International Journal of Pure and Applied Mathematics
Volume 67 No. 3 2011, 237-258

EVOLUTION SYSTEMS IN KINETIC THEORY

Cornelis van der Mee
Dipartimento di Matematica e Informatica
Università degli Studi di Cagliari
Viale Merello, 92, Cagliari, 09123, ITALY
e-mail: cornelis@krein.unica.it

Abstract: In this article we cast the unique solvability of a class of nonautonomous kinetic equations in terms of evolution systems, generalizing semigroup results pertaining to the corresponding autonomous kinetic equations.

AMS Subject Classification: 47N55, 82C70
Key Words: kinetic equation, evolution system

1. Introduction

In this article we study the one-dimensional linear Boltzmann equation

$$\frac{\partial f}{\partial t}(v,t) + a(t)\frac{\partial f}{\partial v}(v,t) + \nu(v)f(v,t) = \int_{-\infty}^{\infty} k(v,\hat{v})\nu(\hat{v})f(\hat{v},t)\,d\hat{v}, \tag{1.1}$$

where $t \geq s$, with initial condition

$$f(v,s) = f_s(v), \tag{1.2}$$

where $f(v,t)$ is a space-averaged distribution function for electrons moving with velocity v at time t in a weakly ionized gas, $a(t)$ is the electrostatic acceleration assumed to depend on time only, $\nu(v)$ is the collision frequency and $k(v,\hat{v})$ is the scattering kernel for collisions with the velocity changing from $\hat{v}$ to v. Because of the hypothesis that ionization and recombination effects balance each other, we have

$$k(v,\hat{v}) \geq 0, \qquad \int_{-\infty}^{\infty} k(v,\hat{v})\,dv \equiv 1.$$

Further, we assume that $\nu(v)$ and $a(t)$ are locally L^1 and $\nu(v)$ is almost every-

Received: November 19, 2010

where positive. Introducing the Banach spaces $L^1(\mathbb{R}, dv)$ and $L^1(\mathbb{R}, \nu dv)$ with their respective norms

$$\|g\|_1 = \int_{-\infty}^{\infty} |g(v)|\, dv, \qquad \|g\|_\nu = \int_{-\infty}^{\infty} \nu(v)|g(v)|\, dv,$$

we can pose the above problem as an initial-value problem for vector functions $f(t)$, $t \geq s$, on $L^1(\mathbb{R}, dv)$ for given initial vector $f_s \in L^1(\mathbb{R}, dv)$. In that case, $(Kg)(v) = \int_{-\infty}^{\infty} k(v, \hat{v})\nu(\hat{v})g(\hat{v})\, d\hat{v}$ is a positive isometry from $L^1(\mathbb{R}, \nu dv)$ into $L^1(\mathbb{R}, dv)$. The physics of the problem demands the existence of a unique solution $f(t)$, $t \geq s$, in $L^1(\mathbb{R}, dv)$ for given $f_s \in L^1(\mathbb{R}, dv)$ which is nonnegative if $f_s \geq 0$.

Equation (1.1) belongs to a class of so-called time dependent kinetic equations which have been studied using the method of characteristics or by the semigroup method [3, 8, 2]. Here we study a particular kinetic equation within the framework of evolution families [10].

Let us describe the contents of the various sections. In Section 2 we prove the existence of a unique solution within the context of bounded evolution families. This means that we prove the existence of a family of bounded linear operators S(t,s), $t \geq s$, on $L^1(\mathbb{R}, dv)$ such that

1. $(t, s) \mapsto S(t, s)g$ is a bounded strongly continuous function on $\{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geq t_2\}$ for every $g \in L^1(\mathbb{R}, dv)$,
2. for $t \geq r \geq s$ we have the product rule $S(t, r)S(r, s) = S(t, s)$, and
3. for $s \in \mathbb{R}$ we have $S(s, s) = I$, the identity operator.

The unique solution can then be written as $f(t) = S(t, s)f_s$. Adopting a procedure reminiscent of the one used in [7] (also [1, 4]), we construct $S(t, s)$ by iterating the Duhamel integral equation

$$S(t, s)g = S_0(t, s)g + \int_s^t S(t, \tau)KS_0(\tau, s)g\, d\tau,$$

where $\hat{f}(t) = S_0(t, s)f_s$, $t \geq s$, is the solution of equations (1.1)-(1.2) with $k(v, \hat{v}) \equiv 0$. The latter solution is easily obtained in closed form. It turns out that, under very weak hypotheses, the norm of the initial distribution $f_s \geq 0$ is preserved, i.e. $\|f(t)\|_1 \equiv \|f_s\|_1$, $t \geq s$, which corresponds physically to the preservation of the number of electrons when ionization and recombination are in dynamic equilibrium. We also give a separate uniqueness proof.

Let us define what we mean by a (mild) solution of equations (1.1)-(1.2). Suppose $a(t)$ is Lipschitz continuous and $q \in L^1(\mathbb{R}\times[s,T];dvdt)$ for every $T > s$. Then the initial-value problem

$$\frac{\partial f}{\partial t}(v,t) + a(t)\frac{\partial f}{\partial v}(v,t) + \nu(v)f(v,t) = q(v,t), \qquad t \geq s, \tag{1.3}$$

$$f(v,s) = f_s(v), \tag{1.4}$$

can be solved uniquely for $f \in L^1(\mathbb{R} \times [s,T];dvdt)$ by integration along characteristics. More precisely (cf. [3]), if we define $W_0(t,s)$ and $S_0(t,s)$ by

$$[W_0(t,s)g](v) = g\left(v - \int_s^t a(\sigma)d\sigma\right), \qquad v \in \mathbb{R}, \tag{1.5}$$

$$[S_0(t,s)g](v) = M_0(t,s;v)[W_0(t,s)g](v), \qquad v \in \mathbb{R}, \tag{1.6}$$

where

$$M_0(t,s;v) = \exp\left(-\int_s^t \nu\left(v - \int_\rho^t a(\sigma)\,d\sigma\right)d\rho\right),$$

then the (distributional) solution is given by

$$f(v,t) = [S_0(t,s)f_s](v) + \int_s^t [S_0(t,\tau)q(\tau)](v)\,d\tau; \tag{1.7}$$

moreover, if $q : [0,T] \to L^1(\mathbb{R},dv)$ is (strongly) continuous, then $f : [0,T] \to L^1(\mathbb{R},dv)$ is (strongly) continuous as well. Then by a mild solution of equations (1.1)-(1.2) on the interval $[s,T]$ we mean a strongly measurable vector function $f \in L^1(\mathbb{R}\times[s,T],\nu dvdt)$ such that (1.7) is satisfied for $q(t) = Kf(t)$. Note that under this definition $q \in L^1(\mathbb{R} \times [s,T],dvdt)$ so that $f \in L^1(\mathbb{R} \times [s,T],dvdt)$. In Section 2 we will, in fact, prove the existence of a mild solution of equations (1.1)-(1.2) under the condition that the quantity

$$M(T,s) = \sup_{-\infty<v<\infty} \int_s^T \nu\left(v + \int_s^\rho a(\sigma)\,d\sigma\right)d\rho \tag{1.8}$$

is finite. At the same time we will prove that $f : [s,T] \to L^1(\mathbb{R},dv)$ is (strongly) continuous, although $f : [s,T] \to L^1(\mathbb{R},\nu dv)$ need not be defined for every $t \in [s,T]$. The solution obtained will turn out to satisfy two forms of the Duhamel equation, namely (2.8) and (2.14). In Section 2 we will actually establish the solvability of (2.8) and (2.14) under much weaker conditions on $a(t)$, $\nu(v)$ and K, although (2.14) must be weakened to (2.9) if $M(t,s)$ is infinite.

The general idea of Section 2 will be to prove the existence of the evolution family $\{S(t,s)\}_{t\geq s}$ pertaining to equations (1.1)-(1.2) under hypotheses on $a(t)$, $\nu(v)$ and K sufficient to define this evolution family but far more general than what is required under practical circumstances.

