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ALBEDO OPERATORS AND H-EQUATIONS FOR GENERALIZED KINETIC MODELS

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ABSTRACT

For abstract kinetic equations on a half-line the albedo operator which gives the initial value as a function of the half-range boundary data, is written in terms of operator-valued generalizations of Chandrasekhar's H-functions, for which coupled non-linear equations are derived. Applications are given to transfer of polarized light, neutron transport and strong evaporation of liquids.

1. INTRODUCTION

In recent years an intensive study has been made of the (unique) solvability and functional-analytic aspects of the abstract boundary value problem

$$(Tf)'(x) = -Af(x) \quad (0<x<\infty) \tag{1.1}$$

$$Q_+ f(0) = f_+ \tag{1.2}$$

$$||f(x)||_H = O(1) \quad (x\to\infty), \tag{1.3}$$

where T is an injective self-adjoint and A a positive self-adjoint operator on the Hilbert space H and Q_+ is the orthogonal projection of H onto the maximal positive T-invariant subspace.[1-4] The solution is usually written in the semigroup form

$$f(x) = e^{-xT^{-1}A} E f_+ \quad (0\leq x<\infty),$$

where E is the Larsen-Habetler[5] albedo operator. In spite of the many applications of Eqs. (1.1) to (1.3) to one-speed and multi-

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group neutron transport,[6,7] radiative transfer without polarization[8-10] and with polarization,[8,11,12] rarefied gas dynamics,[13] phonon transport[14] and evaporating liquids,[15] the functional-analytic theory of these equations so far has fallen short of producing a general procedure of finding explicit expressions for the albedo operator. However, once the albedo operator is calculated Case's method of eigenfunction expansion[6,4] can be applied to find

$$f(x) = \int_0^\infty e^{-\lambda x} \sum_\alpha A(\alpha,\lambda)\phi_{\alpha,\lambda} d\rho(\lambda) \quad (0<x<\infty) ,$$

where $\rho(\cdot)$ is the spectral measure of $T^{-1}A$, $\phi_{\alpha,\lambda}$ are (singular or regular) eigenfunctions of $T^{-1}A$ and $A(\alpha,\lambda)$ are expansion coefficients, which are given by

$$A(\alpha,\lambda) = \frac{1}{\rho(\lambda)} (AEf_+,\phi_{\alpha,\lambda})_H .$$

The main purpose of this article is to derive, for (non-self-adjoint) operators A which are compact perturbations of the identity, an expression for E in terms of operator-valued generalizations of Chandrasekhar's H-function,[8] for which non-linear integral equations are found. We exploit the equivalence[3] of Eqs. (1.1) to (1.3) to the equation

$$f(x) - \int_0^\infty H(x-y)Bf(y)dy = \omega(x) \quad (0<x<\infty) , \tag{1.4}$$

where $B = I - A$ is compact, $\omega(x) = e^{-xT^{-1}}f_+$,

$$H(t) = \begin{cases} +T^{-1}e^{-tT^{-1}}Q_+, & t > 0 ; \\ -T^{-1}e^{-tT^{-1}}Q_-, & t < 0 , \end{cases} \tag{1.5}$$

and $Q_- = I - Q_+$. In many applications B is an operator of finite rank. In order to reduce Eqs. (1.1) to (1.3) to a problem, which is finite-dimensional whenever B has finite rank, we choose a closed subspace $\mathbb{B} \supseteq \operatorname{Ran} B^*$. If j is the natural imbedding of $\mathbb{B}$ into H and π the orthogonal projection of H onto $\mathbb{B}$, then

$$Bj\pi = B \tag{1.6}$$

and Eq. (1.4) can be reduced to the Wiener-Hopf operator integral equation

$$g(x) - \int_0^\infty \pi H(x-y)Bjg(y)dy = \pi\omega(x) \quad (0<x<\infty), \tag{1.7}$$

where $g(x) = \pi f(x)$. If the solution g(x) of (1.7) has been computed, then

$$f(x) = \omega(x) + \int_0^\infty H(x-y)Bjg(y)dy \tag{1.8}$$

will be the solution of (1.4). Now, using the resolution of the identity σ of the self-adjoint operator T, the "symbol" of Eq. (1.8) has the form

$$\Lambda(z) = I - \int_{-\infty}^{\infty} e^{t/z}\pi H(t)Bjdt = I - z\int_{-\infty}^{\infty} \frac{\pi\sigma(dt)Bj}{z-t} . \tag{1.9}$$

This generalization of the dispersion function in neutron transport theory will be the basis on which generalized H-functions will be constructed.

In the so-called *regular case* where A is invertible and $A^{-1}T$ does not have imaginary eigenvalues, Eqs. (1.4) and (1.7) may be solved by Wiener-Hopf factorization.[16] If Eqs. (1.1) to (1.3) are uniquely solvable, there exists the canonical Wiener-Hopf factorization

$$\Lambda(z)^{-1} = H_\ell^+(-z)H_r^+(z) , \quad \operatorname{Re} z = 0 , \tag{1.10}$$

where the factors H_ℓ^+ and H_r^+ are analytic and invertible in the closed right half-plane. The unique solution can be written as

$$g(x) = \pi\omega(x) + \int_0^\infty \gamma(x,y)\pi\omega(y)dy, \quad 0\leqslant x<\infty , \tag{1.11}$$

while the double Laplace transform of $\gamma(x,y)$ is given by[17,18]

$$\int_0^\infty dy \int_0^\infty dz\, e^{y/\mu}e^{-z/\nu}\{\delta(y-z)+\gamma(y,z)\} = \frac{\mu\nu}{\mu-\nu} H_\ell^+(-\mu)H_r^+(\nu). \tag{1.12}$$

Through the relation $Ef_+ = f(0)$ and Eqs. (1.8), (1.11) and (1.12) we express the albedo operator in the functions H_ℓ^+ and H_r^+, and by integrating (1.12) we prove these functions to satisfy a coupled system of non-linear integral equations, which generalize Chandrasekhar's H-equation. For specific transport problems a similar approach was offered by Burniston et al.[19] (for two-group neutron transport) and subsequently by Kelley[20] and Mullikin.[21]

The regular case has applications to neutron transport, radiative transfer and phonon transport in non-conservative media. For conservative media and in rarefied gas dynamics the operator A is not invertible, though $T^{-1}A$ does not have non-zero imaginary eigenvalues. In these "*singular*" *cases* we employ a modified Wiener-Hopf method, where the factors are singular at infinity but analytic and invertible everywhere else in the closed right half-plane. Assuming the (unique or non-unique) solvability of Eqs. (1.1) to (1.3), we choose a corresponding (unique or non-unique) albedo operator E such that $Ef_+ = f(0)$ is the initial value of a solution for every f_+. Putting $E_+ = EQ_+$, we then obtain auxiliary representations of H_ℓ^+ and H_r^+ of the form

$$H_\ell^+(-z) = I - z\pi(T-zA)^{-1}E_+^\dagger Bj \quad ; \tag{1.13}$$

$$H_r^+(z) = I - z\pi(I-E_+)(T-zA)^{-1}Bj \ , \tag{1.14}$$

where $TE_+ = E_+^\dagger T$. (Related representations were found before for the regular case and $\mathbb{B} = H$.[22,23,3]) The inverses we represent by the formulas

$$H_\ell^+(-z)^{-1} = I - z\pi E_+(z-T)^{-1}Bj \ ; \tag{1.15}$$

$$H_r^+(z)^{-1} = I - z\pi(z-T)^{-1}(I-E_+^\dagger)Bj, \tag{1.16}$$

where (1.10) is satisfied. Using (1.13) to (1.16) we derive the previously obtained generalizations of Chandrasekhar's H-equations and a formula for the albedo operator. We have thus accomplished a generalization beyond the regular case. Now applications are given to polarized light transfer, neutron transport and evaporating liquids. Most well-known expressions for the albedo operator obtained by means of resolvent integration[5,24] or Wiener-Hopf equations[17,19-21] could be recovered this way.

Throughout this paper the operator $B = I - A$ is compact and satisfies the weak regularity condition[25]

$$\exists\alpha>0 : \mathrm{Ran}\, B \subseteq \mathrm{Ran}|T|^\alpha \cap D(|T|^{1+\alpha}), \tag{1.17}$$

which is fulfilled for neutron transport[3] with redistribution function $p \in L_{1+\varepsilon}[-1,1]$, radiative transfer[3] with phase function $p \in L_{1+\varepsilon}[-1,1]$ and various BGK models in rarefied gas dynamics. Integrals of vector- and operator-valued functions are to be understood as Bochner integrals.[26,27] We shall prove the equivalence of Wiener-Hopf equation (1.4) and boundary value problem (1.1) to (1.3) in the Appendix. We shall not investigate the uniqueness problem for solutions of our generalized H-equations.

2. ALBEDO OPERATOR: REGULAR CASE

Let A be an invertible and $B = I - A$ a compact operator satisfying (1.17). Suppose that $A^{-1}T$ does not have imaginary eigenvalues. Then the dispersion function $\Lambda(z)$ has a (possibly non-canonical) Wiener-Hopf factorization.[28]

Theorem 2.1. *Assume that* Eqs. (1.1) to (1.3) *are uniquely solvable for all* $f_+ \in Q_+[D(T)]$. *If* σ *denotes the resolution of the identity of* T *and* $f_+ \in \mathrm{Ran}\,\sigma([0,b])$ *for some finite* b, *then the initial value of the solution is given by*

$$Ef_+ = f_+ + \int_{-\infty\,(\mu)}^{0}\int_{0\,(\nu)}^{\infty} \frac{\nu}{\nu-\mu}\,\sigma(d\mu)BjH_\ell^+(-\mu)H_r^+(\nu)\pi\sigma(d\nu)f_+ \tag{2.1}$$

and E *has an extension to a bounded operator on* Ran Q_+.

In the isotropic case of one-speed neutron transport[1,5,6] one has

$$H = L_2[-1,1],\ (Th)(\mu) = \mu h(\mu),\ (Bh)(\mu) = \frac{1}{2}c\int_{-1}^{1} h(\mu')d\mu',$$

where $0<c<1$. Taking $\mathbb{B} = \mathrm{Ran}\,B^* = \mathrm{span}\,\{1\}$, $(\pi h)(\mu) = \frac{1}{2}\int_1^1 h(\mu')d\mu'$ and $\sigma(d\mu) = d\mu$, Eq. (2.1) reduces to the well-known expression

$$(Ef_+)(\mu) = \begin{cases} f_+(\mu) & , \quad \mu>0; \\ \frac{1}{2}c\int_0^1 \nu(\nu-\mu)^{-1}H(-\mu)H(\nu)f_+(\nu)d\nu, & \mu<0, \end{cases} \tag{2.2}$$

where $\Lambda(z) = 1 - \frac{1}{2}cz\int_{-1}^{1}(z-t)^{-1}dt$ and $\Lambda(z)^{-1} = H(-z)H(z)$.

