$

where $h_{0}(\mu)=P_{0}(\mu) / P_{0}(0)=1, \sigma_{c}$ is the Borel measure connected with $\mathrm{c} \mathcal{G}$ and $\mathrm{S}_{\mathrm{c}}^{+}$is the restriction of $\mathrm{TA}_{c}^{-1}$ to $\mathrm{H}_{1}{ }_{\mathrm{c}}^{+}$. For $\mathrm{c}+1$ the above expression tends to $\left\langle A^{-1} p\left(S^{+}\right)\left(I-P_{0}^{*}\right) h_{0},\left(I-P_{0}^{*}\right) h_{0}\right\rangle$. But $h_{s}-h_{0}=\left(P_{S} / P_{S}(0)\right)-\left(P_{0} / P_{0}(0)\right)=T \tilde{n}$ for some polynomial $\tilde{h}$ of degree $s-1$, and so $h_{s}-h_{0} \in \operatorname{span}\left\{T P_{0}, T P_{1}, \ldots, T P_{s-1}\right\}=T\left[H_{0}\right]=\operatorname{Ker}\left(I-P_{0}\right)$. So for every polynomial $p$ one obtains
$\lim _{c \uparrow 1} \int_{N_{c}} p(v) d \sigma_{c}(\nu)=<A^{-1} p\left(S^{+}\right) h_{S}, h_{S}>=\mathcal{K}_{N} p(\nu) d \sigma(v)$.
In the same way, using the stability of the spectral subspace of $S^{+}$corresponding to its spectrum on $[-1,+1]$, one gets

$$
\lim _{c+1} \delta_{[-1,+1]} p(\nu) d \sigma_{c}(v)=\varsigma_{[-1,+1]} p(v) \mathrm{d} \sigma(v) .
$$

But for $0<c<1$ the measure $\sigma_{c}$ is absolutely continuous on $[-1,+1]$. Further, $\lambda(\nu)$ and $\psi(\nu)$ do not have common zeros for $\nu \in(-1,+1)$ and if $\psi( \pm 1)=0$, the limit of $\lambda(\nu)$ as $v \rightarrow \pm 1$ does not vanish (see Proposition 4.1). So the expression $\left[\lambda^{2}(\nu)+\pi^{2} \nu^{2} \psi(\nu)^{2}\right]^{-1}$ is bounded and continuous on $[-1,+1]$ and therefore the measure $\sigma$ is absolutely continuous on $[-1,+1]$ with Radon-Nikodym derivative (5.8). As $\lambda^{2}(v)+\pi^{2} v^{2} \psi(v)^{2}$ appears to be Hölder continuous on ( $-1,+1$ ) and is bounded away from zero, the function (5.8) is Hölder continuous on $(-1,+1)$. Now it is quite trivial to see that for the measure $\sigma$ the operator $\left(\mathrm{F}^{+}\right)^{-1}$ of (5.6a) maps functions on $N$ that are Hölder continuous of exponent $0<\alpha<1$ except for a jump at $v=0$, into functions of the same type. From this and the similar property of $\mathrm{F}^{+}$one easily gets the second part of the theorem. $\square$

To determine the measure $\sigma$ at its discrete points $\nu \in N \backslash[-1,+1]$ one may follow the same method as in [6,11]. We point out that the first statement of Proposition 4.1 plays an indispensabl role in the derivation and so does the simplicity of the zeros $v \not \subset[-1,+1]$ of $\wedge(\nu)$. We get

$$
\sigma(\{v\})=\left\{v \wedge^{\prime}(v) \psi(v)\right\}^{-1}, v \in N \backslash[-1,+1] .
$$

COROLLARY 5.3 Let $0<a_{0} \leq 1$ and $-a_{0} \leq a_{n} \leq a_{0}(n=1,2, \ldots, N)$. Then for any polynomial $p$ the functions ${ }^{R}{ }_{+\tau}^{*} p$ and $T_{+}^{*} p$ are Hölder continuous of exponent $0<\alpha<1$ on $[0,1]$.

PROOF. Let $N=\sigma\left(S^{+}\right)$. Then for every polynomial $p$ one has

$$
\left[F^{+}\left(P_{p}^{*} P_{-}\right) p\right](v)= \begin{cases}--1 s^{0} v(\mu-v)^{-1} \psi(v, \mu) p(\mu) d \mu, v>0 \\ 0 & , v<0\end{cases}
$$

and thus, by the second part of Theorem $5.2, P_{p}^{*} P_{-}$and analogously $P_{m}^{*} P_{+}$extend to a compact operator on $H_{\alpha}[-1,0] \oplus H_{\alpha}[0,1]$, $0<\alpha<1$. Here $H_{\alpha}[a, b]$ is the Banach space of Hölder continuous functions $h:[a, b] \rightarrow \mathbf{C}$ of exponent $\alpha$. Also, by the second part of Theorem $5.2, e^{-t A T^{-1}} P_{p}^{*}$ and $e^{+t A T^{-1}} P_{m}^{*}$ are bounded on $H_{\alpha}[-1,0]$ $\oplus H_{\alpha}[0,1]$. As in the proof of Theorem IV 2.2 of [12] one shows the compactness of an operator; in this specific case we prove that $\mathrm{I}-\mathrm{V}_{\tau}^{+}$, with $\mathrm{V}_{\tau}^{+}$as in (1.9b), is a compact operator on $H_{\alpha}[-1,0] \oplus H_{\alpha}[0,1]$. As $H_{\alpha}[-1,0] \oplus H_{\alpha}[0,1]$ is densely embedded in $L_{2}[-1,+1]$ and $V_{\tau}^{+}$is invertible as an operator on $L_{2}[-1,+1]$, it is also invertible as an operator on $H_{\alpha}[-1,0] \oplus H_{\alpha}[0,1]$.

Next observe that for a polynomial. p the following identity holds:

$$
\left(F^{+} R_{+}^{*} p\right)(v)= \begin{cases}\left(F^{+} p\right)(v)-e^{-\tau / v}\left[F^{+}\left(V_{\tau}^{+}\right)^{-1} P_{-} p\right](v), v>0 \\ \left(F^{+} p\right)(\nu)- & {\left[F^{+}\left(V_{\tau}^{+}\right)^{-1} P_{-} p\right](v), v<0}\end{cases}
$$

(see (2.2c), (2.6a)). Again by the second part of Theorem 5.2, we get $R_{+\tau}^{*} p \in H_{\alpha}[-1,0] \boxplus H_{\alpha}[0,1]$. In the same way one shows that $T_{+}{ }_{\tau}^{*} p \in H_{\alpha}[-1,0] \oplus H_{\alpha}[0,1] . \square$

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C.V.M. van der Mee<br>Department of Physics and Astronomy Vrije Universiteit<br>De Boelelaan 1081<br>1081 HV Amsterdam<br>The Netherlands

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