GENERALIZED KINETIC EQUATIONS
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This paper is dedicated to K.M. Case on the occasion of his sixtieth birthday

We study the abstract differential equation $T \frac{\partial f}{\partial x}+A f=0$ on a Hilbert space $H$, which represents a variety of different kinetic equations. T is assumed bounded and self-adjoint on $H$, and A (unbounded) positive self-adjoint and Fredholm. For partial range boundary conditions and $0 \leqslant x<\infty$, we prove existence and (non-) uniqueness theorems and give representations of the solution. Various examples from neutron transport, radiative transfer of polarized and unpolarized light, and electron transport are given.

## INTRODUCTION

In 1960, K. M. Case[11] introduced the method of singular eigenfunctions for constructing solutions of the neutron transport equation. This method quickly came to supplant the more classical Wiener-Hopf technique[17] (to which it is equivalent[36]) for reasons of convenience and familiarity (the Case method is basically a separation of variables technique, in which boundary data are fitted by singular eigenfunction expansions, so the analogy to the usual theory of partial differential equations is quite close).

Case's second paper (which actually followed two earlier papers by van Kampen and Case[23,10] in which singular eigenfunction techniques had been applied to the study of plasma stability) induced a flurry of activity in which similar ideas were applied to solutions of kinetic equations which arise in many different areas of physics, for example, gas dynamics[14,15], plasma waves[31], radiative transfer[32,33], electron discharge[13], even lattice spin systems[12]. In addition, the one-speed, isotropic scattering neutron transport equation of Case[11] was generalized to models involving anisotropic scattering and energy
dependence. The literature is much too enormous to cite here. However, several review papers exist, in which references abound[27,38].

Despite the popularity of the Case method to physicists and engineers, the mathematicians remained largely unconvinced because of the heuristic nature of many of Case's arguments. (A thoughtful discussion of Case's mathematical irregularities has been given by Hangelbroek[20].) However, in the mid 1970 's Case's formulas were reproduced rigorously by two different methods, resolvent integration[24] and application of the spectral theorem (Hilbert space technique)[20]. Again, a flurry of papers resulted in which functional analytic techniques were applied to a number of problems in kinetic theory, some of which for one reason or another had not been conveniently amenable to singular eigenfunction expansions[19].

It is of comfort to the purist that in mathematics, as in life, virtue need not always be its own reward. Thus, the introduction of mathematically rigorous formulations of Case's ideas made possible, for the first time, the solution of the multigroup neutron transport equation[9] and the correct solution of the initial value problems in which eigenvalues were imbedded in the continuous spectrum[2]. However, perhaps a more important reward was that the understanding introduced by rigorous mathematics into the structure of the Case solutions made it possible to generalize vastly the class of kinetic equations which could be treated. The instigator of this generalization was Richard Beals $[5,6]$.

The generalized kinetic equation considered by Beals can be written in the form

$$
\begin{equation*}
h(\mu) \frac{\partial f}{\partial x}(x, \mu)+A f=0, \quad x \varepsilon[a, b] \subseteq R, \quad \mu \varepsilon S \tag{1}
\end{equation*}
$$

The set $S$ is assumed to be equipped with a measure $d m$, and the operator $A$ is taken to be positive in the Hilbert Space $L^{2}(S, d m)$. The function $h(\mu)$ is essentially bounded and the set $S^{0}=$ $\{\mu \varepsilon S \mid h(\mu)=0\}$ is assumed to have m-measure zero. The following
theorem can be proved by standard methods.
THEOREM 0. Define a solution to equation (1) to be a differentiable map $f:[a, b] \rightarrow L^{2}(S, d m)$ such that $f(a, \mu)=f_{+}, \mu \varepsilon S^{+}=$ $\{\mu \varepsilon S \mid h(\mu)>0\}$, and $f(b, \mu)=f_{f}, \mu \varepsilon S^{-}=\{\mu \varepsilon S \mid h(\mu)<0\}$. Then a solution to Eq. (1) is unique.

This theorem suggests that partial range boundary conditions can lead to well-posed problems. They are the analogues of the standard boundary conditions in neutron transport theory that the incident distribution be specified[13].

In the case that the interval $[a, b]$ is semi-infinite (the situation treated in this paper) the boundary conditions expressed in Theorem I can be weakened. We prove a stronger result for that case in Section $V$.

The operator A is assumed by Beals to be bounded with bounded inverse[5] or to be a specific Sturm-Liouville type appropriate to electron scattering[6]. (Thus, in [6], A has a one-dimensional kernel.) In [5], one section is deyoted to a generalization of the result obtained there to the case that $A$ has a finite-dimensional kernel, but assumptions are introduced which we would like to avoid.)

The purpose of this paper is to generalize and codify Beals' results to the following kinetic equation

$$
\begin{equation*}
\mathrm{T} \frac{\partial f}{\partial \mathrm{X}}+\mathrm{Af}=0, \mathrm{X} \varepsilon[0, \infty) \tag{2}
\end{equation*}
$$

Here $T$ is assumed bounded, injective and self-adjoint on a Hilbert space $H$ and $A$ positive self-adjoint on $H$ with Ker $A$ finite-dimensional and Ran $A$ closed in $H$. We note that since the spectrum of A contains a gap at zero, A restricted to the orthogonal complement of Ker $A$ is strictly positive.

Since this paper is quite long and technical, we shall present here a rather detailed discussion of the content.

Suppose Eq. (2) is rewritten as

$$
\frac{\partial f}{\partial X}+K f=0
$$

with $K=T^{-1} A$. The kernel of $A$ induces a root linear manifold $Z_{0}(K)$ which ( $K$ is not self-adjoint) may contain generalized eigenvectors in addition to the zero eigenvectors of A. Similarly we define $Z_{0}\left(K^{*}\right)$. One sees that Ker $K^{*}=T$ Ker $A$. Given these conditions $H$ can be decomposed as

$$
\begin{align*}
H & =Z_{0}(K) \oplus Z_{0}\left(K^{*}\right)^{\perp}  \tag{3a}\\
& =Z_{0}\left(K^{*}\right) \oplus Z_{0}(K)^{\perp} . \tag{3b}
\end{align*}
$$

On $Z_{0}\left(K^{*}\right)^{\perp}$, $K$ is self-adjoint in a suitable inner product. Straightforward analysis leads to a spectral theorem.

The problem is to apply the spectral theorem to partial range boundary conditions. Recalling Theorem 0 , we can define projection operators $Q_{ \pm}$on $H$ corresponding to the positive and negative parts of $T$ (in Eq. (2), $T$ is the generalization of the multiplication operator $h(\mu)$ of Eq. (l); the obvious generalization of Theorem 0 has not been stated explicitly). Then the boundary data for a half-range problem is $f_{+} \varepsilon \operatorname{Ran}\left(Q_{+}\right)$. In order to use the spectral theorem for $f_{+}$, one must introduce the Larsen-Habetler[24] albedo operator $E: \operatorname{Ran} Q_{ \pm} \rightarrow \operatorname{Ran} P_{ \pm}$where $P_{ \pm}$ are the projections onto the positive and negative parts of $K$. If $Z_{0}(K)$ is trivial, this is all there is to the construction, and the results for bounded $A$ are contained in [5]; the modifications for unbounded A are straightforward[39].

However, if $Z_{0}(K)$ is not trivial, it is necessary to define E such that E maps the subspace Ran $Q_{+}$into the subspace Ran $P_{+}+$Ker A in order to guarantee bounded solutions at infinity. This modification in the definition of $E$ is non-trivial and, in fact, was carried out incorrectly in [6]. Our approach to this problem is to introduce an invertible matrix $\beta$ on $Z_{0}(K)$ and a modified operator $A_{B}$ such that $K_{B}=T^{-1} A_{B}=\left.B \oplus K\right|_{Z_{0}}\left(K^{*}\right)^{\perp}$. (This idea was first exploited in [28]). The problem is now reduced to the previous case since $\operatorname{Ker}\left(A_{\beta}\right)=\{0\}$.

If now $Z_{0}(K)$ is decomposed as $M_{+} \oplus M_{-}$, where $M_{ \pm}$are the positive and negative subspaces of $\beta, E_{\beta}$ is defined such that
$E_{\beta}: \operatorname{Ran} Q_{+} \rightarrow \operatorname{Ran} P_{+} \oplus M_{+}$. Such a construction is always possible; the solution obtained in this way is now projected onto $Z_{0}\left(K^{*}\right)^{\perp}$ and the original equation projected onto $Z_{0}(K)$ (a matrix equation) is trivially solvable. The crux of the matter is that the decomposition $H=Z_{0}(K) \oplus Z_{0}\left(K^{*}\right)^{\perp}$ does not reduce $E$.

The construction above is carried out for the boundary value problem (2) on an enlarged Hilbert space. Under the condition that $I$ - A is a compact operator and $\operatorname{Ran}(I-A) \subseteq \operatorname{Ran}|T|^{\alpha}$ for some $0<\alpha<1$, van der Mee[28] solved the half-space problem and constructed the albedo operator without extending the original Hilbert space (as in [5]).

In Sec. II, we derive the decomposition of $H$ and introduce the matrix $\beta$ and the operator $A_{\beta}$, and prove that there always exists a choice of $\beta$ such that $A_{\beta}$ is strictly positive.

In Sec. III we state the spectral theorem for $K$ (in $H_{A}$ ) and in the case that $K$ admits a rigged extension we state a Case type full-range completeness and orthogonality theorem.

In Sec. IV we consider the solution of half-space problems by proving the existence of the operator E discussed above, and noting that the "half-range"[1],13] expansion of $f_{f}$ corresponds to the full-range expansion (Sec. III) of Ef ${ }_{+}$. The definition of E guarantees that no growing modes occur in the solution and thus we can show (Sec. V) that a bounded solution exists.

Finally in Sec. VI, we discuss several applications.
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## II. DECOMPOSITION

We recall our assumptions (noted throughout this paper) that the (possibly unbounded) operator A is positive self-adjoint and Fredholm on an abstract (complex) Hilbert space $H$, and that the operator $T$ is bounded and self-adjoint with a trivial kernel Ker $T$. Below we analyze the operator $K=T^{-1} A$ and its adjoint $K^{*}$,
which are closed and densely defined by virtue of the Fredholm assumption on $A$.

The zero root linear manifold $Z_{0}(K)$ is defined by

$$
Z_{0}(K)=\left\{f_{0} \varepsilon D(K) \mid f_{0} \in D\left(K^{n}\right) \text { and } K^{n} f_{0}=0 \text { for some } n \varepsilon Z_{+}\right\}
$$

and similarly for $Z_{0}\left(K^{*}\right)$. The next lemma proves that the zero Jordan chains of $K$ have at most length two ([28], Sec. III. 3).