In Section 4 we derive basically the same results in the Banach spaces $L^1_N(\mathbb{R})$, $N \geq 1$, of measurable functions $h : \mathbb{R} \to \mathbb{C}$ bounded with respect to the norm $\|g\|_{1,N} = \int_{-\infty}^{\infty} |g(v)|(1+v^2)^{N/2} dv$. Assuming that $f_s \in L^1_N(\mathbb{R})$, information on the large time behavior of $\|S(t,s)f_s\|_{1,N}$ provides information on the large time behavior of the N-th velocity moment of $f(t)$. For $N=1$, this will give us sufficient conditions in order that the average velocity remains bounded as $t \to +\infty$. The special case $\nu(v) \equiv \nu_0$ is worked out in detail.

2. The Evolution Families Involved

Suppose $a(t)$ is a real function in $L^{1,loc}(\mathbb{R}, dv)$, $\nu(v)$ is a nonnegative function in $L^{1,loc}(\mathbb{R}, dt)$ which is almost everywhere positive, and K is a positive linear operator from $L^1(\mathbb{R}, \nu dv)$ into $L^1(\mathbb{R}, dv)$ such that

$$\|Kg\|_1 = \|g\|_\nu, \qquad g \geq 0 \text{ in } L^1(\mathbb{R}, \nu dv). \tag{2.1}$$

Then, as one easily verifies, the operators $W_0(t,s)$ and $S_0(t,s)$, $t \geq s \geq 0$, defined by (1.5) and (1.6) form an evolution family on $L^1(\mathbb{R}, dv)$, i.e. [10] for $U = W_0$ or $U = S_0$ and $X = L^1(\mathbb{R}, dv)$ the following conditions are fulfilled:

1. The function $(t,s) \mapsto U(t,s)g$ is strongly continuous on $\{(t_1,t_2) \in \mathbb{R}^2 : t_1 \geq t_2\}$ for every $g \in X$;

2. For $t \geq r \geq s$ we have $U(t,r)U(r,s) = U(t,s)$;

3. For $s \in \mathbb{R}$ we have $U(s,s) = I$;

4. The operator norm $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for some $M, \omega \in \mathbb{R}$.

The evolution family is called bounded if the fourth condition is fulfilled for $\omega = 0$.

Proposition 2.1. *For $t \geq s$ we have*

$$\|S_0(t,s)g\|_1 + \int_s^t \|S_0(\tau,s)g\|_\nu \, d\tau = \|g\|_1, \quad g \geq 0 \text{ in } L^1(\mathbb{R}, dv). \tag{2.2}$$

Proof. Writing the second term on the left-hand side of (2.2) as a double integral over $(v,\tau) \in \mathbb{R} \times [s,t]$, transforming the v-variable according to $v \to v + \int_s^t a(\sigma)d\sigma$ and changing the order of integration we get for $g \geq 0$ in $L^1(\mathbb{R}, dv)$

$$\begin{aligned}
&\int_s^t \|S_0(\tau,s)g\|_\nu \, d\tau \\
&= \int_{-\infty}^{\infty} \left[\int_s^t \nu\left(v + \int_s^\tau a(\sigma)\, d\sigma\right) \exp\left(-\int_s^\tau \nu\left(v + \int_s^\rho a(\sigma)\, d\sigma\right) d\rho\right) d\tau \right] g(v) dv \\
&= \int_{-\infty}^{\infty} \left(1 - \exp\left(-\int_s^t \nu\left(v + \int_s^\rho a(\sigma)\, d\sigma\right) d\rho\right)\right) g(v) dv.
\end{aligned}$$

Writing the last line as the difference of two integrals and changing the variable of the second integral via $v \to v - \int_s^t a(\sigma)\, d\sigma$, we obtain (2.2). ☐

From (2.2) it is immediate that, for $t \geq s$, $S_0(t,s)$ is a positive contraction on $L^1(\mathbb{R}, dv)$.

For $n \in \mathbb{N}$, $t \geq s$ and $g \in L^1(\mathbb{R}, dv)$ we define recursively

$$S_n(t,s)g = \int_s^t S_{n-1}(t,\tau) K S_0(\tau,s) g \, d\tau. \tag{2.3}$$

Then an induction argument based on (2.1) and (2.2) shows that each of the operators $S_n(t,s)$ is well-defined and is, in fact, a positive contraction on $L^1(\mathbb{R}, dv)$. By a more involved induction argument one may prove the following

Proposition 2.2. *For $n \in \mathbb{N}$, $t \geq s$ and $g \geq 0$ in $L^1(\mathbb{R}, dv)$ we have*

$$S_n(t,s)g = \int_s^t S_0(t,\tau) K S_{n-1}(\tau,s) g \, d\tau, \tag{2.4}$$

$$\sum_{j=0}^{n} \|S_j(t,s)g\|_1 + \int_s^t \|S_n(\tau,s)g\|_\nu \, d\tau = \|g\|_1 . \tag{2.5}$$

Proof. For the sake of convenience, let us attach the subscript n to equations (2.3)-(2.5). Then $(2.4)_1$ is true by definition and $(2.5)_0$ is a restatement of (2.2). Now let us assume that $(2.4)_n$ and $(2.5)_n$ are true. Since $(2.3)_{n+1}$ is merely the definition of $S_{n+1}(t,s)$, we have by virtue of $(2.3)_{n+1}$ and $(2.4)_n$ and after changing the order of integration

$$S_{n+1}(t,s)g = \int_s^t S_0(t,\rho) K \int_s^\rho S_{n-1}(\rho,\tau) K S_0(\tau,s) g \, d\tau d\rho,$$

which, by $(2.3)_n$, coincides with the right-hand side of $(2.4)_{n+1}$ and hence proves $(2.4)_{n+1}$. Next, using $(2.4)_{n+1}$ and the additivity of the norm on $L^1(\mathbb{R}, \nu dv)$ and changing the order of integration we find

$$\int_s^t \|S_{n+1}(t,s)g\|_\nu \, d\tau = \int_s^t \int_\rho^t \|S_0(\tau,\rho)KS_n(\rho,s)g\|_\nu \, d\tau d\rho.$$

Applying (2.2) and (2.1) we get

$$\int_s^t \|S_{n+1}(t,s)g\|_\nu \, d\tau = \int_s^t \|S_n(t,s)g\|_\nu \, d\tau - \left\| \int_s^t S_0(t,\rho)KS_n(\rho,s)g \, d\rho \right\|_1,$$

which, with the help of $(2.5)_n$ and $(2.4)_{n+1}$, can be rewritten as the right-hand side of $(2.5)_{n+1}$. □

In order to introduce a third evolution family $S(t,s)$ as the sum of the operators $S_n(t,s)$, we need to prove the strong continuity of $S_n(t,s)$. To do so, we notice that the integrals appearing in equations (2.3) and (2.4) are absolutely convergent Bochner integrals [5] with values in $L^1(\mathbb{R}, dv)$ and derive the following lemma.

Lemma 2.3. *Suppose that $G(\tau)$ is Bochner integrable on $[s,t]$ with values in $L^1(\mathbb{R}, dv)$. Then for every $g \in L^1(\mathbb{R}, dv)$ we have*

$$\lim_{h \downarrow 0} \int_s^t \|[S_0(t+h,\tau) - S_0(t,\tau)]G(\tau)\|_1 \, d\tau = 0.$$

Proof. Let $G(\tau)$ be a measurable step function on $[s,t]$ with values in $L^1(\mathbb{R}, dv)$. Then there exists a partition of $[s,t]$ into the finitely many measurable subsets $E_1, \ldots, E_n$ such that $G(\tau) \equiv g_j$ for $\tau \in E_j$ $(j = 1, 2, \ldots, n)$. Then

$$\int_s^t \|[S_0(t+h,\tau) - S_0(t,\tau)]G(\tau)\|_1 d\tau \le \sum_{j=1}^n \int_s^t \|[S_0(t+h,\tau) - S_0(t,\tau)]g_j\|_1 d\tau,$$

which vanishes as $h \downarrow 0$, because $S_0(t,s)g_j$ is strongly continuous in the first time variable.