To prove Theorem 2.1 we consider the equivalent Wiener-Hopf operator integral equation (1.4), reduced to (1.7) by $g(x) = \pi f(x)$. The unique solution of (1.4) can be written as (1.8), where $g(x)$ is given by (1.11). For the resolvent kernel $\gamma(x,y)$ we derive

Lemma 2.2.[17,18,20]

$$\int_0^\infty dy \int_0^\infty dz\ e^{y/\mu}e^{-z/\nu}\{\delta(y-z)+\gamma(y,z)\} = \frac{\mu\nu}{\mu-\nu}H_\ell^+(-\mu)H_r^+(\nu). \tag{2.3}$$

Proof. For the factors in the Wiener-Hopf factorization (1.10) one may find Bochner integrable[28] operator functions α and β such that

$$H_\ell^+(\omega) = I + \int_0^\infty e^{-t/\omega}\,\alpha(t)dt, H_r^+(\omega) = I + \int_0^\infty e^{-t/\omega}\beta(-t)dt,\ \mathrm{Re}\ \omega=0,$$

while the resolvent kernel is given by[16]

$$\gamma(y,z) = \begin{cases} \alpha(y-z) + \int_0^z \alpha(y-u)\beta(u-z)du\ , & 0<z<y<\infty; \quad (2.4a) \\ \beta(y-z) + \int_0^y \alpha(y-u)\beta(u-z)du\ , & 0<y<z<\infty. \quad (2.4b) \end{cases}$$

The left-hand side of (2.3) represents the Laplace transform of the solution of the convolution equation

$$g_\nu(x) - \int_0^\infty \pi\mathcal{H}(x-y)Bjg_\nu(y)dy = e^{-x/\nu}I\ ,\quad 0<x<\infty. \tag{2.5}$$

Extending the equation to the real line and taking Laplace transforms one obtains

$$\Lambda(\mu)\int_0^\infty e^{y/\mu}g_\nu(y)dy + \int_{-\infty}^0 e^{y/\mu}g_\nu(y)dy = \frac{\mu\nu}{\mu-\nu} I,\ \mathrm{Re}\ \mu = 0.$$

Substituting the factorization (1.10) one gets the Riemann-Hilbert problem

$$H_\ell^+(-\mu)^{-1}\int_0^\infty e^{y/\mu}g_\nu(y)dy+H_r^+(\mu)\int_{-\infty}^0 e^{y/\mu}g_\nu(y)dy = \frac{\mu\nu}{\mu-\nu} H_r^+(\mu),\mathrm{Re}\mu =0,$$

whose unique solution

$$\int_0^\infty e^{y/\mu}g_\nu(y)dy = \frac{\mu\nu}{\mu-\nu} H_\ell^+(-\mu)H_r^+(\nu)$$

coincides with the right-hand side of (2.3).

Using $Ef_+ = f(0)$ and Eqs. (1.8) and (1.11) we find

$$Ef_+ = f_+ + \int_0^\infty \mathcal{H}(-y)Bj[\pi e^{-yT^{-1}}f_+ + \int_0^\infty \gamma(y,z)\pi e^{-zT^{-1}}f_+dz]dy. \tag{2.6}$$

We apply the Spectral Theorem to rewrite $\mathcal{H}(-y)$, $e^{-yT^{-1}}f_+$ and $e^{-zT^{-1}}f_+$, and change the order of integration. We obtain

$$Ef_+ = f_+ + \int_{\substack{-\infty\\(\mu)}}^0 \int_{\substack{0\\(\nu)}}^\infty \frac{-1}{\mu}\,\sigma(d\mu)Bj[\int_0^\infty dy\int_0^\infty dze^{y/\mu}e^{-z/\nu}\{\delta(y-z)+\gamma(y,z)\}]\pi\sigma(dv)f_+,$$

where $f_+\ \varepsilon$ Ran $\sigma([o,b])$ for finite b. With the help of (2.3) we get (2.1).

Let us prove that E is bounded on Ran Q_+. Let $L_\infty(H)_0^\infty$ be the Banach space of strongly measurable functions $f:(0,\infty) \to H$, which are bounded with respect to the norm $||f||_\infty = \text{ess sup }\{||f(x)||_H\ |\ x\varepsilon(0,\infty)\}$. Then

$$(\mathcal{L}f)(x) = \int_0^\infty \mathcal{H}(x-y)Bf(y)dy \quad (0<x<\infty)$$

is a bounded operator on $L_\infty(H)_0^\infty$, $I - \mathcal{L}$ is invertible and

$$Ef_+ = [(I-\mathcal{L})^{-1}\omega_{f_+}](0)\ ,\ f_+\ \varepsilon\ Q_+[D(T)]\ ,$$

where $\omega_{f_+}(x)=e^{-xT^{-1}}f_+$. Hence, E is bounded on Ran Q_+ and the proof of Theorem 2.1 is complete.

Theorem 2.3. *The functions* H_ℓ^+ *and* H_r^+ *satisfy the coupled system of equations*

$$H_\ell^+(z)^{-1} = I - z\int_0^\infty (z+t)^{-1}\ H_r^+(t)\pi\sigma(dt)Bj\ ; \tag{2.7a}$$

$$H_r^+(z)^{-1} = I - z\int_0^\infty (z+t)^{-1}\ \pi\sigma(-dt)BjH_\ell^+(t). \tag{2.7b}$$

Proof. Premultiply (2.3) (with μ,ν replaced by $-t,z$) by by $t^{-1}\pi\sigma(-dt)Bj$ and integrate over $(0,\infty)$. Then

$$\int_0^\infty \frac{z}{z+t}\,\pi\sigma(-dt)BjH_\ell^+(t)H_r^+(z) = \int_0^\infty dx\int_0^\infty dye^{-x/t}e^{-y/z}\int_0^\infty \frac{1}{t}\,\pi\sigma(-dt)Bj\ \times$$
$$\times\{\delta(x-y)+\gamma(x,y)\}.$$

Performing the t-integration at the right one finds

$$\int_0^\infty \frac{z}{z+t}\,\pi\sigma(-dt)BjH_\ell^+(t)H_r^+(z) = \int_0^\infty \pi\mathcal{H}(-x)Bj[\int_0^\infty\{\delta(x-y)+\gamma(x,y)\}e^{-y/z}dy]dx.$$

In terms of the solution of Eq. (2.5) (with ν replaced by z) we get

$$\int_0^\infty \frac{z}{z+t}\,\pi\sigma(-dt)BjH_\ell^+(t)H_r^+(z) = \int_0^\infty \pi\mathcal{H}(-x)Bjg_z(x)dx = g_z(0)-I,$$

where (cf. (2.4b))

$$g_z(0) = \int_0^\infty e^{-x/z}\{\delta(x)+\gamma(0,x)\}dx = I + \int_0^\infty e^{-x/z}\beta(-x)dx = H_r^+(z).$$

Thus we have established (2.7b).

Postmultiply (2.3) (with μ,ν replaced by $-z,t$) by $t^{-1}\pi\sigma(dt)Bj$, integrate over $(0,\infty)$ and perform the t-integration. We find

$$\int_0^\infty \frac{z}{z+t} H_\ell^+(z)H_r^+(t)\pi\sigma(dt)Bj = \int_0^\infty[\int_0^\infty\{\delta(x-y)+\gamma(x,y)\}e^{-x/z}dx]\pi H(y)Bj.$$

The expression between square brackets is the solution of the equation

$$h_z(y)-\int_0^\infty h_z(x)\pi H(x-y)Bjdx = e^{-y/z}I,$$

whence

$$\int_0^\infty \frac{z}{z+t} H_\ell^+(z)H_r^+(t)\pi\sigma(dt)Bj = \int_0^\infty h_z(y)\pi H(y)Bjdy = h_z(0)-I.$$

However, (2.4a) implies

$$h_z(0) = \int_0^\infty e^{-x/z}\{\delta(x)+\gamma(x,0)\}dx = I + \int_0^\infty e^{-x/z}\alpha(x)dx = H_\ell^+(z),$$

which proves (2.6a).

The next theorem provides sufficient conditions for the unique solvability of Eqs. (1.1) to (1.3). Part (ii) is known for the case of multigroup neutron transport.[7]

Theorem 2.4 *The boundary value problem* (1.1) *to* (1.3) *and the equivalent Wiener-Hopf operator integral equation* (1.4) *are uniquely solvable in the following cases:*

(i) A *is a strictly positive self-adjoint operator;*

(ii) B *has norm less than unity;*

(iii) A *is invertible and the norm of* $A^{-1}-I$ *is less than unity.*

In these cases the H-*equations* (2.7a) *and* (2.7b) *have solutions.*

Proof. Part (i) was proved by van der Mee[3,23] for bounded T. If T is unbounded, we have to consult the work of Beals[2] to find the unique solvability of (a suitable version of) Eqs. (1.1) to (1.3) on the completion H_T of D(T) with respect to the inner product

$$(h,k)_T = (|T|h,k).$$

From this we derive that Eqs. (1.1) to (1.3) (as stated here) are uniquely solvable for a dense subspace of $f_+ \in Q_+[D(T)]$ in $Q_+[H_T]$, from which the result is immediate.

If Part (ii) is satisfied, then, because π and j have unit norm,

$$||\Lambda(z)-I|| \le ||z(z-T)^{-1}|| \; ||B|| \le ||B|| < 1, \quad \text{Re } z = 0.$$

Invoking a factorization result for Hilbert space operator functions close to the identity,[29] we get the existence of a canonical factorization of $\Lambda(z)$, from which Part (ii) follows. Part (iii) is proved analogously using

$$||\Lambda(z)(\pi Aj)^{-1}-I|| \le ||T(T-z)^{-1}|| \; ||A^{-1}-I|| < 1, \quad \text{Re } z = 0.$$

The statement about the H-functions is immediate from Theorem 2.3.

3. SPECTRAL ANALYSIS

In the next section we shall extend the results of the previous section to non-invertible A. The method of proof will basically consist of replacing A by a finite-dimensional regular perturbation A_β, for which Eqs. (1.1) to (1.3) (with A_β instead of A) are uniquely solvable. This reduction will require a more thorough knowledge of the spectral properties of $T^{-1}A$ and some additional properties of the albedo operators. This knowledge will be provided here.