LEMMA 1. If $f_{0} \in Z_{0}(K)$, there exists $f_{1} \varepsilon D(K)$ such that

$$
\mathrm{Kf}_{0}=\mathrm{f}_{1}, \quad \mathrm{Kf}_{1}=0
$$

PROOF. Assume that $g_{0}, g_{1}, g_{2} \in D(K)$ are chosen in such a way that $K g_{0}=g_{1}, K g_{1}=g_{2}, \mathrm{Kg}_{2}=0$. Then $g_{0}, g_{1}, g_{2} \varepsilon D(A)$, $\mathrm{Ag}_{2}=0$ and

$$
\left(A g_{1}, g_{1}\right)=\left(T g_{2}, g_{1}\right)=\left(g_{2}, T g_{1}\right)=\left(g_{2}, A g_{0}\right)=\left(\mathrm{Ag}_{2}, g_{0}\right)=0
$$

The positivity of the operator $A$ implies $T g_{2}=A g_{1}=0$. As Ker $T$ $=\{0\}$, one gets $g_{2}=0$.

Hangelbroek[20] introduced in his analysis the Hilbert space $H_{A} \subseteq H$ with inner product $(f, g)_{A}=(A f, g)$. Since we do not assume $A$ to be injective (or bounded), we must restrict $A$ to a subspace on which it will be injective, a procedure first followed by Lekkerkerker[25].

PROPOSITION 1. One has.

$$
\begin{equation*}
T Z_{0}(K)=Z_{0}\left(K^{*}\right), A\left\{Z_{0}\left(K^{*}\right)^{\perp} \cap D(A)\right\}=\overline{T\left\{Z_{0}\left(K^{*}\right)^{\perp}\right\}}=Z_{0}(K)^{\perp} \tag{4}
\end{equation*}
$$

and the following decompositions hold true:

$$
\begin{align*}
& Z_{0}(K) \oplus Z_{0}\left(K^{*}\right)^{\perp}=H  \tag{5a}\\
& Z_{0}\left(K^{*}\right) \oplus Z_{0}(K)^{\perp}=H \tag{5b}
\end{align*}
$$

PROOF. Let $A, T$ be chosen. Then Lemma $I$ and dim Ker $A$ finite imply $\operatorname{dim} Z_{0}(K) \leqslant 2 \operatorname{dim} \operatorname{Ker} A<\infty$. On the one hand, if $\mathrm{x} \varepsilon \mathrm{D}(\mathrm{K})$, then $\operatorname{Tx} \in \mathrm{D}\left(\mathrm{K}^{*}\right)$ and $\mathrm{K}^{*} \mathrm{Tx}=\mathrm{TKX}$, implying that $T Z_{0}(K) \subseteq Z_{0}\left(K^{*}\right)$. on the other hand, $Z_{0}\left(K^{*}\right) \subseteq D\left(K^{*}\right) \subseteq \operatorname{Ran} T$. Hence, $Z_{0}\left(K^{*}\right)$ has finite dimension and $T Z_{0}(K)=Z_{0}\left(K^{*}\right)$.

Take $x \in Z_{0}(K) \cap Z_{0}\left(K^{*}\right)^{\perp}$. Then $x \in D(K)$ and $T^{-1} A x=$ $\operatorname{Kx\varepsilon } Z_{0}(K)$. So $A x \varepsilon Z_{0}\left(K^{*}\right)$. Using that $x \in Z_{0}\left(K^{*}\right)^{\perp}$, one gets ( $A x, x$ ) $=0$ and thus $x \varepsilon \operatorname{Ker} A$. As $T$ is bounded and $A$ has closed range, $x=K y$ for some $y \varepsilon Z_{0}(K)$, and $A y=T x \varepsilon Z_{0}\left(K^{*}\right)$. As $T y \varepsilon Z_{0}\left(K^{*}\right)$, one gets

$$
(A y, y)=(T x, y)=(x, T y)=0,
$$

implying that $T x=A y=0$. Thus $x=0$, proving that $Z_{0}(K) \cap Z_{0}\left(K^{*}\right)^{\perp}=\{0\}$.

Next take $y \varepsilon Z_{0}\left(K^{*}\right) \cap Z_{0}(K)^{\perp}$ and $z \varepsilon Z_{0}\left(K^{*}\right)$. Then $y=T x$ for some $x \varepsilon Z_{0}(K), z=T u$ for some $u \varepsilon Z_{0}(K)$, and $(x, z)=(x, T u)=$ $(T x, u)=(y, u)=0$. Thus $x \in Z_{0}(K) \cap Z_{0}\left(K^{*}\right)^{\perp}=\{0\}$ and $y=T x=0$. Hence,

$$
\begin{equation*}
Z_{0}(K) \cap Z_{0}\left(K^{*}\right)^{\perp}=Z_{0}\left(K^{*}\right) \cap Z_{0}(K)^{\perp}=\{0\} \tag{6}
\end{equation*}
$$

However,

$$
\begin{aligned}
& \operatorname{dim} Z_{0}(K)=\operatorname{dim} Z_{0}\left(K^{*}\right)=\operatorname{codim} Z_{0}\left(K^{*}\right)^{\perp} \\
& \operatorname{dim} Z_{0}\left(K^{*}\right)=\operatorname{dim} Z_{0}(K)=\operatorname{codim} Z_{0}(K)^{\perp}
\end{aligned}
$$

which establishes the decompositions (5a) and (5b).
Take $x \varepsilon Z_{0}\left(K^{*}\right)^{\perp}$ and $l e t$ us show that $T x, A x \varepsilon Z_{0}(K)^{\perp}$.
(In the latter case we assume $x \varepsilon D(A)$ ). Let $z \varepsilon Z_{0}(K)$. Then $T z$, $A z \varepsilon Z_{0}\left(K^{*}\right)$ and so $(x, T z)=(x, A z)=0$. Hence, $(T x, z)=0$ for all $z \varepsilon Z_{0}(K)$ and, in case $x \in D(A),(A x, z)=0$ for all $z \varepsilon Z_{0}(K)$, thereby establishing our assertion.

To finish the proof we note that as $T$ is a self-adjoint operator with trivial kernel, its range Ran $T$ is dense in $H$. So $T\left\{Z_{0}\left(K^{*}\right)^{\perp}\right\}$ is dense in $Z_{0}(K)^{\perp}$. Further, as $A$ is assumed to
have closed range and $Z_{0}\left(K^{*}\right)^{\perp}$ has finite codimension (in $H$ ), the restriction of $A$ to $Z_{0}\left(K^{*}\right)^{1} \cap D(A)$ has closed range in $Z_{0}(K)^{1}$. Moreover,

$$
\operatorname{dim} \operatorname{Ker} A=\operatorname{codim} \operatorname{Ran} A=\operatorname{dim} \frac{Z_{0}\left(K^{*}\right)}{A Z_{0}(K)}+\operatorname{dim} \frac{Z_{0}(K)^{\perp}}{A\left\{Z_{0}\left(K^{*}\right)^{1} \cap D(A)\right\}} .
$$

Because Ker $A \subseteq Z_{0}(K)$, also $\operatorname{dim} Z_{0}\left(K^{*}\right) / A Z_{0}(K)=\operatorname{dim} \operatorname{Ker} A$ and thus $A\left\{Z_{0}\left(K^{*}\right)^{1} \cap D(A)\right\}=Z_{0}(K)^{1}$. This completes the proof.
Now that we have constructed the decompositions of the original Hilbert space $H$ given by Eqs. (5), we define the sesquilinear form $(., .)_{A}$ on $Z_{0}\left(K^{*}\right)^{\perp} \cap D(A)$ by putting

$$
\begin{equation*}
(f, g)_{A}=(A f, g) \tag{7}
\end{equation*}
$$

This sesquilinear form is positive and any vector $f \varepsilon Z_{0}\left(K^{*}\right)^{\perp} \cap D(A)$ for which $(f, f)_{A}=(A f, f)=0$ necessarily belongs to Ker $A$ and therefore must vanish. Let us denote by $H_{A}$ the Hilbert space obtained as the completion of $Z_{0}\left(K^{*}\right)^{\perp} \cap D(A)$ with respect to the inner product (.,.) $A$. It is easy to derive that $H_{A}$ will coincide with $Z_{0}\left(K^{*}\right)^{1}$ if and only if $A$ is bounded. For unbounded $A$ the sesquilinear form (.,.) $A$ on $Z_{0}\left(K^{*}\right)^{\perp} \cap D(A)$ is incomplete.

As a corollary of Eq. (4) there exists a unique bounded operator $\hat{K}$ on $Z_{0}\left(K^{*}\right)^{\perp}$ such that

$$
A \hat{K} x=T x, \quad x \in Z_{0}\left(K^{*}\right)^{1} .
$$

Of course, if $\operatorname{Ker} A=\{0\}$, then $\hat{K}=A^{-1}$. It is clear that $Z_{0}\left(K^{*}\right)^{\perp} \cap \mathrm{D}(\mathrm{A})$ is invariant under $\hat{K}$ and

$$
(\hat{K} f, g)_{A}=(f, \hat{K} g)_{A} ; \quad f, g \varepsilon Z_{0}\left(K^{*}\right)^{1} \cap D(A)
$$

Because $\hat{K}$ is bounded on $Z_{0}\left(K^{*}\right)^{\perp} \cap D(A)$ endowed with the norm of Eq. (7), the above equality implies that $\hat{K}$ (restricted to $Z_{0}\left(K^{*}\right)^{1} \cap \mathrm{D}(\mathrm{A})$ ) can be extended in a unique way to a bounded selfadjoint operator on $H_{A}$, also to be denoted by $\hat{K}$.

The next proposition intends to reduce the solution of

Eq. (1) to the solution of the analogous problem for some $A$ with Ker $A=\{0\}$.

PROPOSITION 2. Let $A, T$ be chosen and let $P$ denote the projection of $H$ onto $Z_{0}\left(K^{*}\right)^{1}$ along $Z_{0}(K)$. For some invertible operator $B$ on the finite-dimensional space $Z_{0}(K)$ put

$$
\begin{equation*}
A_{\beta}=A P+T B^{-1}(I-P) \tag{8}
\end{equation*}
$$

Then the operator $A_{\beta}$ is densely defined with bounded inverse and

$$
\begin{equation*}
A_{\beta}^{-1} T=\beta \oplus \hat{K} \tag{9}
\end{equation*}
$$

One may choose $B$ in such a way that $\left(T^{-1}{ }^{-1}, x\right) \geqslant 0$ for all $x \in Z_{0}(K)$, in which case $A_{\beta}$ will be a positive operator. PROOF. Because of Eq. (4) the operator $A_{\beta}$ in (8) is well-defined and has $\left\{Z_{0}\left(K^{*}\right)^{1} \cap D(A)\right\} \oplus Z_{0}(K)=D(A)$ as its domain. As

$$
A\left\{Z_{0}\left(K^{*}\right)^{1} \cap D(A)\right\}=Z_{0}(K)^{1}, \quad T \beta^{-1} Z_{0}(K)=Z_{0}\left(K^{*}\right)
$$

and the decompositions (5a) and (5b) hold true, the operator $A_{B}$ is densely defined and has a bounded inverse. Formula (9) is checked by computation.