Next, consider the operator $\mathcal{L}_h$ defined by

$$\mathcal{L}_h H = \int_s^t [S_0(t+h,\tau) - S_0(t,\tau)] \, H(\tau) d\tau.$$

Then $\mathcal{L}_h$ is a bounded linear operator from $L^1(\mathbb{R} \times [s,t], dv d\tau)$ into $L^1(\mathbb{R}, dv)$ of norm ≤ 2, while $\|\mathcal{L}_h H\|_1$ vanishes as $h \downarrow 0$ if $H(\tau)$ is a measurable step function on $[s,t]$ with values in $L^1(\mathbb{R}, dv)$. Since the measurable step functions on $[s,t]$ with values in $L^1(\mathbb{R}, dv)$ form a dense linear subspace of $L^1(\mathbb{R}\times[s,t], dvd\tau)$, we see that $\|\mathcal{L}_h H\|_1$ vanishes as $h \downarrow 0$ for every $H(\tau)$ in this space. □

Proposition 2.4. *For all $n \in \mathbb{N}$ and $g \in L^1(\mathbb{R}, dv)$, $S_n(t,s)g$ is a strongly continuous function on $\{(t_1,t_2) \in \mathbb{R}^2 : t_1 \geq t_2\}$.*

Proof. Let $h > 0$. Then for $n \geq 0$, $t \geq s$ and $g \in L^1(\mathbb{R}, dv)$ we have

$$\begin{aligned}
&\|[S_{n+1}(t+h,s) - S_{n+1}(t,s)]g\|_1 \\
&\leq \int_t^{t+h} \|S_0(t+h,\tau)KS_n(\tau,s)g\|_1 \, d\tau \\
&+ \int_s^t \|[S_0(t+h,\tau) - S_0(t,\tau)]KS_n(\tau,s)g\|_1 \; d\tau \\
&\leq \int_t^{t+h} \|S_n(\tau,s)|g|\|_\nu \; d\tau \\
&+ \int_s^t \|[S_0(t+h,\tau) - S_0(t,\tau)]KS_n(\tau,s)g\|_1 \; d\tau \\
&\leq \sum_{j=0}^n \|[S_j(t+h,s) - S_j(t,s)]|g|\|_1 \\
&+ \int_s^t \|[S_0(t+h,\tau) - S_0(t,\tau)]KS_n(\tau,s)g\|_1 \; d\tau,
\end{aligned}$$

where we have used $(2.5)_n$ twice. The last term vanishes as a result of Lemma 2.3 and $(2.5)_n$. If we now, inductively, assume that

$$\|[S_j(t+h,s) - S_j(t,s)]g\|_1$$

vanishes as $h \downarrow 0$ for $j = 0, 1, \ldots, n$, then the leftmost side of the above string of inequalities also vanishes as $h \downarrow 0$. Consequently, the expression $\|[S_n(t+h,s) - S_n(t,s)]g\|_1$ vanishes as $h \downarrow 0$, for every $n \in \mathbb{N}$.

Next, for $g \geq 0$ in $L^1(\mathbb{R}, dv)$ we compute

$$\begin{aligned}
&\|[S_{n+1}(t,s+h) - S_{n+1}(t,s)]g\|_1 \\
&\leq \int_s^{s+h} \|S_n(t,\tau)KS_0(\tau,s)g\|_1 \, d\tau \\
&+ \int_{s+h}^t \|S_n(t,\tau)K[S_0(\tau,s+h) - S_0(\tau,s)]g\|_1 \; d\tau
\end{aligned}$$

$$\leq \int_s^{s+h} \|S_0(\tau,s)g\|_\nu \, d\tau + \int_{s+h}^t \|S_0(\tau,s+h)|[I - S_0(s+h,s)]g|\|_\nu \, d\tau$$
$$\leq \int_s^{s+h} \|S_0(\tau,s)g\|_\nu \, d\tau + \|[I - S_0(s+h,s)]g\|_1 ,$$

where we have used (2.2). We now apply (2.2) as well as the strong continuity properties of S_0 to prove that the above expression vanishes as $h \downarrow 0$. □

For $t \geq s$ and $g \geq 0$ in $L^1(\mathbb{R}, dv)$, $\left\{\sum_{j=0}^n S_j(t,s)g\right\}_{n=0}^\infty$ is an increasing sequence of nonnegative functions in $L^1(\mathbb{R}, dv)$ whose norms are bounded above by the norm of g, as one easily verifies from $(2.5)_n$. Hence, this sequence converges in the norm of $L^1(\mathbb{R}, dv)$. As a result, we may define a bounded positive operator $S(t,s)$ on $L^1(\mathbb{R}, dv)$ by the absolutely convergent series

$$S(t,s)g = \sum_{n=0}^\infty S_n(t,s)g. \tag{2.6}$$

By construction, $S(t,s)$ is a contraction on $L^1(\mathbb{R}, dv)$. Moreover, for $g \geq 0$ in $L^1(\mathbb{R}, dv)$ we have $\|S(t,s)g\|_1 = \|g\|_1$ if and only if

$$\lim_{n\to\infty} \int_s^t \|S_n(\tau,s)g\|_\nu \, d\tau = 0. \tag{2.7}$$

Proposition 2.5. *For $t \geq s$ we have the two Duhamel formulas*

$$S(t,s)g = S_0(t,s)g + \int_s^t S(t,\tau)KS_0(\tau,s)g \, d\tau, \tag{2.8}$$

$$< S(t,s)g, \psi >=< S_0(t,s)g, \psi > + \int_s^t < S_0(t,\tau)KS(\tau,s)g, \psi > \, d\tau, \tag{2.9}$$

where $g \in L^1(\mathbb{R}, dv)$ and $\psi \in L^\infty(\mathbb{R}, dv)$. The integral in (2.8) is an absolutely convergent Bochner integral with values in $L^1(\mathbb{R}, dv)$. The integral in (2.9) is absolutely convergent.

Proof. Let $g \geq 0$ in $L^1(\mathbb{R}, dv)$ and $\psi \geq 0$ in $L^\infty(\mathbb{R}, dv)$. Then $(2.3)_n$ and $(2.3)_{n+1}$ imply

$$\sum_{j=0}^{n+1} < S_j(t,s)g, \psi > =< S_0(t,s)g, \psi >$$

$$+\int_s^t \left\langle \left(\sum_{j=0}^{n} S_j(t,\tau) \right) K S_0(\tau,s) g, \psi \right\rangle d\tau,$$

while $(2.4)_n$ and $(2.4)_{n+1}$ imply

$$\sum_{j=0}^{n+1} < S_j(t,s)g,\psi > = < S_0(t,s)g,\psi >$$
$$+\int_s^t \left\langle S_0(t,\tau) K \left(\sum_{j=0}^{n} S_j(\tau,s) \right) g, \psi \right\rangle d\tau.$$

Straight applications of the monotone convergence theorem yield the "weak" version of (2.8) as well as (2.9). In order to prove (2.8) itself, it suffices to show that the integral in (2.8) is an absolutely convergent Bochner integral in $L^1(\mathbb{R}, dv)$, which is clear from (2.2) and the contractivity of $S(t,s)$. □