Formula (2.1) suggests defining the operator

$$E^+f_+ = f_+ + \int\limits_{\substack{-\infty \\ (\mu)}}^{0} \int\limits_{\substack{0 \\ (\nu)}}^{\infty} \frac{\mu}{\nu-\mu}\sigma(d\mu)BjH_\ell^+(-\mu)H_r^+(\nu)\pi\sigma(d\nu)f_+. \qquad (3.1)$$

On its domain $\cup$ {Ran $\sigma([0,b])$ | b finite} one has

$$Ef_+ - E^+f_+ = \left[\int\limits_{\substack{-\infty \\ (\mu)}}^{0} \sigma(d\mu)BjH_\ell^+(-\mu)\pi\cdot\int_0^\infty jH_r^+(\nu)\pi\sigma(d\nu)\right]f_+,$$

so that $E-E^+$ extends to a bounded (and, because B is compact, even a compact) operator on Ran Q_+. Thus E and E^+ both extend to bounded operators on Ran Q_+. Further, $E_+ = EQ_+$ and $E_+^\dagger = E^\dagger Q_+$ are bounded projections on H with kernel Ran Q_-, which satisfy the intertwining property

$$E_+[D(T)] \subseteq D(T),\ TE_+f = E_+^\dagger Tf \text{ for } f\varepsilon D(T). \tag{3.2}$$

Because

$$E_+f = [(I-L)^{-1}\omega_{Q_+f}](0) = Q_+f + \int_0^\infty \Gamma(0,y)e^{-yT^{-1}}Q_+f\,dy,$$

where $\Gamma(0,y)$ is compact and Bochner integrable,[28] the operator E_+-Q_+ is compact. So both E_+ and $E_+^\dagger$ are compact perturbations of Q_+.

Lemma 3.1. *The range of* E_+ *(resp.* $E_+^\dagger$*) is invariant under* $A^{-1}T$ *(resp.* TA^{-1}*), while the restriction of* $A^{-1}T$ *(resp.* TA^{-1}*) to the range of* E_+ *(resp.* $E_+^\dagger$*) has its spectrum in the closed right half-plane.*

Proof. We consider the operator on Ran E_+ defined by

$$U_+(x)Ef_+ = f(x)\ ,\quad 0\leq x<\infty\ ,$$

where f is the unique bounded solution of Eq. (1.4) with right-hand side ω_{f_+}. Then $U_+(x)$ is well-defined, linear, bounded and depends on $x\varepsilon[0,\infty)$ continuously in the strong operator topology. The latter follows, because the solution f of Eq. (1.4) is bounded and continuous on $[0,\infty)$ (see Appendix). Furthermore, if $f_+\varepsilon Q_+[D(T)]$ and therefore $Ef_+\varepsilon\text{Ran } P_+ \cap D(T)$, then $f(x)\varepsilon D(T)\,(0\leq x<\infty)$ and (1.1) holds true. It is straightforward to derive that $U_+(\cdot)$ is a bounded strongly continuous semigroup on Ran E_+ with infinitesimal generator K_+, where

$$D(K_+) = \text{Ran } E_+ \cap D(T^{-1}A)\ ,\quad K_+g = -T^{-1}Ag.$$

Hence, Ran E_+ is $A^{-1}T$-invariant and the restriction of $A^{-1}T$ to Ran E_+ has its spectrum in the right half-plane. Because (3.2) holds true, a similar statement immediately follows for $E_+^\dagger$.

The existence and uniqueness of the solution of Eqs. (1.1) to (1.3) does not play a significant role in the above proof. If one replaces Ran E_+ by the closure of the subspace of the initial values f(0) where f_+ is ranging over $Q_+[D(T)]$, then this subspace is $A^{-1}T$-invariant and the above semigroup can be constructed. If we then replace Ran $E_+^\dagger$ by the closure of the subspace of vectors Tf(0), where f_+ ranges over $Q_+[D(T)]$, we get a TA^{-1}-invariant subspace.

Using Lemma 3.1 the unique solution of Eqs. (1.1) to (1.3) can be written in the familiar[1-4] semigroup form

$$f(x) = e^{-xT^{-1}A}Ef_+\ ,\quad 0<x<\infty\ , \tag{3.4}$$

which converges to zero as $x\to\infty$. Another familiar formula is obtained by writing E as the inverse of the Hangelbroek[1] operator

$$V = Q_+P_+ + Q_-P_-\ , \tag{3.3}$$

where P_+ and P_- are complementary projections commuting with $A^{-1}T$ and Ran P_+ = Ran E_+. In order to do this, we have to prove that Ran E_+ is the maximal $A^{-1}T$-invariant subspace of H such that the restriction of $A^{-1}T$ to it has its spectrum in the closed right half-plane. For strictly positive A this is a well-known fact[1-4].

Theorem 3.2. *There exists a decomposition*

$$Y_+ \oplus Y_- = H$$

of H *into closed* $A^{-1}T$*-invariant subspaces* Y_+ *and* Y_- *such that the restriction of* $A^{-1}T$ *to* $Y_\pm$ *has the property*

$$\sigma(A^{-1}T|_{Y_\pm}) = \{t\varepsilon\sigma(A^{-1}T)/\text{Re } t \gtrless 0\}\ .$$

If Eqs. (1.1) *to* (1.3) *are uniquely solvable, then* Y_+ = Ran E_+.

Proof. The full-line convolution equation

$$f(x) - \int_{-\infty}^{\infty} H(x-y)Bf(y)dy = \omega(x) \quad (-\infty<x<\infty) \tag{3.5}$$

is uniquely solvable. In fact, there exists a Bochner integrable operator function $\ell(\cdot)$ with compact operators as values such that

$$f(x) = \omega(x) + \int_{-\infty}^{\infty} \ell(x-y)\omega(y)dy \quad (-\infty<x<\infty) ,$$

while

$$I + \int_{-\infty}^{\infty} e^{t/\lambda}\ell(t)dt = \left[I-\int_{-\infty}^{\infty} e^{t/\lambda}H(t)Bdt\right]^{-1}, \quad \text{Re}\ \lambda = 0.^{30}$$

If ω is bounded measurable, so will f, and $\omega-f$ will be continuous on $\mathbb{R}$.

Secondly, using the Appendix one sees that for the right-hand sides

$$\omega(x) = \begin{cases} + e^{-xT^{-1}}Q_+h, & x > 0 \\ - e^{-xT^{-1}}Q_-h, & x < 0 \end{cases} \tag{3.6}$$

the bounded solutions of the Wiener-Hopf equation (3.5) satisfy the equations

$$(Tf)'(x) = -Af(x) \quad (0\neq x\in\mathbb{R}) \tag{3.7a}$$

$$f(0^+) - f(0^-) = \omega(0^+) - \omega(0^-) = h; \tag{3.7b}$$

This problem is, in fact, uniquely solvable. Applying Laplace transformation to (3.5) we find for the solution

$$\int_{-\infty}^{\infty} e^{t/\lambda}f(t)dt = \left[I-\int_{-\infty}^{\infty} e^{t/\lambda}H(t)Bdt\right]^{-1} \int_{-\infty}^{\infty} e^{t/\lambda}\omega(t)dt =$$
$$= (T-\lambda A)^{-1}(T-\lambda)\lambda T(\lambda-T)^{-1}h = \lambda A^{-1}T(\lambda-A^{-1}T)^{-1}h,$$

where Re $\lambda = 0$ and Eq. (1.9) is used. Formally we may write

$$f(x) = \begin{cases} + e^{-xT^{-1}A}P_+h, & x > 0 \\ - e^{-xT^{-1}A}P_-h, & x < 0, \end{cases} \tag{3.8}$$

for a suitable pair of complementary projections $P_\pm$ commuting with

$A^{-1}T$. Let us justify Eq. (3.8). Notice that f has a jump discontinuity at $x = 0$, and define

$$P_+h = +f(0^+) , \quad P_-h = -f(0^-) . \tag{3.9a}$$

We also define, for $z > 0$,

$$V_+(z)h = f(z) , \quad V_-(z)h = -f(-z) . \tag{3.9b}$$

Then $V_\pm(z)$ is bounded on $(0,\infty)$ and strongly continuous, while $V_\pm(0^+) = P_\pm$ in the strong operator topology. Surely, $P_\pm$ are bounded projections on H which add up to the identity, $V_\pm(z)$ leaves invariant the range of $P_\pm$, while the restriction of $V_\pm(z)$ to Ran $P_\pm$ induces a bounded C_0-semigroup on Ran $P_\pm$, whose infinitesimal generator $K_\pm$ is given by

$$D(K_\pm) = (\text{Ran}\ P_\pm)\cap D(T^{-1}A) , \quad K_\pm g = \mp T^{-1}AP_\pm .$$

Hence, Ran $P_\pm$ is $A^{-1}T$-invariant and the boundedness of the semigroup implies that the restriction of $A^{-1}T$ to Ran $P_\pm$ has its spectrum in the closed right/left half-plane. The intersection M of Ran P_+ and Ran P_- is a closed $A^{-1}T$-invariant subspace of H such that the restriction of $A^{-1}T$ to M has its spectrum on the imaginary line. As the resolvent set of $A^{-1}T$ does not have bounded connected components, we have $\sigma(A^{-1}T|_M) = \{0\}$. Thus if $M \neq \{0\}$, then $\lambda = 0$ is an isolated point of this spectrum and therefore an eigenvalue, which leads to a contradiction. Hence, $M = \{0\}$ and $P_\pm$ are complementary projections. We have justified Eq. (3.8).

Let $L_\infty(H)_a^b$ denote the Banach space of strongly measurable functions $f:(a,b) \to H$, which are bounded with respect to the norm

$$||f||_\infty = \text{ess sup}\{||f(x)||_H | a<x<b\} .$$

Introduce the operators L, L_+ and L_- on $L_\infty(H)_{-\infty}^{\infty}$, $L_\infty(H)_0^{\infty}$ and $L_\infty(H)_{-\infty}^{0}$, respectively, defined by

$$(\mathcal{L}f)(x) = \int_{\infty}^{\infty} \mathcal{H}(x-y)Bf(y)dy, \quad (\mathcal{L}_{\pm}f)(x) = \pm \int_{0}^{\pm\infty} \mathcal{H}(x-y)Bf(y)dy.$$

Then $(I-\mathcal{L}_+)\oplus(I-\mathcal{L}_-)$ can be identified in a natural way with an operator on $L_\infty(H)_{-\infty}^{\infty}$ and the difference between this operator and the invertible operator $I - \mathcal{L}$ is given by

$$(\mathcal{K}f)(x) = \begin{cases} -\int_{-\infty}^{0} \mathcal{H}(x-y)Bf(y)dy, & x > 0 \\ -\int_{0}^{\infty} \mathcal{H}(x-y)Bf(y)dy, & x < 0 , \end{cases}$$

which is a compact operator. Thus $I - \mathcal{L}_{\pm}$ are Fredholm operators whose indices (i.e., nullity minus deficiency index) add up to zero. In particular, if Eqs. (1.1) to (1.3) are uniquely solvable, then $I - \mathcal{L}_+$ is invertible and $I - \mathcal{L}_-$ is Fredholm of index 0.