Next choose $x \varepsilon H$. Then there exist unique $x_{0} \in Z_{0}(K)$ and $X_{1} \varepsilon Z_{0}\left(K^{*}\right)^{1}$ such that $x=x_{0}+x_{1}$. As $T \beta^{-1} X_{0} \varepsilon Z_{0}\left(K^{*}\right)$, one has $\left(T_{\beta} I_{X_{0}}, X_{1}\right)=0$. Further, if $x \in D\left(A_{B}\right)=D(A)$, also $X_{1} \varepsilon D(A)$, $A x_{1} \varepsilon Z_{0}(K)^{1}(\operatorname{cf.}(4))$ and $\left(A_{1}, x_{0}\right)=0$. Therefore, for $x \in D\left(A_{B}\right)$

$$
\left(A_{\beta} x, X\right)=\left(A X_{1}, X_{1}\right)+\left(T \beta^{-1} X_{0}, x_{0}\right)
$$

This identity proves $A_{\beta}$ to be a positive operator if (and only if) $\left(T \beta^{-1} h, h\right) \geqslant 0$ for all heZ $Z_{0}(K)$.

In case $I$ - A is compact this proposition was derived by van der Mee ([28], Sec. III. 5) and exploited to reduce Eq. (1)
to the case when $\operatorname{Ker} A=\{0\}$. This reduction will be accomp_ lished in Section $V$. It will be convenient to choose $\beta$ in a special way.

There exists a basis $b=\left\{x_{1}, \ldots, x_{r}, y_{1}, \ldots, y_{r_{r}}, z_{1}, \ldots, z_{s}\right\}$ of $Z_{0}(K)$ with respect to which the restriction of $K=T^{-1} A$ to $Z_{0}(K)$ has Jordan normal form. This means that

$$
T^{-1} A y_{j}=x_{j}(j=1, \ldots, r) \text { and Ker } A=\operatorname{span}\left\{x_{1}, \ldots, x_{r}, z_{1}, \ldots, z_{s}\right\}
$$

Note that for $j=1, \ldots, r$ and numbers $\xi_{1}, \ldots, \xi_{r}$,

$$
\begin{align*}
& \left(T\left(y_{j}-\xi_{j} x_{j}\right),\left(y_{j}-\xi_{j} x_{j}\right)\right)=\left(T y_{j}, y_{j}\right)+\left|\xi_{j}\right|^{2}\left(T x_{j}, x_{j}\right)  \tag{10}\\
& -2 \operatorname{Re}\left[\xi_{j}\left(T x_{j}, y_{j}\right)\right]=\left(T y_{j}, y_{j}\right)-2\left(\operatorname{Re} \xi_{j}\right)\left(A y_{j}, y_{j}\right),
\end{align*}
$$

where we used that $T x_{j}=A y_{j}(j=1, \ldots, r)$. Because $A y_{j} \neq 0$, the number $\left(A y_{j}, y_{j}\right)>0$. So one can choose real $\xi_{1}, \ldots, \xi_{r}$ such that for $j=1, \ldots, r$ the expressions (10) are negative. A simple readjustment leads to a Jordan basis b for which ( $\mathrm{Ty}_{\mathrm{j}}, \mathrm{y}_{\mathrm{j}}$ ) < $0(j=l, \ldots, r)$.

On the finite-dimensional space $\mathrm{Z}_{0}(\mathrm{~K})$ one considers the indefinite inner product

$$
\begin{equation*}
[u, v]=(T u, v) \tag{11}
\end{equation*}
$$

which makes $Z_{0}(K)$ a Pontryagin space[8]. The original Jordan basis b can be chosen in such a way that with respect to (ll) vectors of distinct Jordan chains are orthogonal. This requires a repeated application of the reasoning of Eq. (10). Furthermore, the vectors $y_{l}, \ldots y_{r}$ can be chosen strictly negative with respect to (ll) without affecting this orthogonality property. For $j=1, \ldots, r$ and usKer $A$ one has

$$
\begin{equation*}
\left[x_{j}, u\right]=\left(T x_{j}, u\right)=\left(A y_{j}, u\right)=\left(y_{j}, A u\right)=0 . \tag{12}
\end{equation*}
$$

Further, because $N=\operatorname{span}\left\{z_{1}, \ldots, z_{S}\right\}$ is a Pontryagin space, there exist a positive subspace $N_{+}$and a negative subspace $N_{-}$ that are orthogonal with $N_{+} \oplus N_{-}=N$ [8]. This means that $[u, u] \geqslant 0$ for $u \varepsilon N_{+},[v, v] \leqslant 0$ for $v \in N_{-}$and $[u, v]=0$ for $u \varepsilon N_{+}$ and $v \varepsilon \mathrm{~N}_{-}$.

LEMMA 2. The subspace $M_{+}=\operatorname{span}\left\{x_{1}, \ldots, x_{r}\right\} \oplus N_{+}$is a maximal positive subspace of $\mathrm{Z}_{0}(\mathrm{~K})$. With respect to (II) there exists a maximal negative subspace $M_{-}$of $\mathrm{Z}_{0}(\mathrm{~K})$ orthogonal to $\mathrm{M}_{+}$ such that

$$
M_{+} \oplus M_{-}=Z_{0}(K)
$$

PROOF. Because $[u, u] \geqslant 0$ for all uespan $\left\{x_{1}, \ldots, x_{r}\right\}$
(cf. (l2)) and all $u \in N_{+}$, and $[u, v]=0$ for $u \varepsilon \operatorname{span}\left\{x_{1}, \ldots, x_{r}\right\}$ and $v \varepsilon N_{+}$(cf. (12)), the subspace $M_{+}$is positive. Further,

$$
L \equiv \operatorname{span}\left\{y_{1}\right\} \oplus \ldots \Theta \operatorname{span}\left\{y_{r}\right\} \Theta N_{-}, \quad M_{+} \oplus L=Z_{0}(K) .
$$

The orthogonality of vectors of distinct Jordan chains with respect to (II) and the specific choice of $y_{1}, \ldots, y_{r}$ imply the negativity of $L$ with respect to (II). Clearly, $M_{+}$is maximal positive. The second part of the lemma follows immediately from Pontryagin space theory ([8], Sec. IX. I).

For $x=m_{+}+m_{-} \varepsilon Z_{0}(K)$ with $m_{ \pm} \varepsilon M_{ \pm}$put $\beta x=m_{+}-m_{-}$. Then the orthogonality of $M_{+}$and $M_{-}$, the positivity of $M_{+}$and the negativity of $M_{-}$imply that

$$
\left(T \beta^{-1} x, x\right)=\left[m_{+}-m_{-}, m_{+}+m_{-}\right]=\left[m_{+}, m_{+}\right]-\left[m_{-}, m_{-}\right] \geqslant 0 .
$$

We observe

$$
\sigma\left(\beta / M_{ \pm}\right)=\{ \pm 1\}, K\left\{Z_{0}(K)\right\} \subseteq M_{+} \subseteq \operatorname{Ker} A
$$

So for this choice of $\beta$ the operator $A_{\beta}$ in Eq. (8) will be positive. If all Jordan chains of $K$ at $\lambda=0$ have length 2 , then

Ker A is maximal positive and maximal negative at the same time. III. FULL-RANGE EXPANSIONS

We have observed that as a corollary of Proposition l, the operator $\hat{K}$ is a bounded self-adjoint operator on $H_{A}$. Thus $K=\hat{K}^{-1}$ extended to $H_{A}$ has a spectral decomposition

$$
\begin{equation*}
K=\int_{-\infty}^{\infty} \lambda d F(\lambda) \tag{13}
\end{equation*}
$$

Define the (, ) A-orthogonal projections [20]

$$
\begin{equation*}
P_{ \pm}= \pm \int_{0}^{ \pm \infty} d F(\lambda) \tag{14}
\end{equation*}
$$

Since on $P_{ \pm} H_{A}, \mp K$ generate $C_{0}$-semigroups, it is possible to solve the inhomogeneous problem

$$
\begin{equation*}
\frac{\partial}{\partial x}(T \phi)=-A \phi+q \tag{15}
\end{equation*}
$$

for $\mathrm{x} \varepsilon \mathbb{R}$ with appropriate continuity conditions on $\mathrm{q}: \mathbb{R} \rightarrow \mathrm{H}_{\mathrm{A}}$ and boundedness conditions of infinity. Details are given in Appendix A.

In general, for positive $A$, it is only possible to associate a spectral projection $F(\lambda)$ with each value $\lambda$ in the spectrum $\sigma(\mathrm{K})$. The expansion

$$
\begin{equation*}
f=\int_{-\infty}^{\infty} d(F(\lambda) f)+\sum_{i-1}^{\operatorname{dim}}{Z_{i}}_{i}(K) \tag{16}
\end{equation*}
$$

for $f \in H$ with $\operatorname{Pf} \varepsilon H_{A}$ and $\left\{\alpha_{i}\right\}_{i=1}^{\operatorname{dimZ}}{ }_{0}(K)$ a basis of $Z_{0}(K)$ has been called the full-range expansion of $f$. For many physical problems (obviously those which have been solved using orthodox "Caseology"[13]) a description in terms of generalized eigenfunctions has been obtained. A criterion for the existence of such generalized eigenfunction expansions is provided within the framework of rigged Hilbert spaces[7].

We assume that $H_{A}$ contains a dense linear subspace $H_{+}$ which is complete in another inner product (, ) ${ }_{+}$and such that

$$
\mathrm{H}_{+}^{*} \supset \mathrm{H}_{\mathrm{A}} \supset \mathrm{H}_{+}
$$

where the embedding $i: H_{+} \rightarrow H_{A}$ is Hilbert-Schmidt. One says that $K$ admits a rigged extension if there is a dense subspace $D \subseteq H_{+}$ such that $D \subseteq D(K)$ and $K: D \rightarrow H_{+}^{*}$ is continuous.