If $\nu(v)$ is bounded, then both (2.8) and (2.9) may be replaced by corresponding Duhamel formulas where the integrals are Bochner integrals which are absolutely convergent in the operator norm on $L^1(\mathbb{R}, dv)$. On the other hand, if, for $t \geq s$, the quantity $M(t,s)$ defined by (1.7) is finite, then we have for $g \geq 0$ in $L^1(\mathbb{R}, dv)$

$$\begin{aligned} &\|S_0(t,s)g\|_1 \geq e^{-M(t,s)} \|g\|_1 , \\ &\int_s^t \|S_0(\tau,s)g\|_\nu \, d\tau \leq \left(1 - e^{-M(t,s)}\right) \|g\|_1 . \end{aligned} \tag{2.10}$$

The evolution families W_0 and S_0 can then be extended to arbitrary $t, s \in \mathbb{R}$ by extending (1.5) and (1.6) to $t \geq s$ and defining the inverses of the operators in (1.5) and (1.6) by

$$[W_0(s,t)g](v) = g\left(v + \int_s^t a(\sigma)\, d\sigma \right), \qquad v \in \mathbb{R}, \tag{2.11}$$
$$[S_0(s,t)g](v) = M_0(s,t;v)[W_0(s,t)g](v), \qquad v \in \mathbb{R},$$

where

$$M_0(s,t;v) = \exp\left(+ \int_s^t \nu\left(v + \int_s^\rho a(\sigma)\, d\sigma \right) d\rho \right) \tag{2.12}$$

and $t \geq s$. Moreover, we have for $g \geq 0$ in $L^1(\mathbb{R}, dv)$

$$\int_s^t \|S(\tau,s)g\|_\nu \, d\tau \leq \int_s^t \|S_0(\tau,s)g\|_\nu \, d\tau$$

$$+\int_s^t \left(1 - e^{-M(t,\rho)}\right) \|S(\rho, s)g\|_\nu \, d\rho,$$

where $M(t, s)$ is defined by (1.8). Clearly, $M(t, \rho) \le M(t, s)$ for $s \le \rho \le t$. Then

$$\int_s^t \|S(\tau, s)g\|_\nu \, d\tau \le e^{M(t,s)} \int_s^t \|S_0(\tau, s)g\|_\nu \, d\tau \le \left(e^{M(t,s)} - 1\right) \|g\|_1 . \quad (2.13)$$

Hence, $f(t) = S(t, s)f_s$, $t \ge s$, is a mild solution of equations (1.1)-(1.2) on $[s, T]$ if $M(t, s)$ is finite. In that case, the second Duhamel formula (2.9) may be replaced by

$$S(t, s)g = S_0(t, s)g + \int_s^t S_0(t, \tau)KS(\tau, s)g \, d\tau, \quad (2.14)$$

where the integral is an absolutely convergent Bochner integral with values in $L^1(\mathbb{R}, dv)$. If $a(t) \equiv a > 0$ is constant, we have

$$M(t, s) = \sup_{-\infty<v<\infty} \frac{1}{a} \int_v^{v+a(t-s)} \nu(\hat{v}) \, d\hat{v},$$

and hence (2.14) is true in the strong sense if $\nu \in L^p(\mathbb{R}, dv)$ for some $1 \le p \le \infty$ (cf. [6] for $p = 1$).

Theorem 2.6. *The operators $\{S(t, s)\}_{t \ge s}$ form an evolution family consisting of positive contraction operators on $L^1(\mathbb{R}, dv)$. In particular, the following conditions are fulfilled:*

1. *The function $(t, s) \mapsto S(t, s)g$ is bounded and strongly continuous on $\{(t_1, t_2) \in \mathbb{R}^2 : t_1 \ge t_2\}$ for every $g \in L^1(\mathbb{R}, dv)$;*
2. *For $t \ge r \ge s$ we have $S(t, r)S(r, s) = S(t, s)$;*
3. *For $s \in \mathbb{R}$ we have $S(s, s) = I$, the identity operator.*

Proof. Condition 1 is just a restatement of Proposition 2.4 and Condition 3 is obvious. In order to prove Condition 2, we first assume that $\nu(v)$ is bounded and hence that K is a bounded operator on $L^1(\mathbb{R}, dv)$. In that case, the Duhamel formulas (2.8) and (2.14) are valid while the integrals appearing in them are Bochner integrals absolutely convergent in the operator norm on $L^1(\mathbb{R}, dv)$. Condition 2 can then be verified in a completely algebraic fashion. Indeed, for $t \ge r \ge s$ we compute:

$$S(t, r)S(r, s) = \left[S_0(t, r) + \int_r^t S(t, \tau)KS_0(\tau, r) \, d\tau\right]$$

$$
\begin{aligned}
&\times \left[S_0(r,s) + \int_s^r S_0(r,\sigma)KS(\sigma,s)\,d\sigma\right] \\
&= S_0(t,s) + \int_s^r S_0(t,\sigma)KS(\sigma,s)\,d\sigma \\
&+ \int_r^t S(t,\tau)K\left[S_0(\tau,s) + \int_s^r S_0(\tau,\sigma)KS(\sigma,s)\,d\sigma\right]d\tau,
\end{aligned}
$$

where we have applied the product rule for S_0. Now write $\int_s^r = \int_s^t - \int_r^t$ in the second term and $\int_s^r = \int_s^\tau - \int_r^\tau$ in the interior integral of the third term on the right-hand side and apply the Duhamel formulas on $[s,t]$ and $[s,\tau]$, respectively. We get

$$
\begin{aligned}
S(t,r)S(r,s) &= S(t,s) - \int_r^t S_0(t,\sigma)KS(\sigma,s)\,d\sigma \\
&+ \int_r^t S(t,\tau)KS(\tau,s)\,d\tau - \int_r^t\int_\sigma^t S(t,\tau)KS_0(\tau,\sigma)KS(\sigma,s)\,d\tau ds.
\end{aligned}
$$

If we now change the integration variable in the third term from τ to s and change the order of integration in the fourth term, we obtain

$$
\begin{aligned}
S(t,r)S(r,s) &= S(t,s) \\
&+ \int_r^t \left[-S_0(t,\sigma) + S(t,\sigma) - \int_\sigma^t S(t,\tau)KS_0(\tau,\sigma)\,d\tau\right] KS(\sigma,s)\,d\sigma.
\end{aligned}
$$

Since the expression between square brackets vanishes, Condition 2 is clear if $\nu(v)$ is bounded.

To deal with arbitrary $\nu(v)$, we put $\nu_m(v) = \max\{\nu(v), m\}$ and $K_m g = K\left(\frac{\nu_m}{\nu}g\right)$ and define the operators $S_0^{[m]}(t,s)$, $S_n^{[m]}(t,s)$ and $S^{[m]}(t,s)$ by

$$
\begin{aligned}
&S_0^{[m]}(t,s) = S_0(t,s), \qquad S_n^{[m]}(t,s) = \int_s^t S_{n-1}^{[m]}(t,\tau)K_m S_0(\tau,s)\,d\tau, \\
&S^{[m]}(t,s) = \sum_{n=0}^{\infty} S_n^{[m]}(t,s),
\end{aligned}
$$

where $n \in \mathbb{N}$ and $t \geq s$. Then, by the boundedness of $\nu_m(v)$, the previous techniques can be applied to prove that the operators $\left\{S^{[m]}(t,s)\right\}_{t\geq s}$ form an evolution family on $L^1(\mathbb{R}, dv)$ consisting of positive contraction operators and, more precisely, that the following conditions are fulfilled:

1. $S^{[m]}(t,s)g$ is a bounded strongly continuous function on $\{(t_1,t_2) \in \mathbb{R}^2 : t_1 \geq t_2\}$ for every $g \in L^1(\mathbb{R}, dv)$;

2. For $t \geq r \geq s$ we have $S^{[m]}(t,r)S^{[m]}(r,s) = S^{[m]}(t,s)$;

3. For $s \in \mathbb{R}$ we have $S^{[m]}(s,s) = I$, the identity operator.

Moreover, for every $g \geq 0$ in $L^1(\mathbb{R}, dv)$ we have the following monotonicity properties:

1. $0 \leq S_n^{[m]}(t,s)g \leq S_n^{[m+1]}(t,s)g \leq S_n(t,s)g$;