Let us consider the Wiener-Hopf equation (1.4) on $(0,\infty)$ and its counterpart on $(-\infty,0)$, written as

$$(I-\mathcal{L}_+)f = \omega \text{ on } L_\infty(H)_0^\infty, \quad (I-\mathcal{L}_-)f = \omega \text{ on } L_\infty(H)_{-\infty}^{0}, \text{ respectively,}$$

where ω is given by (3.6). Denote by X_+(resp. X_-) the (closed) linear subspace of initial values $f(0^+)$(resp. $f(0^-)$) of solutions, where h ranges over H. For a uniquely solvable right half-space problem one has $X_+ = \text{Ran } E_+$. One easily sees that $X_\pm \subseteq Y_\pm$, but we intend to prove $X_\pm = Y_\pm$. Certainly, by the equivalence theorems, we have

$$\{f(0^+)\,|\,f\varepsilon L_\infty(H)_0^\infty \text{ and } (I-\mathcal{L}_+)f = 0\} = X_+ \cap \text{Ran } Q_- \subseteq D(T)$$

$$\{f(0^+)\,|\,f\varepsilon\ L_\infty(H)_0^\infty\ ;\exists h\varepsilon D(T):[(I-\mathcal{L}_+)f](x) = e^{-xT^{-1}}Q_+h,\ x\geqslant 0\} =$$

$$= [X_+ + \text{Ran } Q_-] \cap D(T),$$

and similar identities for the problem on $L_\infty(H)_{-\infty}^{0}$. Because the Fredholm indices of the Fredholm operators $I - \mathcal{L}_+$ and $I - \mathcal{L}_-$ add up to zero, one has

$$\begin{aligned} &\dim[X_+ \cap \text{Ran } Q_-] - \text{codim}[X_+ + \text{Ran } Q_-] = \\ &= -\dim[X_- \cap \text{Ran } Q_+] + \text{codim}[X_- + \text{Ran } Q_+]. \end{aligned} \tag{3.10}$$

However, $X_\pm \subseteq Y_\pm$ implies

$$\dim[X_\pm \cap \text{Ran } Q_\mp] \leqslant \dim[Y_\pm \cap \text{Ran } Q_\mp]\ ; \tag{3.11}$$

$$\text{codim}[X_\pm + \text{Ran } Q_\mp] \geqslant \text{codim}[Y_\pm + \text{Ran } Q_\mp]. \tag{3.12}$$

We now define V by (3.3) and compute

$$\text{Ker } V = [Y_+ \cap \text{Ran } Q_-]\oplus[Y_- \cap \text{Ran } Q_+]$$

$$\text{Ran } V = [Y_+ + \text{Ran } Q_-]\cap[Y_- + \text{Ran } Q_+].$$

If V would be a compact perturbation of the identity and therefore Fredholm of index 0, we would have dim Ker V = codim Ran V and thus Eq. (3.10) with $X_\pm$ replaced by $Y_\pm$. The latter equation together with (3.10) to (3.12) would imply that equality signs hold in (3.11) and (3.12), and hence that

$$X_\pm \cap \text{Ran } Q_\mp = Y_\pm \cap \text{Ran } Q_\mp,\ X_\pm + \text{Ran } Q_\mp = Y_\pm + \text{Ran } Q_\mp.$$

These identities together with $X_\pm \subseteq Y_\pm$ have as a consequence $X_\pm = Y_\pm$, and the proof of the theorem would be complete.

Because

$$I - V = Q_-P_+ + Q_+P_- = -(Q_+-Q_-)(P_+-Q_+)\ ,$$

it suffices to prove that P_+-Q_+ is compact. Let $f\varepsilon L_\infty(H)_{-\infty}^{\infty}$ be the unique solution of (3.5), where ω is given by (3.6). Using the resolvent kernel $\ell(\cdot)$ one finds

$$(P_+-Q_+)h = \int_0^\infty \ell(-y)e^{-yT^{-1}}Q_+hdy\ ,$$

and the compactness of $P_+ - Q_+$ is clear from the compactness of $\ell(-y)$ and the Bochner integrability of $\ell(-y)e^{-yT^{-1}}Q_+$ on $(0,\infty)$.

The construction and reasoning of the above proof were applied before to one-speed neutron transport in L_p-spaces.[32] We merely have constructed semigroups from solutions, as is usual in some areas. Theorem 3.2 and Lemma 3.1 allow us to extend the Hangelbroek[1] originated semigroup approach to various kinetic models beyond positive (or even self-adjoint)A. The simultaneous unique solvability of Eqs. (1.1) to (1.3) and the analogous left half-space problem is easily seen to be equivalent to the invertibility of the operator V in (3.3), and by putting $E = V^{-1}$ we then define the albedo operator which solves both half-space problems.

4. GENERALIZED H - EQUATIONS: SINGULAR CASE

In this section we assume that $B = I - A$ is a compact operator satisfying (1.17) and that $T^{-1}A$ does not have non-zero imaginary eigenvalues. The invertibility assumption on A is dropped. Though Eqs. (1.4) and (1.7) cannot be solved by Wiener-Hopf factorization, at least not in the usual sense,[16] we shall nevertheless obtain the results of the previous section for this "singular" case. We shall rely on the decompositions and semigroup properties of Section 3.

Theorem 4.1. *Let* E_+ and $E_+^\dagger$ *be projections on* H *with the following properties:*

(i) $E_+[D(T)] \subseteq D(T)$, *while* $TE_+ = E_+^\dagger T$ *on* $D(T)$;

(ii) $A[\mathrm{Ran}\, E_+] \subseteq \mathrm{Ran}\, E_+^\dagger$;

(iii) $\mathrm{Ker}\, E_+$ *and* $\mathrm{Ker}\, E_+^\dagger$ *are coinciding* T-*invariant subspaces.*
Then we have the factorization

$$\Lambda(z)^{-1} = H_\ell^+(-z)H_r^+(z) \; , \tag{4.1}$$

where

$$H_\ell^+(-z) = I - z\pi(T-zA)^{-1}E_+^\dagger Bj \; ; \tag{4.2}$$

$$H_r^+(z) = I - z\pi(I-E_+)(T-zA)^{-1}Bj \; . \tag{4.3}$$

The inverses of these factors are given by

$$H_\ell^+(-z)^{-1} = I - z\pi E_+(z-T)^{-1}Bj \; ; \tag{4.4}$$

$$H_r^+(z)^{-1} = I - z\pi(z-T)^{-1}(I-E_+^\dagger)Bj \; . \tag{4.5}$$

If T is bounded and A (and thus πAj) is invertible, the factors (4.2) to (4.5) can be found using a factorization principle for transfer functions.[22] We then apply it for

$$\Lambda(z) = (\pi Aj) + \pi T(T-z)^{-1}Bj \; ,$$

where $\pi Aj = D_1 D_2$ with $D_1 = I - \pi(I-E_+^\dagger)Bj$ and $D_2 = I - \pi E_+ Bj$, and where E_+ is the "supporting" projection. For strictly positive A and $\mathbb{B} = H$ this was done before,[22,23,3] but the application of the principle given there (for $D_1 = I$ and $D_2 = \pi Aj$) leads to different factors not suitable to our purpose. As we intend a generalization beyond the scope of Ref. 22, we prove (4.1) directly.

Proof of Theorem 4.1. Let us multiply the right-hand sides of (4.2) and (4.4). We get

$$[I-z\pi E_+(z-T)^{-1}Bj][I-z\pi(T-zA)^{-1}E_+^\dagger Bj] = I - z\pi E_+(z-T)^{-1}Bj -$$

$$-z\pi(T-zA)^{-1}E_+^\dagger Bj + z\pi E_+(z-T)^{-1}\{(z-T) + (T-zA)\}(T-zA)^{-1}E_+^\dagger Bj \; .$$

Using that

$$E_+(T-zA)^{-1}E_+^\dagger = (T-zA)^{-1}E_+^\dagger \; , \; E_+(z-T)^{-1}E_+^\dagger = E_+(z-T)^{-1} \; ,$$

on simplifying the above expression one obtains

$$[I-z\pi E_+(z-T)^{-1}Bj][I-z\pi(T-zA)^{-1}E_+^\dagger Bj] = I. \tag{4.6}$$

Let us multiply the right-hand sides of (4.3) and (4.5). As a result we find

$$[I-z\pi(I-E_+)(T-zA)^{-1}Bj][I-z\pi(z-T)^{-1}(I-E_+^\dagger)Bj] =$$

$$= I - z\pi(z-T)^{-1}(I-E_+^\dagger)Bj - z\pi(I-E_+)(T-zA)^{-1}Bj +$$

$$+ z\pi(I-E_+)(T-zA)^{-1}\{(z-T) + (T-zA)\}(z-T)^{-1}(I-E_+^\dagger)Bj \; .$$

Now we use the identities

$$(I-E_+)(T-zA)^{-1}(I-E_+^\dagger) = (I-E_+)(T-zA)^{-1},$$

$$(I-E_+)(z-T)^{-1}(I-E_+^\dagger) = (z-T)^{-1}(I-E_+^\dagger),$$

and derive

$$[I-z\pi(I-E_+)(T-zA)^{-1}Bj][I-z\pi(z-T)^{-1}(I-E_+^\dagger)Bj] = I. \tag{4.7}$$

Let us postmultiply the right-hand side of (4.5) by the one of (4.4). We get

$$[I-z\pi(z-T)^{-1}(I-E_+^\dagger)Bj][I-z\pi E_+(z-T)^{-1}Bj] = I-z\pi E_+(z-T)^{-1}Bj -$$

$$- z\pi(z-T)^{-1}(I-E_+^\dagger)Bj + z\pi(z-T)^{-1}(I-E_+^\dagger)(I-A)E_+z(z-T)^{-1}Bj.$$

Now note that

$$(I-E_+^\dagger)(I-A)E_+ = (I-E_+^\dagger)\{I-(I-E_+)\}-(I-E_+^\dagger)AE_+ = (I-E_+^\dagger)-(I-E_+).$$

Further, condition (i) implies (4.8)

$$z(z-T)^{-1}\{(I-E_+)-(I-E_+^\dagger)\}z(z-T)^{-1} = (I-E_+)z(z-T)^{-1}-z(z-T)^{-1}(I-E_+^\dagger).$$

Using these two identities one simplifies the above product considerably and obtains

$$[I-z\pi(z-T)^{-1}(I-E_+^\dagger)Bj][I-z\pi E_+(z-T)^{-1}Bj] = I-z\pi(z-T)^{-1}Bj=\Lambda(z). \tag{4.9}$$

The latter equality is immediate from (1.9).

The factorizations (4.6) to (4.9) involve factors that are compact perturbations of the identity. Thus the right-hand sides of (4.2) and (4.4) as well as those of (4.3) and (4.5) are inverses of each other, and Eq. (4.1) is clear.

If A is invertible and Eqs. (1.1) to (1.3) are uniquely solvable, then in Theorem 4.1 we could use the operators E_+ and $E_+^\dagger$ of the previous section and in this case the functions in (4.2) and (4.3) coincide with the functions H_ℓ^+ and H_r^+ of Section 2, as we shall see later.