THEOREM 1. (GENERALIZED EIGENVECTOR EXPANSION) (cf.[7])
If $H_{A}$ is as above and $K$ admits a rigged extension, then there exists a set of eigenvectors $\phi_{\alpha, \lambda} \varepsilon H_{+}^{*}$ for $\lambda \varepsilon \sigma(K), \alpha=1, \ldots, N_{\lambda}$, obeying

$$
\widetilde{\mathrm{K}}_{\alpha, \lambda}=\lambda \phi_{\alpha, \lambda}
$$

where $\tilde{\mathrm{K}}$ is understood to represent the rigged extension of K on $\mathrm{H}_{+}^{*}$; equivalently

$$
\left((K-\lambda I) v, \phi_{\alpha, \lambda}\right)_{A}=0, \quad \forall v \varepsilon D .
$$

Moreover, the set of eigenvectors is complete on $H_{+}$:

$$
\begin{aligned}
& v=\int_{-\infty}^{\infty} \sum_{\alpha=1}^{N_{\lambda}} A(\alpha, \lambda) \phi_{\alpha, \lambda} d \rho(\lambda) \\
& A(\alpha, \lambda)=\left(v, \phi_{\alpha, \lambda}\right)_{A}
\end{aligned}
$$

for $\mathrm{V} \varepsilon \mathrm{H}_{+}$and $\rho(\lambda)$ the (Borel) spectral measure of the operator K . Finally, the eigenvectors $\phi_{\alpha, \lambda}$ are "orthogonal" in $H_{A}$ :

$$
\left(\phi_{\alpha, \lambda}, \phi_{\beta, \lambda^{\prime}}\right)_{A}=\frac{1}{\rho(\lambda)} \delta(\alpha, \beta) \delta\left(\lambda-\lambda^{\prime}\right) .
$$

COROLLARY (FULI-RANGE COMPLETENESS) Let the conditions of the previous theorem be fulfilled together with the condition that $K$ admits a rigged extension. Then any function $f \in H$ with $\mathrm{Pf}_{\mathrm{E}}^{\mathrm{H}}+\mathrm{can}$ be expanded as

$$
f=\sum_{i=1}^{\operatorname{dim} Z_{0}(K)} c_{i} \alpha_{i}+\int_{-\infty}^{\infty} \sum_{\alpha=1}^{N_{\lambda}} A(\alpha, \lambda) \phi_{\alpha, \lambda} d \rho(\lambda)
$$

and the expansion coefficients can be computed from the orthogonality relation.
IV. HALF-RANGE EXPANSIONS

We define $Q_{ \pm}$to be the H-orthogonal projections of $H$ onto the maximal T-invariant subspaces on which $T$ is positive/ negative. In analogy with the construction of $H_{A}$ (Sec.II), we define the $(,)_{A_{B}}$ inner product on $H_{A_{B}}$. (Note that $H_{A_{B}} \subseteq H$ densely, and the bounded self-adjoint operator $\hat{K}_{\beta} \equiv \beta \oplus \hat{\mathrm{K}}^{\beta}$ on $H_{A_{\beta}}$ ). Likewise, the $(,)_{A_{\beta}}$-orthogonal projections $P_{ \pm}$are defined as in (14).

Let us introduce two additional inner products on H ; namely,

$$
\begin{equation*}
(., .)_{\mathrm{T}}=(|\mathrm{T}| ., .) \tag{17}
\end{equation*}
$$

with the completion of H denoted $\mathrm{H}_{\mathrm{T}}$, and

$$
\begin{equation*}
(., .)_{K_{\beta}}=\left(\left|\hat{\mathrm{K}}_{\beta}\right| \ldots,\right)_{A_{\beta}} \tag{18}
\end{equation*}
$$

with the completion of $\mathrm{H}_{A_{B}}$ denoted $H_{K_{\beta}}$. The projections $Q_{ \pm}$and $\mathrm{P}_{ \pm}$extend continuously to projection operators on $\mathrm{H}_{\mathrm{T}}$ and $\mathrm{H}_{K_{B}}$, respectively.

The positive operator $A_{\beta}$ has been constructed in such a way that the domain $D\left(A_{\beta}\right)$ is the same as the domain of the original operator $A$, and $A$ and $A_{B}$ coincide on their domains within the subspace $Z_{0}\left(K^{*}\right)^{1}$ of finite codimension. Hence, on $D(A)$ all inner products ( , ) $A_{\beta}$ are equivalent and therefore we may suppress $\beta$ in $H_{A_{B}}$. In a similar way the inner products (, ) $\mathcal{K}_{B}$ are equivalent on $H_{A}$, and thus the subscript $\beta$ may be suppressed in $H_{K}$.

The projection $P$ onto $Z_{0}\left(K^{*}\right)^{1}$ along $Z_{0}(K)$ has been defined on $H$, but, because Ker $P=Z_{0}(K) \subseteq D(A)$, one can extend $P$ continuously from $D(A)$ to bounded projections on the spaces $H_{A}$ and $H_{K}$. In both cases, Ker $P=Z_{0}(K)$.

The Larsen-Habetler[24] albedo operator $E$ is defined by the conditions that, for all $\mathrm{f} \varepsilon \mathrm{H}$,
(i). $Q_{ \pm} E Q_{ \pm} f=Q_{ \pm} f$;
(ii) $P_{\mp} E Q_{ \pm} f=0$.

In transport theory terminology, these conditions imply that if $f \varepsilon \operatorname{Ran} Q_{+}$is an incoming flux for a right half-space problem, then Ef will be the corresponding total (incoming plus reflected) flux, and if feRan $Q_{\text {_ }}$ is an incoming flux for a left half-space problem, then Ef will be the corresponding total flux.

Let us find an explicit representation for $E: H_{T} \rightarrow H_{K}$, which we shall justify later. First we derive the intertwining relation

$$
\begin{equation*}
P_{ \pm} E=E Q_{ \pm} \tag{20}
\end{equation*}
$$

on $H_{T}$. We have

$$
P_{ \pm} E=P_{ \pm} E\left(Q_{+}+Q_{-}\right)=P_{ \pm} E Q_{ \pm}=E Q_{ \pm},
$$

where we have used Eqs. (19). Now by (19), again,

$$
Q_{ \pm} P_{ \pm} E Q_{ \pm}=Q_{ \pm}
$$

whence, by adding the $\pm$ equations,

$$
Q_{+} P_{+} E Q_{+}+Q_{-} P_{-} E Q_{-}=\left(Q_{+} P_{+}+Q_{-} P_{-}\right) E=I .
$$

PROPOSITION 3. There exists a unique albedo operator $\mathrm{E}: \mathrm{H}_{\mathrm{T}} \rightarrow \mathrm{H}_{\mathrm{K}}$ that is bounded, injective and satisfies the conditions (19). Further, E acts as a bounded operator from $\mathrm{H}_{\mathrm{T}}$ into $\mathrm{H}_{\mathrm{T}}$.

PROOF. On $\mathrm{H}_{\mathrm{A}}$ we define the Hangelbroek operators[22]

$$
V=Q_{+} P_{+}+Q_{-} P_{-}: H_{A} \rightarrow H, W=Q_{+} P_{-}+Q_{-} P_{+}: H_{A} \rightarrow H .
$$

A straightforward calculation shows that, for $f \varepsilon H_{A}$,

$$
\left(Q_{+}-Q_{-}\right)\left(P_{+}-P_{-}\right) f=V f-W f=(2 V-I)
$$

(cf.[21]), and therefore

$$
\begin{aligned}
((2 V & -I), f, f)_{T}=\left(|T|\left(Q_{+}-Q_{-}\right)\left(P_{+}-P_{-}\right) f, f\right) \\
& =\left(T\left(P_{+}-P_{-}\right) f, f\right) \\
& =\left(\hat{K}_{\beta}\left(P_{+}-P_{-}\right) f, f\right)_{A_{\beta}}=\left(\left|\hat{K}_{\beta}\right| f, f\right)_{A_{\beta}}=||f||_{K_{\beta}}^{2} .
\end{aligned}
$$

This implies the following identity:

$$
\begin{equation*}
2(V f, f)_{T}=||f||_{T}^{2}+||f||_{K_{B}}^{2}, \quad f \varepsilon H_{A} . \tag{21}
\end{equation*}
$$

Introduce the semi-bounded quadratic form

$$
q(f, g)=2(V f, g)_{T}, \quad f, g \varepsilon H_{A},
$$

on the Hilbert space $H_{T}$. Note that $q$ can be extended to a closed form with domain $D(q)=H_{T} \cap H_{K}$ and $H_{A}$ is a form core for q. Now $q$ is the quadratic form of a unique self-adjoint operator whose domain D satisfies (cf.[30])

$$
\mathrm{H}_{\mathrm{A}} \subseteq \mathrm{D} \subseteq \mathrm{H}_{\mathrm{T}} \cap \mathrm{H}_{\mathrm{K}} \subseteq \mathrm{H}_{\mathrm{T}}
$$

Hence $V$ extends to a self-adjoint operator on $H_{T}$ (with domain $D$ ), and moreover,

$$
2(V f, f)_{T} \geqslant\|f\|_{T}^{2}, \quad f \varepsilon D
$$

From this we find $V$ to have trivial kernel and dense range in $H_{T}$, so that (20) holds for $E=V^{-1}$ on $V\left[H_{A}\right]$. Putting $E$ on $D_{0}(E)$ as

$$
D_{0}(E)=V D \subseteq H_{T}, \quad E(V f)=f \varepsilon D
$$

E extends to a bounded operator on $H_{T}$, thereby proving the second part of the proposition.

Since $V_{A}$ is dense in $H_{P}$, we may consider $E$ as a densely defined operator on $D_{1}(E)$ :

$$
D_{1}(E)=\left\{V f \mid f \varepsilon H_{A}\right\}, \quad E(V f)=f \varepsilon H_{A}
$$

Following an argument of Beals[5] one computes that, for $f \varepsilon H_{A}$,

This formula implies

$$
||E g||_{K_{B}}^{2} \leqslant||g||_{T}^{2}, \quad g \varepsilon D_{1}(E) \equiv H_{T}
$$

which establishes the existence of $E$ as a bounded operator from $\mathrm{H}_{\mathrm{T}}$ into $\mathrm{H}_{\mathrm{K}}$ satisfying (20).

One might be interested in knowing whether the total
(i.e., incoming plus reflected) fluxes Ef with feRan $Q_{ \pm}$make up the whole subspace Ran $P_{ \pm}$and not just some dense subspace of it. In general this is not the case. However, for bounded and injective $A$, and for non-injective $A$ with certain restrictions, Beals[5] succeeded in proving the invertibility of the Hangelbroek operator $V$ from $H_{K}$ into $H_{T}$ and in establishing the equivalence of the inner products $(,)_{K_{B}}$ and $(,)_{T}$ on $H$, after which he could simply define $E=V^{-1}$. Unfortunately, in a recent paper[6] he misstated some steps in a derivation conceming non-injective A, and thus arrived at an incorrect result (cf.[6], first paragraph of Sec. IV). Using the operator $A_{\beta}$ of $\sec .2$, one may extend Beals' earlier work to all bounded positive Eredholm A. In Appendix $B$ we give a new proof of the equivalence of $H_{T}$ and $H_{K}$ for this case.

Earlier, Hangelbroek[20] proved the invertibility of $V$ as an operator from $H$ into $H$ for neutron transport with isotropic (and later also for some anisotropic) scattering kernels. In that work I - A was assumed compact. Under the conditions that $I-A$ is compact and $\operatorname{Ran}(I-A) \subseteq \operatorname{Ran}|T|^{\alpha}$ for some $0<\alpha<1$, van der Mee[28] proved the invertibility of $V$ and of $\mathrm{TVT}^{-1}$ on $H$, which in this case implies Beals' result on $H_{T}$.