2. $0 \leq S^{[m]}(t,s)g \leq S^{[m+1]}(t,s)g \leq S(t,s)g$,

where the right-hand side belongs to $L^1(\mathbb{R}, dv)$. Thus there exist bounded positive operators $\widetilde{S}_n(t,s)$ and $\widetilde{S}(t,s)$ which arise as the strong limits of $S_n^{[m]}(t,s)$ and $S^{[m]}(t,s)$, respectively, as $m \to \infty$. Straightforward induction with respect to n and summation over n then imply that $\widetilde{S}_n(t,s) = S_n(t,s)$ and $\widetilde{S}(t,s) = S(t,s)$. Passing to the limit as $m \to \infty$ in the product rule for $S^{[m]}$ yields Condition 2 for arbitrary $\nu(v)$. □

Suppose there exists a strongly measurable function $f : [s,t] \to L^1(\mathbb{R}, dv)$ such that $\int_s^t \|f(\tau)\|_\nu \, d\tau$ is finite and

$$f(t) = S_0(t,s)g + \int_s^t S_0(t,\tau)Kf(\tau)\,d\tau \tag{2.15}$$

for some $g \in L^1(\mathbb{R}, dv)$. Then, iterating (2.15) n times we get [cf. (2.4)]

$$f(t) = \sum_{j=0}^{n} S_j(t,s)g + \int_s^t S_n(t,\tau)Kf(\tau)\,d\tau, \tag{2.16}$$

where the finiteness of $\int_s^t \|f(\tau)\|_\nu \, d\tau$ justifies the changes in the order of integration required to obtain the integral term in (2.16). Since

$$\begin{aligned}\sum_{n=0}^{\infty} \left\| \int_s^t S_n(t,\tau)Kf(\tau)\,d\tau \right\|_1 &\leq \int_s^t \sum_{n=0}^{\infty} \|S_n(t,\tau)K|f(\tau)|\|_1 \, d\tau \\ &\leq \int_s^t \|S(t,\tau)K|f(\tau)|\|_1 \, d\tau \leq \int_s^t \|f(\tau)\|_\nu \, d\tau < +\infty,\end{aligned}$$

the second term on the right-hand side of (2.16) vanishes in the norm of $L^1(\mathbb{R}, dv)$ as $n \to \infty$. As a result,

$$f(t) = \sum_{j=0}^{\infty} S_j(t,s)g = S(t,s)g,$$

so that the Duhamel equation (2.14) is satisfied. Consequently, if $M(t,s)$ is finite, then $f(\tau) = S(\tau,s)g$ is the only strongly measurable function $f : [s,t] \to L^1(\mathbb{R}, dv)$ such that $\int_s^t \|f(\tau)\|_\nu \, d\tau$ is finite and (2.15) holds true. Another corollary of the above reasoning is that $f(t) = S(t,s)f_s$ is the unique mild solution on $[s,T]$ of equations (1.1)-(1.2) if $a(t)$ is Lipschitz continuous on $[s,T]$ and $M(T,s)$ is finite.

Now suppose there exists a strongly measurable function $F : [s,t] \to \mathcal{L}(L^1(\mathbb{R}, dv))$, the Banach algebra of bounded operators on $L^1(\mathbb{R}, dv)$, such that $\|F(\cdot)\|_{\mathcal{L}(L^1(\mathbb{R},dv))}$ is essentially bounded on $[s,t]$ and for all $g \in L^1(\mathbb{R}, dv)$

$$F(\sigma)g = S_0(t,\sigma)g + \int_\sigma^t F(\tau)KS_0(\tau,\sigma)g \, d\tau, \qquad s \le \sigma \le t. \tag{2.17}$$

Then, iterating (2.17) n times we obtain

$$F(\sigma)g = \sum_{j=0}^{n} S_j(t,\sigma)g + \int_\sigma^t F(\tau)KS_n(\tau,\sigma)g \, d\tau, \tag{2.18}$$

where the essential boundedness of $F(\tau)$ justifies the changes in the order of integration required to find the integral term in (2.18). Let us denote the essential supremum of $\|F(\cdot)\|_{\mathcal{L}(L^1(\mathbb{R},dv))}$ by γ. Because [cf. (2.5)]

$$\begin{aligned} &\left\| \int_\sigma^t F(\tau)KS_n(\tau,\sigma)g \, d\tau \right\|_1 \\ &\le \gamma \int_\sigma^t \|S_n(\tau,\sigma)|g|\|_\nu \, d\tau = \gamma \left\{ \|g\|_1 - \sum_{j=0}^{n} \|S_j(t,\sigma)|g|\|_1 \right\}, \end{aligned}$$

the second term on the right-hand side of (2.18) vanishes in the norm of $L^1(\mathbb{R}, dv)$ as $n \to \infty$ and uniformly in σ on $[s,t]$, provided $\|S(t,\sigma)h\|_1 = \|h\|_1$ for $s \le \sigma \le t$ and all $h \ge 0$ in $L^1(\mathbb{R}, dv)$. In that case, $F(\sigma) = S(t,\sigma)$ for $s \le \sigma \le t$. Thus, under the above provision, $F(\sigma) = S(t,\sigma)$ is the only essentially bounded function $F : [s,t] \to \mathcal{L}(L^1(\mathbb{R}, dv))$ satisfying (2.17) for all $g \in L^1(\mathbb{R}, dv)$. Necessary and sufficient conditions as well as sufficient conditions for this provision to be true are given in Proposition 2.7 below. Two of the sufficient conditions are the finiteness of $M(t,s)$ and the boundedness of $\nu(v)$.

The physics of the problem suggests that the total number of electrons is independent of time. In mathematical terms this means that $\|S(t,s)g\|_1 = \|g\|_1$

for every $g \geq 0$ in $L^1(\mathbb{R}, dv)$. This property is indeed true under weak assumptions on $\nu(v)$ and K, as indicated by the following result. In the statement of this result, we need the operator $\mathcal{L}_{(t,s)}$ defined by

$$(\mathcal{L}_{(t,s)}F)(\rho) = \int_s^\rho KS_0(\rho, \tau)F(\tau)\, d\tau, \qquad s \leq \rho \leq t. \tag{2.19}$$

It is immediate that $\mathcal{L}_{(t,s)}$ is a positive contraction on $L^1(\mathbb{R}\times[s,t]; dvd\rho)$. Moreover, if 1 were to be an eigenvalue of $\mathcal{L}_{(t,s)}$, then there would exist a nontrivial function $G \geq 0$ in $L^1(\mathbb{R} \times [s,t]; dvd\rho)$ such that $\mathcal{L}_{(t,s)}G = G$. Then a simple calculation would give [cf. (2.2)]

$$\left\|\mathcal{L}_{(t,s)}G\right\|_1 = \int_s^t \left(\|G(\tau)\|_1 - \|S_0(t,\tau)G(\tau)\|_1\right) d\tau,$$

which would be strictly less than the norm of G. Hence, 1 cannot be an eigenvalue of $\mathcal{L}_{(t,s)}$.