If T is bounded, then the operators

$$P_0 = (-2\pi i)^{-1}\int_\Gamma(A-\zeta T)^{-1}Td\zeta \ , \quad P_0^\dagger = (-2\pi i)^{-1}\int_\Gamma T(A-\zeta T)^{-1}d\zeta \ ,$$

where Γ is a small positively oriented circle which separates $\lambda = 0$ from the non-zero part of the (identical) spectra of $T^{-1}A$ and AT^{-1}, are bounded projections[3] onto the finite-dimensional subspaces

$$Z_0 = \bigcup_{n=1}^{\infty} \mathrm{Ker}(T^{-1}A)^n \ , \qquad Z_0^\dagger = \bigcup_{n=1}^{\infty} \mathrm{Ker}(AT^{-1})^n \ , \tag{4.10}$$

where $Z_1 = \mathrm{Ker}\ P_0$ and $Z_1^\dagger = \mathrm{Ker}\ P_0^\dagger$ are invariant under $T^{-1}A$ and AT^{-1}, respectively. Furthermore,

$$T[Z_0] = Z_0^\dagger \ , \quad A[Z_0] \subseteq Z_0^\dagger \ , \ \overline{T[Z_1]} = A[Z_1] = Z_1^\dagger \ ; \tag{4.11}$$

$$Z_0 \oplus Z_1 = H \ , \quad Z_0^\dagger \oplus Z_1^\dagger = H \ . \tag{4.12}$$

If T is unbounded, however, we assume that the subspaces in (4.10) have a finite dimension, that $Z_0 \subseteq D(|T|^{2+\alpha})$ and Eqs. (4.11) and (4.12) are fulfilled. If A is positive self-adjoint, conditions of this kind were previously known.[33]

Lemma 4.2. *Assume that to every* $f_+ \varepsilon Q_+[D(T)]$ *there exists a unique solution of* Eqs. (1.1) *to* (1.3). *Then there exist bounded projections* E_+ *and* $E_+^\dagger$ *with kernel* Ran Q_- *such that* $f(0) = E_+ f_+$ *is the initial value of the solutions and conditions* (i) *to* (iii) *of Theorem* 4.1 *are fulfilled.*

Proof. On D(T) we define the operator E_+ by $E_+h = f(0)$, where f is the unique solution of Eqs. (1.1) to (1.3) with initial value $f_+ = Q_+h$. Let us choose a subspace N such that

$$N \oplus \{Z_0 \cap E_+[D(T)]\} = Z_0 \ . \tag{4.13}$$

Choose a "matrix" β on Z_0 without imaginary eigenvalues such that β is reduced by the decomposition (4.13), $\sigma(\beta|N) \subseteq \{\lambda|\mathrm{Re}\ \lambda<0\}$ and $\sigma(\beta|\{Z_0 \cap E_+[D(T)]\}) \subseteq \{\lambda|\mathrm{Re}\ \lambda>0\}$. Put

$$A_\beta = T\beta^{-1}P_0 + A(I-P_0).$$

Then A_β is invertible, the operator $I-A_\beta$ satisfies (1.17), the operator

$$A_\beta^{-1}T = \beta \oplus (T^{-1}A|_{Z_1})^{-1}$$

does not have imaginary eigenvalues and the function

$$g(x) = e^{-xT^{-1}A}P_0E_+f_+ + (I-P_0)f(x) \quad (0\leq x<\infty)$$

is a solution of the "regular" boundary value problem

$$(Tg)'(x) = -A_\beta g(x) \quad (0\leq x<\infty) \tag{4.14}$$

$$Q_+g(0) = f_+ \ , \ ||g(x)||_H = 0(1)(x\to\infty) \ , \tag{4.15}$$

to which the theory of Section 3 applies.[34] The roles of Y_+ and Y_- are now played by Ran E_+ and an extension of N, respectively, so that Eqs. (4.14) and (4.15) are uniquely solvable with E_+ as the albedo operator which maps the boundary data f_+ into the initial value $g(0)$. Thus E_+ is a bounded projection on H with kernel Ran Q_-, whose range is invariant under $A_\beta^{-1}T$ and $T^{-1}A_\beta$. As obviously $Z_0 \cap \mathrm{Ran}\ E_+ \subseteq \mathrm{Ker}\ A$ (for otherwise g might be unbounded as $x \to \infty$), (3.12) implies that Ran E_+ is $T^{-1}A$-invariant. Moreover, there also exists a bounded projection $E_+^\dagger$ on H with kernel Ran Q_-, such that Ran $E_+ \subseteq D(T)$ and $TE_+ = E_+^\dagger T$ on D(T). Hence, the conditions of Theorem 4.1 are fulfilled.

The unique solvability of problem (1.1) to (1.3) really needs not be assumed. It suffices to assume that to every $f_+ \varepsilon Q_+[D(T)]$ there exists at least one solution which is representable as

$f(0) = Ef_+$ for some linear operator E on D(T). This applies to positive self-adjoint A.[4,33] The remark we may also make as to the next two results.

Theorem 4.3. *Assume that to every* $f_+ \varepsilon Q_+[D(T)]$ *there exists a unique solution of* Eqs. (1.1) *to* (1.3), *and put* $E_+h = f(0)$ *with* $Q_+h = f_+$. *Then the functions* $H^+_\ell(z)^{-1}$ *and* $H^+_r(z)^{-1}$ *in* (4.4) *and* (4.5) *are continuous and invertible on the closed right half-plane and analytic on the open right half-plane, while* $H^+_\ell(z)^{-1}$ *and* $H^+_r(z)^{-1}$ *are bounded for* $z \to \infty$ (Re $z \geq 0$). *Further,*

$$\lim_{z\to\infty,\mathrm{Re}\ z\geq 0} z^{-1}H^+_\ell(z) = -\pi T^{-1}P_0^\dagger E_+^\dagger Bj\ , \quad \lim_{z\to\infty,\mathrm{Re}\ z\geq 0} z^{-1}H^+_r(z) = \tag{4.16}$$
$$= -\pi(I-E_+)T^{-1}P_0^\dagger Bj,$$

where the projection of H *onto* Z_0^+ *along* Z_1^+ *is denoted by* $P_0^\dagger$. *The functions* H^+_ℓ *and* H^+_r *satisfy the generalizations* (2.7) *of Chandrasekhar's* H*-equations.*

Proof. Note that $(I-E_+)(T-zA)^{-1}$ and $(z-T)^{-1}(I-E_+^\dagger)$ extend to analytic functions on the right half-plane, while $(T-zA)^{-1}E_+^\dagger$ and $E_+(z-T)^{-1}$ extend analytically to the left half-plane. The analyticity of the factors (4.2) to (4.5) on the respective half-planes then is immediate. Their continuity one only has to prove for $z \to 0$ and (if possible) for $z \to \infty$. Since $z(z-T)^{-1}Q_\pm \to 0$ in the strong operator topology if $z \to 0$ from the left/right half-plane, we have [23,22,3]

$$||z(z-T)^{-1}Q_\pm K|| \to 0 \quad (z \to 0,\ \mathrm{Re}\ z \lesseqgtr 0)$$

for all compact operators K. Thus $H^+_\ell(0^+)^{-1} = I$ in the norm, whence $H^+_\ell(0^+)=H^+_r(0^+)=I$ in the norm. In a similar way, using $z(z-T)^{-1}Q_\pm \to Q_\pm$ strongly if $z\to\infty$ from the left/right-half plane,

$$H^+_\ell(\infty)^{-1} = I-\pi E_+Bj\ ,\ H^+_r(\infty)^{-1} = I-\pi(I-E_+^\dagger)Bj. \tag{4.17}$$

With the help of (4.11) and (4.12) and the projection $P_0^\dagger$ of H onto Z_0^+ along $Z_1^\dagger$, we easily prove that

$$\lim_{z\to\infty,\mathrm{Re}\ z\leq 0} ||\pi(T-zA)^{-1}(I-P_0^\dagger)E_+^\dagger Bj|| = 0,$$

$$\lim_{z\to\infty,\mathrm{Re}\ z\geq 0} ||\pi(I-E_+)(T-zA)^{-1}(I-P_0^\dagger)Bj|| = 0.$$

However, as T-zA maps Z_0 onto $Z_0^\dagger$, one has

$$(T-zA)^{-1}Th = h\ ,\quad h\varepsilon\mathrm{Ker}\ A.$$

Since Ran $E_+ \cap Z_0 \subseteq$ Ker A and $TE_+ = E_+^\dagger T$ on D(T), one eventually obtains (4.16).

Put

$$\phi_\ell(z) = H^+_r(z)^{-1} + z\int_0^\infty (z+t)^{-1}\pi\sigma(-dt)BjH^+_\ell(t).$$

This function is well-defined and continuous on the closed right half-plane, is analytic on the open right half-plane and satisfies $\phi_\ell(0^+) = I$. The limit of $\phi_\ell(z)$ as $z \to \infty$ in the closed right half-plane exists (cf. (4.16)). Using (1.9) and (4.1) one writes

$$\phi_\ell(z) = H^+_\ell(-z)-z\int_0^\infty (z-t)^{-1}\pi\sigma(dt)BjH^+_\ell(-z) -$$
$$-z\int_0^\infty (z+t)^{-1}\pi\sigma(-dt)BjH^+_\ell(-z)+z\int_0^\infty(z+t)^{-1}\pi\sigma(-dt)BjH^+_\ell(t) =$$
$$= H^+_\ell(-z)+z\int_0^\infty \frac{\pi\sigma(dt)Bj}{t-z} H^+_\ell(-z)+z\int_0^\infty\pi\sigma(-dt)Bj\ \frac{H^+_\ell(t)-H^+_\ell(-z)}{t+z}\ ,$$

which is continuous on the closed and analytic on the open left half-plane. This expression is O(z) as $z \to \infty$ in the left half-plane. Using Liouville's theorem and $\phi_\ell(0^+) = I$, one finds $\phi_\ell(z) \equiv I$, which implies (2.7b). Equation (2.7a) is proved analogously with the aid of the auxiliary function

$$\phi_r(z) = H^+_\ell(z)^{-1}+z\int_0^\infty (z+t)^{-1}H^+_r(t)\pi\sigma(dt)Bj.$$

Theorem 4.4. *Assume that to every* $f_+\varepsilon Q_+[D(T)]$ *there exists a unique solution of* Eqs. (1.1) *to* (1.3), *and put* $E_+h = f(0)$ *with* $Q_+h = f_+$. *Then the albedo operator* E *is given by* (2.1), *where* H^+_ℓ *and* H^+_r *are the functions in* (4.2) *and* (4.3).