Implications of the boundedness of $V$ are given in the following lemma.

LEMMA 3. Let V be the previous extension of $\mathrm{V}: \mathrm{H}_{\mathrm{A}} \rightarrow \mathrm{H}_{\mathrm{T}}$. Then the following five statements are equivalent:
(a) $V \varepsilon L\left(H_{K}, H_{T}\right)$
(b) $V \varepsilon L\left(\mathrm{H}_{\mathrm{K}}, \mathrm{H}_{\mathrm{T}}\right)$ invertible
(c) $\|\mid\|_{T}$ and $\left\|\|_{\mathrm{K}_{B}}\right.$ are equivalent
(d) $V \varepsilon L\left(H_{T}, H_{T}\right)$ invertible
(e) $H_{K} \supseteq H_{T} \supseteq H \supseteq H_{A}$.

PROOF. (b) => (a): Trivial.
(a) $=$ (b): Combining Eq. (22) with $V \varepsilon L\left(H_{K}, H_{T}\right)$ we find, for some constant $M$,
$M||f||_{K_{B}}^{2} \geqslant||V f||_{T}^{2} \geqslant||V f||_{T}^{2}-\left|\left|W f^{2}\right|_{T}^{2}=\left||f|_{K}^{2}\right.\right.$,
which proves (b) (a) $\underset{\Rightarrow(c): ~ F r o m ~ E q . ~(2 l) ~ w e ~ h a v e ~ i m m e d i a t e l y ~}{\text { ( }}$

$$
\begin{array}{ll}
||f||_{T}^{2} \leqslant 2(V f, f)_{T} \leqslant 2| | V| | & ||f||_{K_{B}}| | f| |_{T} \\
||f||_{K_{\beta}}^{2} \leqslant 2(V f, f)_{T} \leqslant 2| | V| | & ||f||_{K_{B}}| | f| |_{T}
\end{array}
$$

where $||V||$ indicates the norm of $V \varepsilon L\left(H_{K}, H_{T}\right)$, and (c) follows.
$(c) \Rightarrow(a): \quad$ Recall that $Q_{ \pm}$are bounded on $H_{T}$ and $P_{ \pm}$are bounded on $H_{K}$. The equivalence of the norms obviously implies (a).
$(c)=(d)(c f .[21]):$ From (c) follows the estimate

$$
c_{1}| | f\left|\left\|_{T}^{2} \leqslant\right\| f\left\|_{K_{\beta}}^{2} \leqslant c_{2}| | f\right\|_{T}^{2}\right.
$$

for $f \varepsilon H_{T}$, or

$$
c_{1}|T| \leqslant T\left(P_{+}-P_{-}\right) \leqslant c_{2}|T| .
$$

But

$$
T\left(P_{+}-P_{-}\right)=|T|\left(Q_{+}-Q_{-}\right)\left(P_{+}-P_{-}\right)=|T|(2 V-I)
$$

whence

$$
\frac{1}{2}\left(1+c_{1}\right) \leqslant V \leqslant \frac{1}{2}\left(1+c_{2}\right)
$$

on $H_{T}$. Now clearly (c) implies (d).

$$
\begin{aligned}
& (d) \Leftrightarrow(e) \text { : From (22) and (d), it follows that } \\
& \|f\|_{K_{\beta}}^{2} \leqslant\left\{2\|V\|_{H_{T}} \rightarrow H_{T}-1\right\} \quad\|f\|_{T}^{2}
\end{aligned}
$$

for $f_{A} H_{A}$. Conversely, Eq. (22), (e) and Proposition 3 imply (d).
(e) $\Rightarrow$ (c) [Pointed out by R. Beals]: Following pre-
cisely [5], one may estimate, for feD(A):

$$
\begin{aligned}
\|f\|_{T}^{2} & =\left(\operatorname{Tf},\left(Q_{+}-Q_{-}\right) f\right)=\left(\left|\hat{K}_{\beta}\right|\left(P_{+}-P_{+}\right) f,\left(Q_{+}-Q_{-}\right) f\right)_{A_{\beta}} \\
& =\left(\left(P_{+}-P_{-}\right) f,\left(Q_{+}-Q_{-}\right) f\right)_{K_{\beta}} \\
& \leqslant\|f\|_{K_{\beta}}\left\|\left(Q_{+}-Q_{-}\right) f\right\|_{K_{\beta}} \leqslant\left. c\|f\|\right|_{K_{\beta}}\|f\|_{T} .
\end{aligned}
$$

The last estimate follows from (e), and completes the proof of the lemma.

Under any of the (equivalent) conditions of Lemma 3, the half-range expansion of $f_{\varepsilon} H_{T}$ is the full-range expansion of $E f \varepsilon H_{K}$. However, even if none of these conditions is satisfied, we may formulate such a statement. Let us denote by $F$ the resolution of the identity of $K_{B}$ as a self-adjoint operator on $H_{K}$. For $f \in \operatorname{Ran} Q_{ \pm}$,

$$
f= \pm Q_{ \pm} \int_{0}^{ \pm \infty} d(F(\lambda) E f)
$$

Under the conditions of Theorem 1 , one can write

$$
E f= \pm \int_{0}^{ \pm \infty} \sum_{\alpha} A(\alpha, \lambda) \phi_{\alpha, \lambda} d \rho(\lambda)
$$



## V. EXISTENCE AND UNIQUENESS THEORY FOR HALF-SPACE PROBLEMS

To solve the half-space problem, one seeks a solution of Eq. (I), $f:[0, \infty) \rightarrow H_{K}$, subject to

$$
\begin{align*}
& f(0)=f_{+}, \quad f_{+} \varepsilon Q_{+}\left(H_{T}\right)  \tag{23a}\\
& \operatorname{Lim} \sup _{x \rightarrow \infty}| | f(x)| | \quad \text { finite. } \tag{23b}
\end{align*}
$$

Because the albedo operator $E$ acts from $H_{T}$ into $H_{K}$, a statement of this type is required. Below, we give a more precise statement of the problem.

The decomposition of $H$ into reducing subspaces of $K$, Proposition 1 , decouples the half-space problem, into a halfspace problem on PH (with a different $f_{+}$) and a finite-dimensional first order system on (I - P)H. However, the use of a suitable operator $A_{\beta}$ makes it possible to extend the half-space problem on PH to one on $H$ of a simpler structure than the original problem, the simplicity stemming from the injectivity of $A_{\beta}$. The main difficulty of the newly obtained half-space problem is that the albedo operator $E$ acts from $H_{T}$ into $H_{K}$ and might not act from $H$ into $H$. For this reason we state the following weakened version of the half-space problem:

Given $\mathrm{f}_{+} \varepsilon_{Q_{+}}\left[\mathrm{H}_{\mathrm{T}}\right]$, construct a continuous function $\phi:[0, \infty) \rightarrow H_{K}$ with KP $\phi$ and (I - P) $\phi$ differentiable on $(0, \infty)$, such that

$$
\begin{align*}
& \frac{d}{d x} K^{-1} P \phi=-P \phi \quad\left(\text { on } P H_{K}\right)  \tag{24a}\\
& \frac{d}{d x}(I-P) \phi=-T^{-1} A \phi \quad\left(\text { on } Z_{0}(K)\right)  \tag{24b}\\
& \phi(0) \varepsilon H_{T} \text { and } Q_{+} \phi(0)=f_{+}  \tag{24c}\\
& \|P \phi(x)\|_{K}=0(I),\|(I-P) \phi(x)\|=0(I)(x \rightarrow \infty) .
\end{align*}
$$

We did not use $\beta$ in this statement of the half-space problem. In Equation (24d) it is immaterial which $\beta$ one applies in the $K_{B}$-norm.

The decompositions of $H_{K}$ into reducing subspaces of $K$, Proposition 1 extended to $H_{K}$, decouples the weak half-space problem (24) into an infinite dimensional evolution equation $\mathrm{PH}_{\mathrm{K}}$ (namely, (24a) with initial value $\mathrm{PEf}_{+}$) and a finite-dimensional first order system on $(I-P) H_{K}=Z_{0}(K)$. On $P H_{K}$, the weak halfspace problem is equivalent to the semigroup problem

$$
\begin{aligned}
& \frac{\partial}{\partial \mathrm{x}} \mathrm{~T} \phi=-\mathrm{A} \phi \\
& \phi(0)=\mathrm{PEf}_{+} ; \\
& \| \phi(\mathrm{x})| |_{K_{\beta}}=0(1)(\mathrm{x} \rightarrow \infty),
\end{aligned}
$$

which has a unique solution once $\phi(0)=P E f_{+}$is specified uniquely. The albedo operator E satisfies conditions (19). On (I - P) $H_{K}=$ $Z_{0}(K)$, boundedness at infinity requires that (I - P)Ef ${ }_{+} \varepsilon \operatorname{Ker} A$, after which the solution on $Z_{0}(K)$ can be written as a constant; more precisely,

$$
(I-P) \phi(x)=e^{-x T^{-1} A}(I-P) E f_{+} \equiv(I-P) E f_{+} .
$$

Recalling the remark at the end of sec. II, we have

THEOREM 2. For every $\mathrm{f}_{+} \varepsilon Q_{+}\left(\mathrm{H}_{\mathrm{T}}\right)$, the half-space problem has a unique (differentiable) solution if and only if Ker $A$ is positive definite with respect to the indefinixe inner product (11). This will be the case if each $\lambda=0$ eigenvector of K has a corresponding generalized eigenvector. If Ker A is not positive, there exist non-trivial solutions with incoming flux $\mathrm{f}_{+}=0$ (non-uniqueness), and at least one solution for every $f_{+} \varepsilon_{+} Q_{+}\left[H_{T}\right]$. on $P_{K},\left.\lim _{x \rightarrow \infty}| | P \phi(x)\right|_{K_{\beta}}=0$.

The theorem follows immediately from standard semigroup theory, assuming the construction of $E$ (which depends on $\beta$ ) gives a unique albedo operator $E$. We observe that $P E: H_{T} \rightarrow H_{K}$ is independent of the choice of $\beta$. So we must investigate the dependence
of (I - P)EQ $: H_{T} \rightarrow Z_{0}(K)$ on $\beta$. Recall that $\beta$ was constructed from a decomposition $M_{+} \oplus M_{-}=Z_{0}(K)$ into subspaces $M_{ \pm}$that are positive/negative with respect to the inner product (ll). However, $E$ has a dense range in $H_{K}$ and maps Ran $Q_{ \pm}$into Ran $P_{ \pm}$, where $\operatorname{Ran} P_{ \pm}=\operatorname{Ran} P_{ \pm} \oplus M_{ \pm}$. Thus ( $I-P$ ) EQ ${ }_{+} \operatorname{maps} H_{T}$ onto $M_{+}$, and we conclude that ( $I-P$ ) $E Q_{+}$depends on the choice of the maximal positive subspace $\mathrm{M}_{+}$only. In order that the weak halfspace problem (24) has a (bounded) solution, one has to be able to take ( $I-P) E f+$ Ker A. Hence, problem (24) has a (bounded) solution for every $f_{+} \varepsilon_{+} Q_{+}\left(H_{T}\right)$, if and only if there exists a maximal positive subspace $M_{+}$of $Z_{0}(K)$ with respect to (ll) with the property $M_{+} \subseteq$ Ker $A$. The existence of such $M_{+}$follows from Lemma 2. The uniqueness statement in Theorem 1 will be proved shortly.