Proposition 2.7. *Let $t \geq s$. The following three assertions are equivalent:*

1. $\|S(t,s)g\|_1 = \|g\|_1$ *for every $g \geq 0$ in $L^1(\mathbb{R}, dv)$;*
2. *For each $g \geq 0$ in $L^1(\mathbb{R}, dv)$, the integral $\int_s^t \|S_n(\tau,s)g\|_\nu\, d\tau$ vanishes as $n \to \infty$;*
3. 1 *does not belong to the residual spectrum of $\mathcal{L}_{(t,s)}$.*

In particular, these three assertions are true if either one of the following conditions is satisfied:

1. *$\nu(v)$ is bounded;*
2. *$\mathcal{L}_{(t,s)}$ is a weakly compact operator on $L^1(\mathbb{R} \times [s,t]; dvd\rho)$;*
3. *The quantity $M(t,s)$ defined by* (1.8) *is finite.*

Proof. The equivalence of assertions 1 and 2 is given by (2.7). To prove that these two assertions are equivalent to the third assertion, we observe that $\left\{\left\|[\mathcal{L}_{(t,s)}]^n F\right\|_1\right\}_{n=0}^\infty$ is a decreasing sequence of nonnegative numbers whenever $F \geq 0$ in $L^1(\mathbb{R} \times [s,t]; dvd\rho)$. The limit of this sequence can be represented in the form $< F, \Psi >= \int_{-\infty}^\infty \int_s^t F(v,\rho)\Psi(v,\rho)\, d\rho dv$ where $\Psi \geq 0$ in $L^\infty(\mathbb{R} \times [s,t]; dvd\rho)$. In terms of the adjoint $[\mathcal{L}_{(t,s)}]^*$ of $\mathcal{L}_{(t,s)}$ on $L^\infty(\mathbb{R} \times [s,t]; dvd\rho)$, we have $[\mathcal{L}_{(t,s)}]^*\Psi = \Psi$.

Now, note that $\int_s^t \|S_n(\tau,s)g\|_\nu \, d\tau = \left\|[\mathcal{L}_{(t,s)}]^n G\right\|_1$ with $G(\tau) \equiv g$, whenever $g \geq 0$ in $L^1(\mathbb{R}, dv)$. Hence, if the second assertion is not true, then the corresponding $G(\tau) \equiv g \geq 0$ has the property that $< G, \Psi >$ is strictly positive. As a consequence, Ψ is nontrivial and hence 1 is in the point spectrum of the adjoint $[\mathcal{L}_{(t,s)}]^*$. Because 1 cannot be in the point spectrum of $\mathcal{L}_{(t,s)}$, it must belong to its residual spectrum, thus contradicting the third assertion.

If 1 belongs to the residual spectrum of $\mathcal{L}_{(t,s)}$, then 1 also belongs to the point spectrum of the adjoint $[\mathcal{L}_{(t,s)}]^*$ and hence there is a nontrivial $\Psi \geq 0$ in $L^\infty(\mathbb{R} \times [s,t]; dvd\rho)$ such that $[\mathcal{L}_{(t,s)}]^*\Psi = \Psi$. It is possible to choose some $F \geq 0$ in $L^1(\mathbb{R} \times [s,t]; dvd\rho)$ with $< F, \Psi >$ strictly positive, and for this F one may, in fact, choose a measurable step function on $[s,t]$ with the finitely many "values" $g_1, \ldots, g_m$ in $L^1(\mathbb{R}, dv)$. Putting $g = g_1 + \cdots + g_m$ and replacing F by $F(\tau) \equiv g \geq 0$, the quantity $< F, \Psi >$ remains strictly positive, but it is now seen to arise also as the limit of $\int_s^t \|S_n(\tau,s)g\|_\nu \, d\tau$ as $n \to \infty$, which contradicts the second assertion. Consequently, the three assertions are equivalent, as claimed.

Let us now consider the three sufficient conditions for assertions 1-3. If $\nu(v)$ is bounded, then it is immediate from (2.19) that for all $F \in L^1(\mathbb{R} \times [s,t]; dvd\rho)$

$$\left\|[\mathcal{L}_{(t,s)}]^n F\right\|_1 \leq \frac{(t-s)^n \ \|\nu\|_\infty^n}{n!} \ \|F\|_1 , \tag{2.20}$$

so that in this case $\mathcal{L}_{(t,s)}$ is quasinilpotent and hence 1 does not belong to its spectrum. Next, if $\mathcal{L}_{(t,s)}$ is weakly compact, its square is compact and thus 1 cannot be in its residual spectrum. Finally, using (2.1), (1.8), (2.10) and (2.19) we have

$$\left\|\mathcal{L}_{(t,s)}F\right\|_1 \leq \int_s^t \left(1 - e^{-M(t,\tau)}\right) \|F(\tau)\|_1 \, d\tau \leq (1 - \exp\{-M(t,s)\}) \ \|F\|_1 \tag{2.21}$$

if $M(t,s)$ is finite. Hence 1 cannot belong to the spectrum of $\mathcal{L}_{(t,s)}$. □

If $a(t) \equiv a > 0$ is constant, then $S(t,s) = S(t-s)$. In this case the sufficient condition 3 is satisfied if $\nu \in L^p(\mathbb{R}, dv)$ for some $1 \leq p \leq \infty$, in accordance with [7].

3. Using the Evolution Semigroup

If $\{U(t,s)\}_{t \geq s}$ is an evolution family on a Banach space X, then $\{e^{t\Gamma_U}\}_{t \geq 0}$ defined by

$$\left(e^{t\Gamma_U} f\right)(\tau) = U(\tau, \tau - t) f(\tau - t), \qquad \tau \in \mathbb{R}, \tag{3.1}$$

is a strongly continuous semigroup on the Banach space $L^p(\mathbb{R};X)$ of strongly measurable functions $f:\mathbb{R}\to X$ bounded with respect to the norm $\|f\|_p = \left[\int_{-\infty}^{\infty}\|f(t)\|^p\,dt\right]^{1/p}$, $1\le p<+\infty$. On the other hand, if $\{U(t,s)\}_{t\ge s}$ is a strongly measurable family of bounded linear operators on X such that (3.1) defines a strongly continuous semigroup on X, then the algebraic properties $U(t,r)U(r,s)=U(t,s)$ for $t\ge r\ge s$ and $U(s,s)=I$ for $s\in\mathbb{R}$ are satisfied but $U(t,s)$ need not be strongly continuous in (t,s). However, $\{U(t,s)\}_{t\ge s}$ is an evolution family on X if and only if $\{e^{t\Gamma_U}\}_{t\ge 0}$ is a strongly continuous semigroup on the Banach space $C_0(\mathbb{R};X)$ of strongly continuous functions $f:\mathbb{R}\to X$ such that $\lim_{\tau\to\pm\infty}\|f(\tau)\|=0$. For evolution semigroups we refer to [9, 11].

In the present situation we may view $e^{t\Gamma_U}$, with $p=1$ and U any of W_0, S_0 and S, as an operator on $L^1(\mathbb{R}^2,dvdt)$. We then have

$$\left(e^{t\Gamma_{W_0}}f\right)(\tau,v) = f\left(v-\int_{\tau-t}^{\tau}a(\sigma)\,d\sigma,\tau-t\right);$$
$$\left(e^{t\Gamma_{S_0}}f\right)(\tau,v) = M_0(\tau,\tau-t;v)f\left(v-\int_{\tau-t}^{\tau}a(\sigma)\,d\sigma,\tau-t\right).$$

Then for $f\ge 0$ in $L^1(\mathbb{R}^2,dvdt)$ we get

$$\|e^{t\Gamma_{W_0}}f\|_1 = \|f\|_1;$$
$$\|e^{t\Gamma_{S_0}}f\|_1 = \|f\|_1 - \int_{-\infty}^{\infty}\int_{\tau}^{\tau+t}\|S_0(\sigma,\tau)f(\tau)\|_\nu\,d\sigma d\tau;$$
$$\|e^{t\Gamma_{S}}f\|_1 = \|f\|_1 - \lim_{n\to\infty}\int_{-\infty}^{\infty}\int_{\tau}^{\tau+t}\|S_n(\sigma,\tau)f(\tau)\|_\nu\,d\sigma d\tau.$$

The evolution semigroup $\{e^{t\Gamma_S}\}_{t\ge 0}$ has the property

$$\|e^{t\Gamma_S}f\|_1 = \|f\|_1,\qquad f\ge 0 \text{ in } L^1(\mathbb{R}^2,dvdt),\tag{3.2}$$

if and only if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\int_{\tau}^{\tau+t}\|S_n(\sigma,\tau)f(\tau)\|_\nu\,d\sigma d\tau = 0.\tag{3.3}$$

One may consider $f(v,\tau,t) = \left(e^{t\Gamma_S}f_0\right)(v,\tau)$ as the unique solution of the kinetic equation

$$\frac{\partial f}{\partial t}(v,\tau,t) + \frac{\partial f}{\partial \tau}(v,\tau,t) + a(\tau)\frac{\partial f}{\partial v}(v,\tau,t) + \nu(v)f(v,\tau,t)\tag{3.4}$$
$$= \int_{-\infty}^{\infty}k(v,\hat{v})\nu(\hat{v})f(\hat{v},\tau,t)\,d\hat{v},\tag{3.5}$$

with initial condition

$$f(v,\tau,0)=f_0(v,\tau), \tag{3.6}$$

where $\mathbb{R}^2$ is the phase space of velocity-time points (v,τ) and semigroup methods on the function space $L^1(\mathbb{R}^2, dvdt)$ are to be applied.