Proof. Using (4.3) we compute, for $\mu\varepsilon(-\infty,0)$,

$$\int_0^\infty \frac{\nu}{\nu-\mu}\ H^+_r(\nu)\pi\sigma(d\nu) = \pi T(T-\mu)^{-1}Q_+ - \int_0^\infty \frac{\nu}{\nu-\mu}\ \pi(I-E_+)(T-\nu A)^{-1}(\nu B)\sigma(d\nu).$$

We now substitute the identity

$$\nu B = (\nu-T) + (T-\nu A), \tag{4.18}$$

and, using $(\nu-T)\sigma(d\nu) = 0$, we obtain

$$\int_0^\infty \frac{\nu}{\nu-\mu} H_r^+(\nu)\pi\sigma(d\nu) = T(T-\mu)^{-1}Q_+ - \pi(I-E_+)T(T-\mu)^{-1}Q_+ = \pi E_+T(T-\mu)^{-1}Q_+ .$$

The integral term in (2.1) now reads (cf. (4.2)):

$$\int_{-\infty}^0 \sigma(d\mu)BjH_\ell^+(-\mu)\ E_+T(T-\mu)^{-1}Q_+ = \int_{-\infty}^0 \sigma(d\mu)BE_+T(T-\mu)^{-1}Q_+ -$$
$$- \int_{-\infty}^0 \sigma(d\mu)(\mu B)(T-\mu A)^{-1}E_+^\dagger BE_+T(T-\mu)^{-1}Q_+ .$$

Again we involve (4.18) and get for the integral term in (2.1)

$$\int_{-\infty}^0 \sigma(d\mu)(I-E_+^\dagger)BE_+T(T-\mu)^{-1}Q_+ . \tag{4.19}$$

We now employ the identity

$$(I-E_+^\dagger)BE_+ = (I-E_+^\dagger)\{I-(I-E_+)\}+(I-E_+^\dagger)AE_+ = E_+ - E_+^\dagger ,$$

together with (4.8), to simplify (4.19). As a result we get

$$\underset{(\mu)}{\int_{-\infty}^0} \underset{(\nu)}{\int_0^\infty} \frac{\nu}{\nu-\mu} \sigma(d\mu)BjH_\ell^+(-\mu)H_r^+(\nu)\pi\sigma(d\nu) = Q_-E_+ ,$$

which yields (2.1).

We have generalized the Wiener-Hopf factorization, the H-equations and the expression for the albedo operators to the case when A does not have an inverse. If Eqs. (1.1) to (1.3) are non-uniquely solvable, several of these factorizations exist and the behaviour of $H_\ell^+(z)$ and $H_r^+(z)$ for $z \to \infty$ must be used to single out the factors appropriate to the problem. In this article we shall not elaborate upon the uniqueness problem for solutions of the generalized H-equations. We remark that generalized H-equations and a formula for E in terms of H-equations have been found by many authors. (The literature is too enormous to cite here). A more or less abstract procedure, with T still a multiplication operator symmetric with respect to $\lambda = 0$, was given by Kelley.[20]

5. SOME APPLICATIONS

(a) *Transfer of polarized light* in a semi-infinite homogeneous planetary atmosphere is described by the equation[8,12]

$$u \frac{d\underset{\sim}{I}(\tau,u,\phi)}{d\tau}+\underset{\sim}{I}(\tau,u,\phi) = \frac{a}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \underset{\sim}{Z}(u,u',\phi-\phi')\underset{\sim}{I}(\tau,u',\phi')d\phi'du', \tag{5.1}$$

where $\underset{\sim}{I}(\tau,u,\phi)$ is the Stokes (four} vector describing the intensity and state of polarization of a beam with directional parameters $u\varepsilon[-1,1]$ and $\phi\varepsilon[0,2\pi]$ at optical depth $\tau\varepsilon[0,\infty)$. Here $\underset{\sim}{Z}(u,u',\phi-\phi')$ is the phase matrix and $0<a\leq 1$ the albedo of single scattering. Throughout we use the conventions of Ref. 12.

On applying Fourier decomposition and symmetry properties[35,also12] the full equation (5.1) can be decomposed into twice the set of component equations

$$u \frac{d\underset{\sim}{X}^j(\tau,u)}{d\tau} + \underset{\sim}{X}^j(\tau,u) = \frac{1}{2}\ a \int_{-1}^{1} \underset{\sim}{W}^j(u,u')\underset{\sim}{X}^j(\tau,u')du', \tag{5.2}$$

where $u\varepsilon[-1,1]$, $\tau\varepsilon[0,\infty)$ and $j = 0,1,2,\ldots$. Here the kernel is given by

$$\underset{\sim}{W}^j(u,u') = \sum_{\ell=j}^{L} \underset{\sim}{\Pi}_\ell^j(u)\underset{\sim}{B}_\ell^j\underset{\sim}{\Pi}_\ell^j(u') , \tag{5.3}$$

where for certain special functions and expansion coefficients

$$\underset{\sim}{\Pi}_\ell^j(u) = \begin{bmatrix} P_\ell^j(u) & 0 & 0 & 0 \\ 0 & R_\ell^j(u) & -T_\ell^j(u) & 0 \\ 0 & -T_\ell^j(u) & R_\ell^j(u) & 0 \\ 0 & 0 & 0 & P_\ell^j(u) \end{bmatrix}, \underset{\sim}{B}_\ell^j = \frac{(\ell-j)!}{(\ell+j)!} \begin{bmatrix} \beta_\ell & \gamma_\ell & 0 & 0 \\ \gamma_\ell & \alpha_\ell & 0 & 0 \\ 0 & 0 & \zeta_\ell & -\varepsilon_\ell \\ 0 & 0 & \varepsilon_\ell & \delta_\ell \end{bmatrix}$$

The physical requirement that the degree of polarization of a beam does not exceed unity, necessitates imposing the condition that $\underset{\sim}{Z}(u,u',\phi-\phi')$ maps real vectors $\underset{\sim}{I} = (I,Q,U,V)$ satisfying

$$I \geqslant \sqrt{Q^2+U^2+V^2} \geqslant 0$$

into vectors of the same type.

Under the boundary conditions

$$\underset{\sim}{X}^j(0,u) = \underset{\sim}{X}_+^j(u)\,(0\leqslant u\leqslant 1),\ \int_{-1}^{1} ||\underset{\sim}{X}^j(\tau,u)||^2 du = 0(1)\,(\tau\to\infty) \qquad (5.4)$$

Equation (5.2) is uniquely solvable if the summation in (5.3) is finite,[36] which we shall henceforth assume. Let H be the Hilbert space of measurable L_2-functions $\underset{\sim}{I}:[-1,1]\to \mathbb{C}^4$, and define T, B, A and Q_+ by the equations

$$(T\underset{\sim}{I})(u) = u\underset{\sim}{I}(u),\ (B\underset{\sim}{I})(u) = \frac{1}{2}\,a \int_{-1}^{1} \underset{\sim}{W}^j(u,u')\underset{\sim}{I}(u')du',$$

$$(A\underset{\sim}{I})(u) = \underset{\sim}{I}(u) - \frac{1}{2}\,a \int_{-1}^{1} \underset{\sim}{W}^j(u,u')\underset{\sim}{I}(u')du',\ (Q_+\underset{\sim}{I})(u) = \begin{cases} \underset{\sim}{I}(u), & u>0 \\ 0 & , u<0. \end{cases}$$

Then Eqs. (5.2) and (5.4) are an example of problem (1.1) to (1.3), which is uniquely solvable and for which $T^{-1}A$ does not have non-zero imaginary eigenvalues.[36] Therefore, there is an albedo operator E_+ which maps $\underset{\sim}{X}_+^j$ into $\underset{\sim}{X}^j(0)$ uniquely.

Let $\mathbb{B} = \text{span}\,\{e^j_{\ell,k}\,|\,\ell=j,j+1,\dots,L;\ k=1,2,3,4\}$, where $L\geqslant \max(j,2)$ and

$$e^j_{\ell,1} = (P^j_\ell,0,0,0),\ e^j_{\ell,2} = (0,R^j_\ell,\,-\,T^j_\ell,0)$$

$$e^j_{\ell,3} = (0,-T^j_\ell,R^j_\ell,0),\ e^j_{\ell,4} = (0,0,0,P^j_\ell).$$

Then $\mathbb{B} \supseteq \text{Ran}\ B^*$, and in the usual inner product of H we have,[37, also 12]

(i) $\qquad (e^j_{\ell,k},e^j_{r,i})_H = 0$ if either $\ell \neq r$ or $k \neq i$;

(ii) $$||e^j_{\ell,k}||^2_H = \begin{cases} \dfrac{2}{2\ell+1}\,\dfrac{(\ell+j)!}{(\ell-j)!} & \text{if } \begin{cases}\text{either } \ell \geqslant j \text{ and } k = 1,4, \\ \text{or } \ell \geqslant \max(j,2) \text{ and } k = 2,3;\end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathbb{B}$ has dimension $4(L-j) + 2\min(j,2)$. Further, the imbedding previously devoted by j is the natural imbedding of $\mathbb{B}$ into H, while π is given by

$$\pi\underset{\sim}{I} = \sum_{\substack{\ell=j \\ k=1,4}}^{L} \frac{2\ell+1}{2}\,\frac{(\ell-j)!}{(\ell+j)!}\,(\underset{\sim}{I},e^j_{\ell,k})_H e^j_{\ell,k} \,+$$

$$+ \sum_{\substack{\ell=\max(j,2) \\ k=2,3}}^{L} \frac{2\ell+1}{2}\,\frac{(\ell-j)!}{(\ell+j)!}\,(\underset{\sim}{I},e^j_{\ell,k})_H e^j_{\ell,k}\ ,$$

while

$$(2\ell+1)Be^j_{\ell,k} = a\,\frac{(\ell+j)!}{(\ell-j)!}\sum_{m=1}^{4}\,[\underset{\sim}{B}^j_\ell]_{mk}\,e^j_{\ell,m}\ .$$

With respect to the ordered basis $\{e^j_{\ell,k}\,|\,(\ell,k)=(j,1),\dots,(j,4),\dots,(L,1),\dots,(L,4)\}$ (with $e^j_{\ell,2}$ and $e^j_{\ell,3}$ left out for $j \leqslant \ell < \max(j,2)$) the dispersion matrix has the form

$$[\Lambda(z)]_{(\ell,k),(r,i)} = \delta_{\ell r}\delta_{ki} - \frac{1}{2}\,az\,\frac{2\ell+1}{2r+1}\,\frac{(\ell-j)!}{(\ell+j)!}\,\frac{(r+j)!}{(r-j)!}\ \times$$

$$\times \int_{-1}^{1} \frac{\sum_{m=1}^{4} [\underset{\sim}{B}^j_\ell]_{mi}\{e^j_{\ell,k}(t)\cdot e^j_{r,m}(t)\}}{z-t}\,dt\ .$$