A measure of non-uniqueness is given by the number

$$
\delta=\operatorname{dim}\left[\operatorname{Ran} \mathrm{PP}_{+} \oplus \operatorname{Ran} Q_{-}\right] \cap \operatorname{Ker} A
$$

Indeed, let $\phi:[0, \infty) \rightarrow H_{K}$ be any solution of the weak half-space problem (24) with incoming flux $f_{+}=0$. Then $\phi(0) \varepsilon Q_{-}\left(H_{T}\right)$, and (I - P) $\phi(0) \varepsilon$ Ker A (so that the solution $\phi$ will be bounded at $+\infty)$. Therefore, $(I-P) \phi(0) \varepsilon\left[\operatorname{Ran} P_{+} \oplus \operatorname{Ran} Q_{-}\right] \cap \operatorname{Ker} A$. Conversely, note that any vector $f_{0}=f_{p}+f_{f}$ with $f_{p} \varepsilon \operatorname{Ran} P P_{+}$, $\mathrm{f}_{-} \varepsilon \operatorname{Ran} \mathrm{Q}_{-}$and $\mathrm{f}_{0} \varepsilon \operatorname{Ker} \mathrm{~A}$ satisfies

$$
0 \geqslant\left[f_{-}, f_{-}\right]=\left[f_{p}, f_{p}\right]+\left[f_{0}, f_{0}\right]=\left(\hat{K}_{\beta} f_{p}, f_{p}\right)_{A_{\beta}}+\left[f_{0}, f_{0}\right],
$$

and, as $\left(\hat{K}_{\beta} f_{p}, f_{p}\right)_{A_{\beta}} \geqslant 0$, the space $N_{-}=\left[\operatorname{Ran} P P_{+} \oplus \operatorname{Ran} Q_{-}\right]$K Ker $A$ is strictly negative with respect to (II), and thus $M_{+} \mathrm{N}_{N_{-}}=$ $\{0\}$ for every choice of $\mathbb{M}_{+}, \beta$ or $E$. Thus the incoming flux $f_{+}=$ $Q_{+} \phi(0)$ is identically zero, which proves our assertion. As in the previous paragraph we prove that
$Z_{0}^{ \pm}=\left[\operatorname{Ran} P_{F} \oplus \operatorname{Ran} Q_{ \pm}\right] \cap Z_{0}(K)$
is strictly positive/negative with respect to (11). Using the
computational rules for orthogonal complements of sums and intersections of linear subspaces, together with the self-adjointness of $Q_{ \pm}$and $T P P_{\mp}$, one finds

$$
\left(z_{0}^{ \pm}\right)^{1}=T\left[Z_{0}^{\mp}\right] \oplus Z_{0}(K)^{1}
$$

This equation implies (together with $\mathrm{Z}_{0}^{+} \cap \mathrm{Z}_{0}^{-}=(0)$ ):

$$
z_{0}^{+} \oplus z_{0}^{-}=\left(Z_{0}^{+}+z_{0}^{-}\right)^{1}=\left(\left(z_{0}^{+}\right)^{1} \cap\left(z_{0}^{-}\right)^{1}\right)^{1}=Z_{0}(K)
$$

Thus $Z_{0}^{ \pm}$is a maximal strictly positive/negative subspace of $Z_{0}(K)$ (cf.[28], Sec. III. 5). But then
$\left[\operatorname{Ran} P P_{\mp} \oplus \operatorname{Ran} Q_{ \pm}\right] \cap \operatorname{Ker} A=Z_{0}^{ \pm} \cap \operatorname{Ker} A$.
is a maximal strictly positive/negative subspace of Ker A (endowed with (11)). Using the measure of non-uniqueness $\delta$ of the previous paragraph we prove the uniqueness statement of Theorem 1.

THEOREM 3. Any solution of the half-space problem
(24) has the form

$$
\phi(x)=\sum_{i=1}^{m} c_{i} \alpha_{i}+\int_{0}^{\infty} e^{-\lambda x_{d}\left(F(\lambda) P E f_{+}\right)},
$$

where $c_{1}, \ldots, c_{m_{+}}$are the expansion coefbicients of (I $\left.-P\right) E f_{+}$ with respect to a basis $\left\{\alpha_{1}, \ldots, \alpha_{m_{+}}\right\}$of $M_{+}=\operatorname{Ran}(I-P) P_{+}$.

Under the conditions of Theorem 1 , one can expand $P \phi(x)$ in terms of Case's eigenfunctions as follows:

$$
P_{\phi}(x)=\int_{0_{+}}^{\infty} e^{-\lambda x} \sum_{\alpha} A(\alpha, \lambda) \phi_{\alpha, \lambda} d \rho(\lambda) .
$$

The functions $A(\alpha, \lambda)$ are computed from full-range orthogonality:

$$
A(\alpha, \lambda)=\frac{I}{\rho(\lambda)}\left(\operatorname{PEf}_{+}, \phi_{\alpha}, \lambda\right)_{A}
$$

## VI. APPLICATIONS

This section contains several physical models leađing to an equation of the form (I). All models involve a timeindependent one-dimensional transport problem in a semi-infinite medium and in all cases the spatial variable $x \in(0, \infty)$. For all these models we shall specify the Hilbert space $H$, the operators $T$ and $A$ and the structure of the zero root linear manifold $Z_{0}(K)$.

1. ONE-SPEED NEUTRON TRANSPORT (of. [13,20,5,28])

$$
\begin{align*}
\mu \frac{\partial f}{\partial x}=-f(x, \mu) & +c \int_{-1}^{+1} p\left(\mu, \mu^{\prime}\right) f\left(x, \mu^{\prime}\right) d \mu^{\prime} \\
& +g(x, \mu), \quad(-1 \leqslant \mu \leqslant+1) \tag{25}
\end{align*}
$$

where

$$
p\left(\mu, \mu^{\prime}\right)=\sum_{n=0}^{\infty} a_{n}\left(n+\frac{1}{2}\right) P_{n}(\mu) P_{n}\left(\mu^{\prime}\right),
$$

the $P_{n}$ are the usual Legendre polynomials, $a_{0}=1,-1 \leqslant a_{n} \leqslant 1$ and $\lim _{n \rightarrow \infty} a_{n}=0$. In this case we take $H=L_{2}[-1,+1]$ and define $A$ and $T$ by

$$
\begin{align*}
(A f)(\mu)= & f(\mu)-c \int_{-1}^{+1} p\left(\mu, \mu^{\prime}\right) f\left(\mu^{\prime}\right) d \mu^{\prime},  \tag{26}\\
& (T f)(\mu)=\mu f(\mu) .
\end{align*}
$$

Obviously $T$ is bounded self-adjoint, $A$ positive and I - A is compact (cf.[35]). For $0 \leqslant c<l$ the operator $A$ is invertible. For $c=1$, however, Ker $A$ is non-trivial and equals the span of $\left\{P_{n} \mid a_{n}=1\right\}$. The zero root linear manifold $Z_{0}(K)$ is well-known and its dimension is even [28]. In fact, $K=T^{-1} A$ has zero Jordan chains of length 2 only, if and only if the finite set $M=$ $\left\{n \geqslant 0 \mid a_{n}=I\right\}$ does not contain consecutive integers. This situation is fulfilled if $p\left(\mu, \mu^{\prime}\right) \geqslant 0, c=I$ and $\sum_{n=I}^{\infty}\left|a_{n}\right|^{2}<\infty$, in which case $M=\{0\}$ and $Z_{0}(K)=\{a+b \mu \mid a, b \in \mathbb{C}\}[26]$.

> 2. RADIATIVE TRANSFER OF UNPOLARIZED LIGHT $(c f \cdot[16,26,28])$
> $(\cos \theta) \frac{\partial f}{\partial x}(x, \omega)=-f(x, \omega)+\frac{c}{4 \pi} \int_{\Omega} p\left(\omega, \omega^{\prime}\right) f\left(x, \omega^{\prime}\right) d \omega^{\prime}+g(x, \omega)$.

Here $\omega$ is a point of the unit sphere $\Omega$ in $\mathbb{R}^{3}$ with polar coordinates $(\theta, \phi)$. We assume that $0 \leqslant c \leqslant I$ and that the phase function $p$ is nonnegative, belongs to $L_{1}[-1,+1]$ and satisfies $\int_{-1}^{+1} p(t) d t=2$. For this model we take $H=L_{2}(\Omega)$ and define $A$ and $T$ by

$$
\begin{aligned}
& (\mathrm{Af})(\omega)=f(\omega)-\frac{c}{2} \int_{\Omega} p\left(\omega \cdot \omega^{\prime}\right) f\left(x, \omega^{\prime}\right) d \omega^{\prime}, \\
& (\mathrm{Tf})(\omega)=(\cos \theta) f(\omega) .
\end{aligned}
$$

Certainly $T$ is bounded self-adjoint, A positive and I - A compact[35]. For $0 \leqslant c<1$ the operator $A$ invertible. For $c=1$ and $p \varepsilon L_{2}[-1,+1]$ Maslennikov[26] proved that $\operatorname{Ker} A=\operatorname{span}\{I\}$ and $Z_{0}(K)=\operatorname{span}\{1, \cos \theta\}$.
3. RADIATIVE TRANSFER OF UNPOLARIZED IIGHT (FOURIER DECOMPOSED)
Writing

$$
f(x, \omega)=f_{0}(x, \cos \theta)+2 \sum_{m=1}^{\infty} f_{m}(x, \cos \theta) \cos m \phi
$$

Eq. (27) can be reduced to a sequence of equations of the form [1,26,34]

$$
\begin{aligned}
\mu \frac{\partial f_{m}}{\partial x}(x, \mu)= & -f_{m}(x, \mu)+\frac{c}{2} \int_{-1}^{+l} p_{m}\left(\mu, \mu^{\prime}\right) f_{m}\left(x, \mu^{\prime}\right) d \mu^{\prime} \\
& +q_{m}(x, \mu), \quad \mu \varepsilon[-1,+l]
\end{aligned}
$$

where $0 \leqslant c \leqslant l$ and $p_{m}\left(\mu, \mu^{\prime}\right)$ is the kernel

$$
p_{m}\left(\mu, \mu^{\prime}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} p\left(\mu \mu^{\prime}+\sqrt{1-\mu^{2}} \sqrt{1-\mu^{\prime 2}} \cos \alpha\right) \cos m \alpha d \alpha .
$$