Equations (3.5)-(3.6) may be studied using the methods of [7] without relying on evolution families. The equivalence of (3.2) and (3.3) and Proposition 2.7 are then virtually immediate. Indeed, as in [7] one defines the evolution semigroup as follows:

$$e^{t\Gamma_S}f=\sum_{n=0}^{\infty}\left[e^{t\Gamma_S}\right]_n f, \qquad f\geq 0 \text{ in } L^1(\mathbb{R}^2, dvdt),$$

where one establishes inductively

$$\left(\left[e^{t\Gamma_S}\right]_0 f\right)(\tau)=\left(e^{t\Gamma_{S_0}}f\right)(\tau)=S_0(\tau,\tau-t)f(\tau-t);$$

$$\left(\left[e^{t\Gamma_S}\right]_n f\right)(\tau)=\int_0^t\left[e^{(t-\sigma)\Gamma_S}\right]_{n-1}K\left[e^{\sigma\Gamma_S}\right]_0 f\,d\sigma=S_n(\tau,\tau-t)f(\tau-t).$$

Here we have extended K to an operator from $L^1(\mathbb{R}^2,\nu dvdt)$ into the space $L^1(\mathbb{R}^2, dvdt)$. Then

$$\int_0^t\left\|\left[e^{\sigma\Gamma_S}\right]_n f\right\|_\nu d\sigma=\int_{-\infty}^{\infty}\int_0^t\|S_n(\tau,\tau-\sigma)f(\tau-\sigma)\|_\nu\, d\sigma d\tau$$

$$=\int_{-\infty}^{\infty}\int_0^t\|S_n(\tau+\sigma,\tau)f(\tau)\|_\nu\, d\sigma d\tau=\int_{-\infty}^{\infty}\int_\tau^{\tau+t}\|S_n(\sigma,\tau)f(\tau)\|_\nu\, d\sigma d\tau,$$

which implies the equivalence of (3.2) and (3.3) as a consequence of the corresponding result in [7].

4. Evolution Families in Weighted L^1-Spaces

In this section we will change the Banach space setting of the problem from $L^1(\mathbb{R}, dv)$ to $L^1_N(\mathbb{R})$ for some $N\geq 1$ where $L^1_N(\mathbb{R})$ is the Banach space of measurable functions $h:\mathbb{R}\to\mathbb{C}$ which are bounded with respect to the norm

$$\|h\|_{1,N}=\int_{-\infty}^{\infty}|h(v)|(1+v^2)^{N/2}\,dv. \tag{4.1}$$

We will study the three evolution families $\{W_0(t,s)\}_{t\geq s}$, $\{S_0(t,s)\}_{t\geq s}$, and $\{S(t,s)\}_{t\geq s}$ on the Banach space $L^1_N(\mathbb{R})$ which, for any $N\geq 1$, is continuously and densely imbedded in $L^1(\mathbb{R}, dv)$. We will prove that (1) $W_0(t,s)g$,

$S_0(t,s)g$ and $S(t,s)g$ belong to $L^1_N(\mathbb{R})$ if $g \in L^1_N(\mathbb{R})$, and (2) these vectors depend continuously on t and s in the norm of $L^1_N(\mathbb{R})$. Throughout this section we assume that

$$\kappa = \operatorname*{ess\,sup}_{\widehat{v}\in\mathbb{R}} \int_{-\infty}^{\infty} k(v,\hat{v})(1+v^2)^{N/2}\,dv$$

is finite. As a result, K defined by $(Kg)(v) = \int_{-\infty}^{\infty} k(v,\hat{v})\nu(\hat{v})g(\hat{v})\,d\hat{v}$ for some nonnegative measurable function $k(v,\hat{v})$ acts as a bounded operator from $L^1(\mathbb{R},\nu dv)$ into $L^1_N(\mathbb{R})$ and its norm is given by κ.

Proposition 4.1. *$\{W_0(t,s)\}_{t\geq s}$ is an evolution family on $L^1_N(\mathbb{R})$ which extends to all $t,s\in\mathbb{R}$. The norm of the operator $W_0(t,s)$ on $L^1_N(\mathbb{R})$ is given by $\Phi(|\int_s^t a(\sigma)\,d\sigma|)^N$ where*

$$\Phi(\alpha) = \frac{1}{2}\left(\alpha + (\alpha^2+4)^{1/2}\right), \qquad \alpha \geq 0. \tag{4.2}$$

Proof. From (1.5) we have for $g \geq 0$ in $L^1_N(\mathbb{R})$ after a simple change of variable

$$\|W_0(t,s)g\|_{1,N} = \int_{-\infty}^{\infty} \left(\frac{1+\left(v+\int_s^t a(\sigma)\,d\sigma\right)^2}{1+v^2}\right)^{N/2} \times g(v)(1+v^2)^{N/2}\,dv,$$

so that the operator norm of $W_0(t,s)$ on $L^1_N(\mathbb{R})$ is the N-th power of the maximum of the function

$$\varphi_\alpha(v) = \left(\frac{1+(v+\alpha)^2}{1+v^2}\right)^{1/2}, \qquad v\in\mathbb{R},$$

where $\alpha = \int_s^t a(\sigma)\,d\sigma$. This maximum equals $\Phi(|\alpha|)$. The same result can be derived from (2.11) if $t \leq s$. Applying the principle of dominated convergence, we see that the expression $\|[W_0(t_1,s_1) - W_0(t,s)]g\|_{1,N}$ vanishes as $(t_1,s_1) \to (t,s)$, which proves the strong continuity of W_0. □

Proposition 4.2. *$\{S_0(t,s)\}_{t\geq s}$ is an evolution family on $L^1_N(\mathbb{R})$ which extends to all $t,s\in\mathbb{R}$ if the quantity $M(t,s)$ defined by (1.8) is finite for $t \geq s$. The norm of the operator $S_0(t,s)$ on $L^1_N(\mathbb{R})$ is given by*

$$N(t,s) = \operatorname*{ess\,sup}_{v\in\mathbb{R}} \left(\frac{1+\left(v+\int_s^t a(\sigma)\,d\sigma\right)^2}{1+v^2}\right)^{N/2}$$

$$\times \exp\left(-\int_s^t \nu\left(v+\int_s^\rho a(\sigma)\,d\sigma\right)\,d\rho\right), \tag{4.3}$$

which is bounded above by $\Phi(|\int_s^t a(\sigma)\,d\sigma|)^N$ *with* $\Phi(\alpha)$ *as in* (4.2).