This square matrix function of order $4(L-j) + 2\min(j,2)$ has Wiener-Hopf factorization

$$\Lambda(z)^{-1} = H^+_\ell(-z)H^+_r(z),\ \text{Re}\ z = 0,$$

where H^+_ℓ and H^+_r are continuous and invertible on the closed right half-plane and analytic in the open right half-plane, $H^+_\ell(0^+) = H^+_r(C^+) = I$ and $H^+_\ell(w) = 0(w)$ and $H^+_r(w) = 0(w)\,(w\to\infty,\ \text{Re}\ w \geqslant 0)$. The H-equations are now given by

$$[H^+_\ell(z)^{-1}]_{(\ell,k),(r,i)} = \delta_{\ell r}\delta_{ki} - \frac{1}{2}\,az\ \times$$

$$\times \int_0^1 \frac{du'}{u'+z} \sum_{(s,n)} \sum_{m} \frac{2s+1}{2r+1}\,\frac{(r+j)!}{(r-j)!}\,\frac{(s-j)!}{(s+j)!}\ x$$

$$\times\ [H_r^+(u')]_{(\ell,k),(s,n)}[\underset{\sim}{B}_r^j]_{mi}\{e_{r,m}^j(u')\cdot e_{s,n}^j(u')\}\ ;$$

$$[H_r^+(z)^{-1}]_{(\ell,k),(r,i)} = \delta_{\ell r}\delta_{ki} - \frac{1}{2}\,az\ \times$$

$$\times\int_{-1}^{0}\frac{du'}{z-u'}\sum_{(s,n)}\sum_{m}\frac{2\ell+1}{2s+1}\,\frac{(s+j)!}{(s-j)!}\,\frac{(\ell-j)!}{(\ell+j)!}\ \times$$

$$\times\ [\underset{\sim}{H}_\ell^+(-u')]_{(s,n),(r,i)}[\underset{\sim}{B}_s^j]_{mn}\{e_{\ell,k}^j(u')\cdot e_{s,m}^j(u')\}\ .$$

Finally, the albedo operator E is given by $(E\underset{\sim}{X}_+^j)(u) = \underset{\sim}{X}_+^j(u)$ for $u > 0$, while for $u < 0$

$$(E\underset{\sim}{X}_+^j)(u) = \frac{1}{2}\,a\int_0^1\frac{u'}{u'-u}\sum_{(\ell,k)}\sum_{(r,i)}\sum_{(s,n)}\sum_{m}\frac{2s+1}{2\ell+1}\,\frac{(s-j)!}{(s+j)!}\,\frac{(\ell+j)!}{(\ell-j)!}\ \times$$

$$\times[\underset{\sim}{B}_\ell^j]_{mk}[H_\ell^+(-u)]_{(\ell,m),(r,i)}[H_r^+(u')]_{(r,i),(s,n)}e_{\ell,m}^j(u)$$

$$\{e_{s,n}^j(u')\cdot\underset{\sim}{X}_+^j\ (u')\}du'.$$

For $j = 0$ the above expressions decouple into pairs of expressions involving matrices of order 2L.

(b) *Neutron transport* with angularly dependent cross-sections[38] may be described by

$$\mu\,\frac{\partial f}{\partial x}\,(x,\mu) + \Sigma(\mu)f(x,\mu) = \frac{1}{2}\int_{-1}^{1}\Sigma_s(\mu')f(x,\mu')d\mu'\ ,$$

where $\mu\varepsilon[-1,1]$ and $x\varepsilon[0,\infty)$ are the angular and position variable, respectively. This equation is also encountered in phonon transport in crystalline solids.[14] We assume that Σ and Σ_s are measurable and satisfy $\Sigma \geqslant \Sigma_s \geqslant \varepsilon > 0$. Write $g = (\Sigma\Sigma_s)^{\frac{1}{2}}f$, then

$$(\mu/\Sigma(\mu))\ \frac{\partial g}{\partial x}(x,\mu)+g(x,\mu)= \frac{1}{2}\,v(\mu)\int_{-1}^{1}v(\mu')g(x,\mu')d\mu'\ ,$$

where $v = (\Sigma_s/\Sigma)^{\frac{1}{2}}$. We analyze this equation in $H = L_2[-1,1]$ and define

$$(Th)(\mu) = \mu\Sigma(\mu)^{-1}h(\mu)\ ,\ (Bh)(\mu) = \frac{1}{2}\,v(\mu)\int_{-1}^{1}v(\mu')h(\mu')d\mu'$$

$$(Ah)(\mu)=h(\mu)-\frac{1}{2}\,v(\mu)\int_{-1}^{1}v(\mu')h(\mu')d\mu'\ ,\ (Q_\pm h)(\mu)=\begin{cases}h(\mu) & ,\ \mu \gtrless 0\\ 0 & ,\ \mu \lessgtr 0\ .\end{cases}$$

Condition (1.17) is satisfied if and only if $\int_{-1}^{1}\Sigma^{2\alpha-1}\Sigma_s|\mu|^{-2\alpha}d\mu<\infty$ for some $\alpha > 0$. Under this condition we shall study the boundary value problem

$$(Tg)'(x) = -Ag(x)\quad (0<x<\infty) \tag{5.5}$$

$$\lim_{x\downarrow 0}||Q_+g(x)-g_+||_H = 0\ ,\ ||g(x)||_H = O(1)(x\to\infty). \tag{5.6}$$

Then A is positive self-adjoint, B compact and Ker A $\subseteq$ span{v}. There are three cases to be considered: (i) $\Sigma_s(\mu)\not\equiv\Sigma(\mu)$, where Ker A = {0}, (ii) $\Sigma_s(\mu)\equiv\Sigma(\mu)$ and $\int_{-1}^{1}\mu\Sigma(\mu)d\mu\neq 0$, where Z_0=Ker A= span{v}, and (iii) $\Sigma_s(\mu)\equiv\Sigma(\mu)$ and $\int_{-1}^{1}\mu\Sigma(\mu)d\mu=0$, where Z_0 = span{v,$\mu\Sigma$}. Then Eqs. (5.5) and (5.6) are uniquely solvable, unless $\Sigma_s(\mu)\equiv\Sigma(\mu)$ and $\int_1^1\mu\Sigma(\mu)d\mu<0$, in which case there always exists a solution with measure of non-uniqueness one.[4] So there exists an albedo operator (unique except for this exceptional case) and the H-functions can be found.

Let us choose $\mathbb{B}$ = span{v}. Then $\pi h = \{(h,v)_H/||v||_H^2\}v$ and

$$\Lambda(z) = 1 - \frac{1}{2}\,z\int_1^1\frac{\Sigma_s(\mu')d\mu'}{z\Sigma(\mu')-\mu'}\ .$$

This dispersion function we factorize as

$$\Lambda(z)^{-1} = H_\ell^+(-z)H_r^+(z)\ ,\ \mathrm{Re}\ z = 0,$$

where H_ℓ^+ and H_r^+ satisfy the H-equations

$$H_\ell^+(z)^{-1} = 1 - \frac{1}{2}\,z\int_0^1\frac{\Sigma_s(\mu')H_r^+(\mu'/\Sigma(\mu'))}{z\Sigma(\mu') + \mu'}\,d\mu'$$

$$H_r^+(z)^{-1} = 1 - \frac{1}{2}\,z\int_{-1}^{0}\frac{\Sigma_s(\mu')H_\ell^+(-\mu'/\Sigma(\mu'))}{z\Sigma(\mu') - \mu'}\,d\mu'\ .$$

The albedo operator then is given by

$$(E_+g_+)(\mu) = \begin{cases} g_+(\mu) \ , \quad \mu > 0 \\ \frac{1}{2} \int_0^1 \frac{\mu'\Sigma(\mu)}{\mu'\Sigma(\mu)-\mu\Sigma(\mu')} \left[\frac{\Sigma_s(\mu')}{\Sigma(\mu')}\right]^{\frac{1}{2}} H_\ell^+\left(\frac{-\mu}{\Sigma(\mu)}\right) H_r^+\left(\frac{\mu'}{\Sigma(\mu')}\right) g_+(\mu')d\mu', \\ \hfill \mu < 0. \end{cases}$$

If Σ_s and Σ are even functions, then $H_\ell = H_r$. For this case related results are obtained by Williams[38] using his Wiener-Hopf method.

(c) *Strong evaporation* of a liquid into a half-space vacuum, with a drift velocity $d > 0$ at infinity, is described by the equation[15,39]

$$(v+d) \frac{df}{dx}(x,v)+f(x,v) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \{1+2vv'+2(v^2-\tfrac{1}{2})(v'^2-\tfrac{1}{2})\}f(x,v')e^{-v'^2}dv', \tag{5.7}$$

where $f(x,v)$ is the deviation from the drift Maxwellian and v the velocity. Effects transverse to the x-direction are neglected. The boundary conditions

$$f(0,v) = a_0+a_1 v\sqrt{2} + a_2(v^2-\tfrac{1}{2})\sqrt{2}\ (v>-d), \quad f(\infty,v) \equiv 0 \tag{5.8}$$

are imposed. Note that $f(0,v)$ is required to be a collision invariant.

Let H be the Hilbert space of measurable functions $h:\mathbb{R} \to \mathbb{C}$ with inner product

$$(h,k) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} h(v')\overline{k(v')}e^{-v'^2}dv' \ ,$$

and define on H the operators T, $Q_\pm$, A and B by

$$(Th)(v) = (v+d)h(v) \ , \qquad (Q_\pm h)(v) = \begin{cases} h(v) \ , & v > -d \\ 0 \ , & v < -d \end{cases}$$

$$(Ah)(v) = h(v) - (Bh)(v), \quad (Bh)(v) = \sum_{i=0}^{2} (h,e_i)e_i(v),$$

where we have the orthonormal set

$$e_0(v) = 1 \ , \ e_1(v) = v\sqrt{2}, \ e_2(v) = (v^2-\tfrac{1}{2})\sqrt{2}.$$

Then $\operatorname{Ker} A = \operatorname{Ran} B = \operatorname{span}\{e_0,e_1,e_2\}$ is the set of collision invariants. Choose $\mathbb{B} = \operatorname{span}\{e_0,e_1,e_2\}$. With respect to the basis $\{e_0,e_1,e_2\}$ we have

$$[\Lambda(z)]_{ij} = \delta_{ij} - \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e_i(v')e_j(v'-d)}{z - v'} e^{-v'^2} dv'.$$

It is known[40] that for this problem Eqs. (1.1) to (1.3) are uniquely solvable for $d \geqslant \frac{1}{2}\sqrt{6}$ and are solvable with measure of non-uniqueness one for $0<d<\frac{1}{2}\sqrt{6}$. So there exists an albedo operator E_+ (unique for $d \geqslant \frac{1}{2}\sqrt{6}$, and depending on one parameter for $0<d<\frac{1}{2}\sqrt{6}$) such that E_+f_+ is the initial value of a solution of Eqs. (1.1) to (1.3), given $f_+\epsilon Q_+[D(T)]$. There are corresponding H-functions satisfying

$$\Lambda(z)^{-1} = H_\ell^+(-z)H_r^+(z), \ \operatorname{Re} z = 0.$$

The corresponding albedo operator is given by

$$(E_+f_+)(v) = \begin{cases} f_+(v) \ , \quad v > -d \ ; \\ \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{v'}{v'-v} \sum_{i,j=0}^{2} e_i(v)[H_\ell^+(-v-d)H_r^+(v')]_{ij}e_j(v')f_+(v'-d)e^{-v'^2}dv', \\ \hfill v < -d. \end{cases}$$