If $p \varepsilon L_{1}[-1,+1]$, then $[35,34]$

$$
p_{m}\left(\mu, \mu^{\prime}\right)=\sum_{n=m}^{\infty} a_{n}(2 n+1) \frac{(n-m)!}{(n+m)!} P_{n}^{m}(\mu) P_{n}^{m}\left(\mu^{\prime}\right)
$$

where $P_{n}^{m}(\mu)=\left(1-\mu^{2}\right)^{\frac{1}{2} m}\left(\frac{d}{d \mu}\right)^{m_{n}} P_{n}(\mu)$ is the associated Legendre function and $a_{0}=1,-1 \leqslant a_{n} \leqslant+1$ and $\lim _{n \rightarrow \infty} a_{n}=0$. For $m=0$ and $p\left(\mu, \mu^{\prime}\right)=\frac{1}{2} p_{m}\left(\mu, \mu^{\prime}\right)$ one gets Eq. (25). Now take $H=L_{2}[-1,+1]$ and define $A$ and $T$ to be

$$
\begin{aligned}
(A f)(\mu)= & f(\mu)-\frac{c}{2} \int_{-1}^{+l} p_{m}\left(\mu, \mu^{\prime}\right) f\left(\mu^{\prime}\right) d \mu^{\prime} \\
& (T f)(\mu)=\mu f(\mu) .
\end{aligned}
$$

Clearly T is bounded self-adjoint, A is positive and I - A compact. If $m=0$, the structure of $Z_{0}(K)$ is basically described at the first application. If $p \varepsilon L_{2}[-1,+1]$ (or equivalently if
$\left.\sum_{n=0}^{\infty}\left|a_{n}\right|^{2}<\infty\right)$, the result of Maslennikov implies that $A$ is invertible whenever $m \geqslant 1$, because $-1 \leqslant a_{n}<1(n \geqslant m \geqslant 1)$.
4. SYMMETRIC MULTIGROUP NEUTRON TRANSPORT (cf.[18])

$$
\begin{aligned}
\mu \frac{\partial f_{i}}{\partial x}(x, \mu)= & -\sigma_{i} f_{i}(x, \mu)+\frac{1}{2} \sum_{j=1}^{N} C_{i j} \int_{-1}^{+1} f_{j}\left(x, \mu^{\prime}\right) d \mu^{\prime} \\
& +q_{i}(x, \mu) \quad(i=1, \ldots, N ;-1 \leqslant \mu \leqslant+1)
\end{aligned}
$$

where $\sigma_{1}, \ldots, \sigma_{N}$ are positive constants and $C$ is a real symmetric matrix. Writing $\Sigma$ for the diagonal matrix of order $N$ with diagonal entries $\sigma_{1}, \ldots, \sigma_{N}$ we assume that $\Sigma-C$ is positive, possibly with a non-trivial kernel. Now consider the Hilbert space $H=$ N $\underset{i=1}{\oplus} L_{2}[-l,+1]$, the direct sum of $N$ copies of $L_{2}[-1,+l]$. Define $A$ and $T$ by

$$
(A f)_{i}(\mu)=\sigma_{i} f_{i}(\mu)-\frac{1}{2} \sum_{j=1}^{N} C_{i j} \int_{-l}^{+1} f_{j}\left(\mu^{\prime}\right) d \mu^{\prime}
$$

$$
(T f)_{i}(\mu)=\mu f_{i}(\mu) \quad(i=1, \ldots, N ;-1 \leqslant \mu \leqslant+1) .
$$

Then $T$ is bounded self-adjoint, $A$ positive and $I-\Sigma^{-1} A$ compact. The kernel of $A$ is given by

$$
\operatorname{Ker} A=\left\{f=\left(f_{i}\right)_{i=1}^{N} \mid f_{i}(\mu) \equiv \xi_{i},(\Sigma-C) \xi=0\right\} .
$$

Solving the equation $A g=T f$ for $f \in K e r A$ and exploiting Lemma 1 one obtains

$$
\begin{aligned}
& Z_{0}(K)=\left\{f=\left(f_{i}^{\prime}\right){ }_{i=1}^{N} \mid f_{i}(\mu)=\xi_{i}+\sigma_{i}^{-1} \eta_{i} \mu,\right. \\
& (\Sigma-C) \xi=(\Sigma-C) n=0\} .
\end{aligned}
$$

Hence, $\operatorname{dim} Z_{0}(K)=2 \operatorname{dim} \operatorname{Ker} A=2 \operatorname{dim} \operatorname{Ker}(\Sigma-C)$ and so all zero Jordan chains of $K=T^{-1} A$ have length 2 .
5. TRANSFER OF POLARIZED LIGHT WITH RAYLEIGH SCATTERING (ef.[16])

$$
\begin{aligned}
& \mu \frac{d}{d x}\binom{f_{\ell}}{f_{r}}=-\binom{f_{\ell}(x, \mu)}{f_{r}(x, \mu)} . \\
& +\frac{3}{8} \int_{-1}^{+1}\left(\begin{array}{cc}
2\left(1-\mu^{2}\right)\left(1-\mu^{\prime}\right. & \left(-1 \leqslant \mu^{2} \mu^{\prime 2}\right. \\
\mu^{\prime}, 2 \\
1
\end{array}\right)\binom{f_{\ell}\left(x, \mu^{\prime}\right)}{f_{r}\left(x, \mu^{\prime}\right)} d \mu^{\prime}
\end{aligned}
$$

where we have omitted the internal source term $q(x, \mu)$. This problem is considered on the Hilbert space $H=L_{2}[-1,+1] \oplus L_{2}[-1,+1]$ and the operators $A$ and $T$ are defined by

$$
\begin{aligned}
& \binom{(A f)_{\ell}(\mu)}{(A f)_{r}(\mu)}=\binom{f_{\ell}(\mu)}{f_{r}(\mu)} \\
& -\frac{3}{8} \int_{-1}^{+1}\binom{2\left(1-\mu^{2}\right)\left(1-\mu^{\prime}{ }^{2}\right)+\mu^{2} \mu^{\prime} \mu^{2} \mu^{2}}{\mu^{\prime}}\binom{f_{\ell}\left(\mu^{\prime}\right)}{f_{r}\left(\mu^{\prime}\right)} d \mu^{\prime} ; \\
& \quad(T f)_{\ell}(\mu)=\mu f_{\ell}(\mu), \quad(T f)_{r}(\mu)=\mu f_{r}(\mu) .
\end{aligned}
$$

Certainly $T$ is bounded self-adjoint, A positive and $I$ - A compact. It is straightforward to show that

$$
\text { Ker } A=\operatorname{span}\left\{\binom{I}{I}\right\}, \quad Z_{0}(K)=\operatorname{span}\left\{\binom{I}{I},\binom{\mu}{\mu}\right\}
$$

Hence, $K=T^{-1} A$ has one zero Jordan chain, which has length 2.
6. NEUTRON TRANSPORT WITH ANGULARLY DEPENDENT CROSS SECTIONS (cf.[29,37])

$$
\begin{equation*}
\mu \frac{\partial f}{\partial x}(x, \mu)+\Sigma(\mu) f(x, \mu)=\frac{1}{2} \int_{-1}^{+1} \Sigma_{S}\left(\mu^{\prime}\right) f\left(x, \mu^{\prime}\right) d \mu^{\prime}+g(x, \mu) \tag{28}
\end{equation*}
$$

We assume that $\Sigma$ and $\Sigma_{s}$ are measurable, $\Sigma_{s}$ is bounded (whereas $\Sigma$ is not assumed to be so) and $\Sigma \geqslant \Sigma_{S} \geqslant \varepsilon>0$. Now premultiply Eq. (28) by $\Sigma_{S}(\mu)$ and consider the new equation on $H=L_{2}[-1,+1]$. Put

$$
\begin{aligned}
& \left(A f^{\prime}\right)(\mu)=\Sigma_{S}(\mu)\left\{\Sigma(\mu) f(\mu)-\frac{1}{2} \int_{-1}^{+1} \Sigma_{S}\left(\mu^{\prime}\right) f\left(\mu^{\prime}\right) d \mu^{\prime}\right\} \\
& \left(T f^{\prime}\right)(\mu)=\mu \Sigma_{S}(\mu) f(\mu)
\end{aligned}
$$

Then $T$ is bounded self-adjoint, $A$ is self-adjoint with closed range. Schwarz's inequality implies that

$$
\frac{1}{2}\left|\int_{-1}^{+1} \Sigma_{s} f^{2} d \mu\right|^{2} \leqslant \frac{1}{2} \int_{-1}^{+1} \Sigma_{s}^{2}|f|^{2} d \mu \cdot \int_{-1}^{+1} d \mu \leqslant \int_{-1}^{+1} \Sigma_{s} \Sigma|f|^{2} d \mu
$$

and therefore $A$ is positive. Note that $A$ is bounded if and only if $\Sigma$ is bounded.

Let us determine the zero root linear manifold $Z_{0}(K)$. First observe that $\Sigma^{-1}$ belongs to $H=L_{2}[-1,+1]$ and that every function in Ker $A$ is proportional to $\Sigma^{-1}$. However, in order that Ker $A \neq\{0\}$ it is necessary and sufficient that

$$
\frac{1}{2} \int_{-1}^{+1} \frac{\Sigma-\Sigma_{S}}{\Sigma} d \mu=1-\frac{1}{2} \int_{-1}^{+1} \frac{\Sigma_{S}}{\Sigma} d \mu=\Sigma_{S}(\mu)^{-1}\left(A \Sigma^{-1}\right)(\mu)=0
$$

We conclude that $\operatorname{Ker} A=\{0\}$ unless $\Sigma(\mu)=\Sigma_{S}(\mu)$ for almost every $\mu \varepsilon[-1,+1]$, in which case Ker $A=\operatorname{span}\left\{\Sigma^{-1}\right\}$. If this is fulfilled, then

$$
Z_{0}(K)=\left\{\begin{array}{l}
\operatorname{span}\left\{\Sigma^{-1}, \mu \Sigma(\mu)^{-2}\right\} \text { if } \int_{-1}^{+1} \mu \Sigma(\mu)^{-1} d \mu=0 ; \\
\operatorname{span}\left\{\Sigma^{-1}\right\}=\text { Ker A if } \int_{-1}^{+1} \mu \Sigma(\mu)^{-1} d \mu \neq 0 .
\end{array}\right.
$$

Since for non-trivial Ker A one has $\left[\Sigma^{-1}, \Sigma^{-1}\right]=\int_{-1}^{+i} \mu \Sigma(\mu)^{-1} \mathrm{~d} \mu$ (cf.(11)), the half-space problem (24) has a unique solution for $\delta_{-1}^{+1} \mu \Sigma(\mu)^{-1} \mathrm{~d} \mu \geqslant 0$ and measure of non-uniqueness one for negative values of this integral.