Proof. The proof of this proposition is the same as the proof of Proposition 4.1, except for using (1.6) and (2.12) instead of (1.5) and (2.11). □

If $M(t,s)$ defined in (1.8) is finite, the norm of $S_0(s,t)$ on $L^1_N(\mathbb{R})$ for $t \geq s$ is given by

$$N(s,t) = \operatorname*{ess\,sup}_{v\in\mathbb{R}} \left(\frac{1+\left(v-\int_s^t a(\sigma)\,d\sigma\right)^2}{1+v^2}\right)^{N/2}$$
$$\times \exp\left(+\int_s^t \nu\left(v-\int_\rho^t a(\sigma)\,d\sigma\right)\,d\rho\right),$$

which is bounded above by $\exp\{M(t,s)\}\,\Phi(|\int_s^t a(\sigma)\,d\sigma|)^N$. We also have for $t \geq s$

$$\exp[-M(t,s)]\Phi\left(|\int_s^t a(\sigma)\,d\sigma|\right)^N$$
$$\leq \|S_0(t,s)\|_{L^1_N(\mathbb{R})} \leq \Phi\left(|\int_s^t a(\sigma)\,d\sigma|\right)^N. \tag{4.4}$$

Moreover, for $\nu(v) \equiv \nu_0$ we have

$$N(t,s) = \Phi\left(\left|\int_s^t a(\sigma)\,d\sigma\right|\right)^N \times e^{-\nu_0(t-s)}.$$

The next two propositions involve the operators $S(t,s)$ instead of $S_0(t,s)$.

Theorem 4.3. *Suppose the quantity* $M(t,s)$ *defined by* (1.8) *is finite for* $t \geq s \geq 0$. *Then* $\{S(t,s)\}_{t\geq s}$ *is an evolution family on* $L^1_N(\mathbb{R})$.

Proof. Let $g \geq 0$ in $L^1_N(\mathbb{R})$. From (2.14) we obtain with the help of (2.13)

$$\begin{aligned}\|S(t,s)g\|_{1,N} &\leq \|S_0(t,s)g\|_{1,N} + \kappa\int_s^t \|S_0(t,\tau)\|_{L^1_N(\mathbb{R})}\,\|S(\tau,s)g\|_{1,N}\,d\tau \\ &\leq \sup_{s\leq\tau\leq t} N(t,\tau)\left(\|g\|_{1,N} + \kappa\left(\exp\{M(t,s)\}-1\right)\|g\|_1\right)\end{aligned}$$

$$\leq \sup_{s\leq\tau\leq t} N(t,\tau)\,(1+\kappa\,(\exp\{M(t,s)\}-1))\;\|g\|_{1,N}\,. \tag{4.5}$$

Put $F(t,s)g = S(t,s)g - S_0(t,s)g$. Then $F(t,s)g = \int_s^t S_0(t,\tau)KS(\tau,s)g\,d\tau$ while for $h\geq 0$ we have

$$\begin{aligned}[F(t+h,s)-F(t,s)]g &= \int_t^{t+h} S_0(t+h,\tau)KS(\tau,s)g\,d\tau \\ &\quad + \int_s^t [S_0(t+h,\tau)-S_0(t,\tau)]KS(\tau,s)g\,d\tau.\end{aligned}$$

The first term on the right-hand side vanishes in the norm of $L^1_N(\mathbb{R})$ as $h\downarrow 0$, as a result of (2.13) and the boundedness of K as an operator from $L^1(\mathbb{R},\nu dv)$ into $L^1_N(\mathbb{R})$. The second term vanishes as a consequence of Proposition 2.3 phrased in the norm of $L^1_N(\mathbb{R})$. Also, for $h\geq 0$ we have from (2.8)

$$\begin{aligned}&[F(t,s+h)-F(t,s)]g \\ &= \int_{s+h}^t S(t,\tau)K[S_0(\tau,s+h)-S_0(\tau,s)]g\,d\tau - \int_s^{s+h} S(t,\tau)KS_0(\tau,s)g\,d\tau.\end{aligned}$$

In the norm of $L^1_N(\mathbb{R})$, the second term on the right-hand side vanishes as $h\downarrow 0$, as a result of (4.5) and (2.2). In the norm of $L^1_N(\mathbb{R})$, the first term has as an upper bound some multiple of the expression

$$\int_{s+h}^t \|S_0(\tau,s+h)|[I-S_0(s+h,s)]g|\|_\nu\,d\tau \leq \|[I-S_0(s+h,s)]g\|_1\,,$$

which vanishes as $h\downarrow 0$. Hence, $S(t,s)g$ depends continuously on t and s in the norm of $L^1_N(\mathbb{R})$. □

Theorem 4.4. *Let γ be a constant such that $\nu(v)\leq\gamma(1+v^2)^{N/2}$. Then $\{S(t,s)\}_{t\geq s}$ is an evolution family on $L^1_N(\mathbb{R})$.*

Proof. First, note that

$$\|Kg\|_{1,N}\leq\kappa\,\|g\|_\nu\leq\kappa\gamma\,\|g\|_{1,N}\,,\qquad g\geq 0 \text{ in } L^1_N(\mathbb{R}).$$

Then K is a bounded operator on $L^1_N(\mathbb{R})$ and we can apply a standard perturbation result [10]. □

If (1) $N(t,s) \leq c_0 e^{-\lambda(t-s)}$, (2) $M(t,s)$ is finite, and (3) K is a bounded operator from $L^1(\mathbb{R}, dv)$ into $L^1_N(\mathbb{R})$ of norm κ_1, then the norm of $S(t,s)$ as an operator acting on $L^1_N(\mathbb{R})$ is bounded above by

$$\begin{aligned} & c_0 \left(e^{-\lambda(t-s)} + \kappa_1 \int_s^t e^{-\lambda(t-\tau)}\, d\tau \right) = c_0 \left(e^{-\lambda(t-s)} + \frac{\kappa_1}{\lambda} \left\{ 1 - e^{-\lambda(t-s)} \right\} \right) \\ & \leq c_0 \max \left(1, \frac{\kappa_1}{\lambda} \right). \end{aligned}$$

This situation occurs if (1) $\nu(v) \equiv \nu_0$, (2) $|\int_s^t a(\sigma)\, d\sigma| \leq C$ for $t \geq s$, and (3) K is a bounded linear operator from $L^1(\mathbb{R}, \nu dv)$ into $L^1_N(\mathbb{R})$. In that case, we get from (4.5) (with $c_0 = \Phi(C)^N$, $\kappa_1 = \kappa\nu_0$ and $\lambda = \nu_0$) the upper bound $\kappa\Phi(C)^N$.

References

[1] L. Arlotti, G. Frosali, Runaway electrons in a Boltzmann-like transport equation, *Math. Meth. Appl. Sci.*, **2** (2992), 203-221.

[2] J. Banasiak, L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer, New York (2006).

[3] R. Beals, V. Protopopescu, Abstract time-dependent transport equations, *J. Math. Anal. Appl.*, **121** (1987), 370-405.

[4] G. Busoni, G. Frosali, Asymptotic behaviour for a charged particle transport problem with time-varying acceleration field, *Transp. Theory Stat. Phys.*, **21** (2992), 713-732.

[5] J. Diestel, J.J. Uhl Jr., *Vector Measures*, Math. Surveys, **15**, Amer. Math. Soc., Providence (1977).

[6] G. Frosali, C.V.M. van der Mee, Scattering theory in the linear transport theory of particle swarms, *J. Stat. Phys.*, **56** (1989), 139-148.

[7] G. Frosali, C.V.M. van der Mee, S.L. Paveri-Fontana, Conditions for runaway phenomena in the kinetic theory of particle swarms, *J. Math. Phys.*, **30** (1989), 1177-1186.

[8] W. Greenberg, C. van der Mee, V. Protopopescu, *Boundary value problems in abstract kinetic theory*, Birkhäuser, Basel, **23** (1987).

[9] R. Nagel, Semigroup methods for non-autonomous Cauchy problems, In: *Tübinger Berichte zur Funktionalanalysis*, Heft 2, Jahrgang 92/93, Tübingen (1993), 103-118.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., **44**, Springer, New York (1983).

[11] R.T. Rau, Hyperbolic evolution groups and dichotomic evolution families, *J. Dynam. Differential Equations*, **6** (1994), 335-350.