Finally, as pointed by R. Beals, we note that in $L_{2}\left([-1, I] ; \Sigma_{S}(\mu) \Sigma(\mu) d \mu\right)$ the operator $A_{1}=\Sigma(\mu)^{-1} \Sigma_{S}(\mu)^{-1} A$ is a (self-adjoint) compact perturbation of the identity, and the bounded A theory may also be applied to (28). In this way one may specify the domain of $V$, but the set of allowable incident fluxes is smaller.
7. $2 n^{\text {th }}$ ORDER STURM-LIOUVILLE DIFFUSION (cf. $[3,4,6]$ )

$$
h(\mu) \frac{\partial f}{\partial x}=\frac{-d^{n}}{d \mu^{n}}\left(p(\mu) \frac{d^{n} f}{d \mu^{n}}\right)+q(x, \mu), \mu \varepsilon J \subseteq \mathbb{R}
$$

for $(-1)^{n+1} p(\mu) \geqslant 0$ on a compact interval $J$ and $p$ n-times continuously differentiable on $J$. Of course, $p$ may not vanish on a set of positive measure. The function $h$ is assumed to be bounded measurable and the set $S_{0}=\{\mu \varepsilon J \mid h(\mu)=0\}$ has measure zero. Now consider $H=L_{2}(J)$ and define $A$ and $T$ by

$$
(A f)(\mu)=\frac{-d^{n}}{d \mu^{n}}\left(p(\mu) \frac{d^{n} f}{d \mu}\right), \quad(T f)(\mu)=h(\mu) f(\mu),
$$

with self-adjoint boundary conditions assumed for $A$. Then $T$ is bounded self-adjoint with Ker $T=\{0\}$, A positive with closed range and the point $\lambda=0$ does not belong to the spectrum of $A$ or is an isolated eigenvalue of finite multiplicity. For $J=[-1,1]$, $\mathrm{n}=1, \mathrm{~h}(\mu)=\mu, \mathrm{p}(\mu)=1-\mu^{2}$ with $\mathrm{D}(\mathrm{A})$ consisting of bounded func-
tions, one gets an application to electron transport, where $T^{-1} A$ has one zero Jordan chain, which has length 2 [5]. Thus this problem is uniquely solvable.
APPENDIX A: INFINITE MEDIUM INHOMOGENEOUS EQUATION
The inhomogeneous equation

$$
\begin{equation*}
\frac{\partial T f}{\partial X}=-A f+T q \tag{A.1}
\end{equation*}
$$

on $H$ can be solved in terms of the full-range expansion. We assume that $\mathrm{Pq}(\mathrm{x})$ is uniformly Hölder continuous in x as a function from $R$ to $H_{A}$ with bound

$$
\begin{equation*}
\left|\mid P q(x) \|_{A} \leqslant c e^{-\delta|x|}\right. \tag{A.2}
\end{equation*}
$$

for some $\delta>0$, and ( $I-P$ ) $q(x)$ is continuous in $x$ as a function from $\mathbb{R}$ to $H$.

On (I - P)H, (A.l) is an elementary first order system of linear differential equations. Denoting the basis of $Z_{0}(K)$ by $x_{i}, y_{i}, z_{j}$ as in the proof of Proposition 2, we may write the solution of the homogeneous equation on $Z_{0}(K)$ as

$$
(I-P) f(x)=\sum_{i=1}^{r}\left\{c_{i} x_{i}+d_{i}\left(I-x T^{-1} A\right) y_{i}\right\}+\sum_{j=1}^{S} e_{j} z_{j}
$$

for constants $c_{i}, d_{i}, e_{j}$. Then the variation of parameters formula may be used to obtain a particular solution of the inhomogeneous equation. Expanding (I-P)q as

$$
(I-P) q(x)=\sum_{i=1}^{r}\left\{\ell(x) T^{-l} A+m_{i}(x)\right\} y_{i}+\sum_{j=1}^{S} n_{j}(x) z_{j},
$$

we find

$$
\begin{align*}
(I-P) f(x)= & \sum_{i=1}^{r} \int_{-\infty}^{x}\left\{\ell_{i}(\hat{x}) T^{-1} A+m_{i}(\hat{x})(x-\hat{x}) T^{-1} A\right. \\
& \left.+m_{i}(\hat{x}) I\right\} d \hat{x}_{i}+\sum_{j=1}^{S} \int_{-\infty}^{x} n_{i}(\hat{x}) d \hat{x}_{j} \tag{A.3}
\end{align*}
$$

is the solution of (A.l) on $Z_{0}(K)$ which vanishes at $x=-\infty$.

On $H_{A}$, a solution of (A.l) may be written

$$
\begin{align*}
(P f)(x)= & \int_{x}^{\infty} \int_{-\infty}^{0} e^{-\lambda(x-\hat{x})} d\left(F^{\prime}(\lambda) P_{-} q\right)(\hat{x}) d \hat{x} \\
& +\int_{-\infty}^{x} \int_{0}^{\infty} e^{-\lambda(x-\hat{x})} d\left(F(\lambda) P_{+} q\right)(\hat{x}) d \hat{x} \tag{A.4}
\end{align*}
$$

THEOREM A.I If Pq is uniformly $\mathrm{H}_{\mathrm{A}}$-Hölder continuous in $x$ with bound. (A.2) and (I-P)q is $L^{2}$-continuous, then the solution (A.I) with the conditions that $\|P f(x)\|_{A}$ will be bounded for $x \in \mathbb{R}$ and $|\mid(I-P) f(x) \| \rightarrow 0$ as $x \rightarrow-\infty$ is given by Eqs. (A.3) and (A.4) and is unique.

PROOF. Let $U_{ \pm}(x)$ denote the holomorphic semigroups generated by $\mp K$ on $\mathrm{P}_{ \pm} \mathrm{H}_{\mathrm{A}}{ }^{-}$. The convergence of the integrals

$$
\int_{ \pm \infty}^{x} U_{ \pm}(x-x) P_{ \pm} P q(x) d x
$$

and the bound on $||P f||_{A}$ result from the inequality (A.2).
To prove that Pf is a solution of Equation (A.l) on the subspace $H_{A}$, we fix $x \in \mathbb{R}$ and define $g_{ \pm}(x)=P_{ \pm} f\left(x_{0} \pm x\right)$. Then $g_{ \pm}$satisfies the semigroup equation

$$
\frac{\partial g_{ \pm}}{\partial x}=P_{ \pm} K g_{ \pm}(x)=f\left(x_{0} \pm x\right), x>0
$$

The proof of the theorem follows from standard results in semigroup theory (on $\mathrm{P}_{ \pm} \mathrm{H}_{\mathrm{A}}$ ) and the existence theorem for solutions of finite systems of first order linear equations (on $Z_{\theta}(K)$ ).

The solution is also unique. On $Z_{0}(K)$ this is immediate. On $P_{+} H_{A}$, for example, the Lebesque Dominated Convergence Theorem applied to $P_{+} f(x)=\int_{-\infty}^{x} U_{+}(x-x) P_{+} P q(x) d x$, along with Eq. (A.2), yields $\left.\lim _{x \rightarrow-\infty}| | P_{+} f(x)\right|^{-\infty}=0$. If $\phi_{I}$ and $\phi_{2}$ are both solutions, then semigroup arguments show that $U_{+}(x-\hat{x})\left(\phi_{1}(\hat{x})-\phi_{2}(\hat{x})\right)$ is constant for $x \leqslant x$, hence $\phi_{1}(x)=\phi_{2}(x)$. These arguments follow the uniqueness proof for subcritical neutron transport; see Ref. [20].

COROLLARY. If the conditions of Corollary 1 are fulfilled, then a solution of Eq. (A.1) may be written

$$
\begin{aligned}
\phi(x)=\left((I-P)_{\phi}\right)(x) & +\int_{x}^{\infty} \int_{-\infty}^{0} \sum_{\alpha=1}^{N_{\lambda}} e^{-\lambda(x-\hat{x})} A(\hat{x}, \alpha, \lambda) \phi_{\alpha, \lambda} d p(\lambda) d \hat{x} \\
& +\int_{-\infty}^{x} \int_{0}^{\infty} e^{-\lambda(x-\hat{x})_{A}(\hat{x}, \alpha, \lambda) \phi_{\alpha, \lambda} d \rho(\lambda) d \hat{x}}
\end{aligned}
$$

where $A(x, \alpha, \lambda)$ are the expansion coefficients of $q(x, \cdot)$ given by the full-range expansion formula and (I-P) $\phi$ is given by Eq. (A.3).

APPENDIX B: EQUIVALENCE OF INNER PRODUCTS
For bounded positive A with bounded inverse a new proof is given of Beals' result [5] that the inner products (, ) $T_{T}$ and ( , ) ${ }_{K}$ (the subscript $B$ will be suppressed) are equivalent.

Observe that $\mathrm{TA}^{-1}$ is self-adjoint in the completion $\mathrm{H}_{\mathrm{A}}-1$ of H with respect to the inner product

$$
\begin{equation*}
(h, k)_{A^{-1}}=\left(A^{-1} h, k\right) \tag{B.I}
\end{equation*}
$$

By assumption, $H_{A}=H=H_{A}-1$. Let $P_{ \pm}^{+}$be the (, ) $A^{-l \text {-orthogonal }}$ projection of $H_{A}-1(=H)$ onto the maximal ( , ) $A^{-l-p o s i t i v e /-n e g a-~}$ tive $T A^{-1}$-invariant subspace. Then $T\left(A^{-1} T\right)=\left(\mathrm{TA}^{-1}\right) T$ gives

$$
T P_{ \pm}=P_{ \pm}^{+} T
$$

Therefore, if

$$
V^{+} \stackrel{\text { dee }}{=} \cdot Q_{+} P_{+}^{+}+Q_{-} P_{-}^{+},
$$

then

$$
\begin{equation*}
T V=V^{\dagger} T \tag{B.2}
\end{equation*}
$$

where $V$ and $V^{\dagger}$ are bounded on $H$. By (B.2), the operators $V$ and $\left(Q_{+}-Q_{-}\right)\left(V^{\dagger}\right) *\left(Q_{+}-Q_{-}\right)$are (possibly unbounded) adjoints in $H_{T}$, which are bounded on $H$. According to Theorem I 1.2 of [7], V extends to a bounded operator on $H_{T T}$, and thus (see Lemma 3) the inner products $(,)_{T}$ and ( , $)_{K}$ are equivalent.

If $T$ is unbounded, the reasoning is the same with the following modifications: (i) $V$ leaves invariant $D(T)$ and (B.2) holds true on $D(T)$, and (ii) $V$ and $\left(Q_{+}-Q_{-}\right)\left(V^{\dagger}\right) *\left(Q_{+}-Q_{-}\right)$leave invariant $D(T)$ and their restrictions to $D(T)$ are bounded with respect to the graph norm for $T$. Using [7] again, we may then continuously extend $V$ from $D(T)$ to $H_{T}$, and the equivalence of the inner products follows as in the proof of Lemma 3.

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