The H-equations have the form

$$[H_\ell^+(z)^{-1}]_{ij} = \delta_{ij} - \frac{z}{\sqrt{\pi}} \int_0^{\infty} \sum_{k=0}^{2} [H_r^+(v')]_{ik} \frac{e_k(v')e_j(v'-d)}{z + v'} e^{-v'^2}dv';$$

$$[H_r^+(z)^{-1}]_{ij} = \delta_{ij} - \frac{z}{\sqrt{\pi}} \int_0^{\infty} \sum_{k=0}^{2} [H_\ell^+(v')]_{kj} (-1)^{k+j} \frac{e_k(v'+d)e_i(v')}{z - v'} e^{-v'^2}dv' .$$

For $d \geqslant \frac{1}{2}\sqrt{6}$ Eqs. (5.7) and (5.8) do not have non-zero solutions, whereas for $0 < d < \frac{1}{2}\sqrt{6}$ there is a one-dimensional subspace of boundary data of collision invariants for which a (unique) solution exists.[15,39,40] We may use any of the non-unique above albedo operators and find this subspace by solving the linear system

$$\{(E_+e_i,e_j)_H\}_{i,j=0}^{2}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix} = 0,$$

which yields a one-dimensional subspace of $\mathbb{C}^3$. The admissible boundary data then are $a_0e_0+a_1e_1+a_2e_2$ and the initial value of the solution $E_+(a_0e_0+a_1e_1+a_2e_2)$.

We remark that in Refs. 15 and 39 a scalar dispersion function is used, which is the determinant of our matrix dispersion function.

Appendix

Let T be injective self-adjoint on H with orthogonal projection $Q_\pm$ onto its maximal positive/negative invariant subspace, and define $H(t)(0\neq t\varepsilon\mathbb{R})$ by (1.5). Assume that B is bounded and satisfies (1.17), and put $A = I - B$. We define $L_\infty(H)_0^\infty$ as in Section 2. We have

Theorem A.1. *Let* $\omega:[0,\infty) \to H$ *be bounded and continuous, let* $\omega(x)\varepsilon D(T)$ *for* $0<x<\infty$, *and suppose that* $T\omega$ *is strongly differentiable on* $(0,\infty)$. *Then a function* $f:(0,\infty) \to D(T)$ *is a solution to the boundary value problem*

$$(Tf)'(x) = -Af(x) + (T\omega)'(x) + \omega(x) \quad (0<x<\infty) \tag{A.1}$$

$$\lim_{x\downarrow 0} ||Q_+f(x)-Q_+\omega(0)||_H = 0, \quad ||f(x)||_H = O(1) \quad (x\to\infty), \tag{A.2}$$

if and only if $f\varepsilon L_\infty(H)_0^\infty$ *and satisfies the Wiener-Hopf equation*

$$f(x) - \int_0^\infty H(x-y)Bf(y)dy = \omega(x), \quad 0<x<\infty. \tag{A.3}$$

Any such solution is bounded and continuous on $[0,\infty)$.

For bounded T this theorem was proved in Ref. 3. A preliminary version appeared in Ref. 23. The generalization to unbounded T causes additional technicalities to resolve, and for this reason we give a full proof. We need three lemmas.

Lemma A.2. $\int_{-\infty}^{\infty} ||H(t)B||dt<\infty, \int_{-\infty}^{\infty} ||TH(t)B|| dt<\infty$.

Proof. Note the following norm estimates:

$$||\,|T|^{\alpha}H(t)|| = O(|t|^{\alpha-1}), \quad ||TH(t)|| = O(1) \qquad (t\to 0)$$

$$||\,|T|^{-1-\alpha}H(t)|| = O(|t|^{-2-\alpha}), \quad ||\,|T|^{-\alpha}H(t)\,|| = O(|t|^{-1-\alpha}). \qquad (t\to\pm\infty)$$

With the help of (1.17) we find $B = |T|^{\alpha}D_1$ and $|T|^{1+\alpha}B = D_2$ for bounded D_1 and D_2, and the lemma follows.

Lemma A.3.[41] *If* (E,ρ) *is a measure space,* T *a closed linear operator, and if* $f:E\to D(T)$ *and* Tf *are Bochner integrable, then* $\int_E f d\rho\varepsilon D(T)$ *and*

$$T\int_E fd\rho = \int_E (Tf)d\rho.$$

Lemma A.4. *Let* $f:(0,\infty) \to H$ *be bounded continuous, and let* T *be an injective self-adjoint operator. Then* $\lim_{y\to\infty} e^{(y-x)T^{-1}}Q_-f(y) = 0$ *in the weak sense.*

Proof. Using the resolution of the identity $\sigma(.)$ of T, Ker T = {0} (and thus $\sigma(\{0\}) = 0$) and dominated convergence we have immediately

$$\lim_{y\to\infty} e^{(y-x)T^{-1}}Q_-f(y) = \lim_{y\to\infty}\int_0^\infty e^{(x-y)/t}\sigma(-dt)f(y) = 0,$$

where the limit is taken in the weak sense.

Proof of Theorem A.1. Let $f:(0,\infty)\to D(T)$ be a solution to (A.1) and (A.2), and put $\chi = f-\omega$. Choose $0<x<\infty$, and take $0<x_1<x<x_2<x_3<\infty$. Then

$$\int_0^{x_1} H(x-y)Bf(y)dy = \int_0^{x_1} H(x-y)\{(T\chi)'(y)+\chi(y)\}dy = [e^{-(x-y)T^{-1}}Q_+\chi(y)]_0^{x_1}$$

$$\int_{x_2}^{x_3} H(x-y)Bf(y)dy = \int_{x_2}^{x_3} H(x-y)\{(T\chi)'(y)+\chi(y)\}dy = [-e^{-(x-y)T^{-1}}Q_-\chi(y)]_{x_2}^{x_3} .$$

The left-hand sides have strong limits for $x_1\uparrow x$, $x_2\downarrow x$ and $x_3\to\infty$ (see Lemma A.2, together with the boundedness of f). Thus the right-hand sides have strong limits for $x_1\uparrow x$, $x_2\downarrow x$ and $x_3\to\infty$. As $Q_+\chi(y)\to 0$ for $y\downarrow 0$ strongly and Lemma A.4 holds, one obtains (A.3) as a result, where $f \in L_\infty(H)_0^\infty$.

Conversely, let f be a solution in $L_\infty(H)_0^\infty$ of (A.3). Because $f-\omega$ is the convolution product of the Bochner integrable function $\mathcal{H}(\cdot)B$ and a function $f \in L_\infty(H)_0^\infty$, it is bounded and continuous on $[0,\infty)$. Using Lemma A.3 we prove that

$$g(x) = \int_0^\infty \mathcal{H}(x-y)Bf(y)dy \in D(T) \text{ (see also Lemma A.2)},$$

while

$$T\int_0^\infty \mathcal{H}(x-y)Bf(y)dy = \int_0^\infty T\mathcal{H}(x-y)Bf(y)dy, \quad 0 \leqslant x < \infty.$$

Repeatedly using Lemma A.3 we get for all $\varepsilon > 0$:

$$T\{g(x+\varepsilon)-g(x)\}/\varepsilon = h_1 + h_2 + h_3 + h_4 ,$$

where

$$h_1 = \varepsilon^{-1}[e^{-\varepsilon T^{-1}}Q_+ - Q_+]T\int_0^x \mathcal{H}(x-y)Bf(y)dy$$

$$h_2 = -\varepsilon^{-1}[e^{\varepsilon T^{-1}}Q_- - Q_-]T\int_x^\infty \mathcal{H}(x-y)Bf(y+\varepsilon)dy$$

$$h_3 = \varepsilon^{-1}\int_x^{x+\varepsilon} T\mathcal{H}(x+\varepsilon-y)Bf(y)dy$$

$$h_4 = -\varepsilon^{-1}\int_x^{x+\varepsilon} T\mathcal{H}(x-y)Bf(y)dy.$$

Let us take $\varepsilon \downarrow 0$. Using simple semigroup theory, $h_1 \to -\int_0^x \mathcal{H}(x-y)Bf(y)dy$. By the continuity of f, dominated convergence[26] and the same semigroup property, $h_2 \to -\int_x^\infty \mathcal{H}(x-y)Bf(y)dy$. By the continuity of the integrands, $h_3 \to Q_+Bf(x)$ and $h_4 \to Q_-Bf(x)$. Thus Tg is strongly differentiable on $(0,\infty)$ from the right and

$$(Tg)'(x) = -g(x)+Bf(x). \qquad (A.4)$$

Similarly, one proves strong differentiability from the left. Hence, Tg is strongly differentiable on $(0,\infty)$ and (A.4) holds true. However, (A.3) implies $g = f-\omega$, and the function f therefore satisfies (A.1). The first one of Eqs. (A.2) follows by substitution of $x = 0$ into (A.3).

If $f_+ \in Q_+[D(T)]$ and $\omega(x) = e^{-xT^{-1}}f_+$, then (A.3) is equivalent to problem (1.1) to (1.3), as one easily sees.

The next theorem is stated without proof. For ω as in (3.6) we find that Eq. (3.5) implies problem (3.7).

Theorem A.5. *Let* $\omega:\mathbb{R} \to H$ *be bounded and continuous, except for a possible jump discontinuity at* $x=0$. *Let* $\omega(x) \in D(T)$ *and* $T\omega$ *be strongly differentiable on* $\mathbb{R}\backslash\{0\}$. *Then a solution* $\psi \in L_\infty(H)_{-\infty}^\infty$ *of the Wiener-Hopf equation*

$$f(x) - \int_{-\infty}^\infty \mathcal{H}(x-y)Bf(y)dy = \omega(x) \quad (x \in \mathbb{R}) \qquad (A.5)$$

is bounded and continuous on R, *except possibly for a jump discontinuity at* $x = 0$ *of size*

$$\psi(0^+) - \psi(0^-) = \omega(0^+) - \omega(0^-),$$

and satisfies the vector-valued differential equation

$$(Tf)'(x) = -Af(x)+(T\omega)'(x)+\omega(x) \quad (0 \neq x \in \mathbb{R}).$$

This result can easily be deduced from Theorem A.4 by decomposing (A.5) in equations on $(0,\infty)$ and $(-\infty,0)$.

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ERROR BOUNDS FOR SOME CHARACTERISTIC METHODS IN AN EXCEPTIONAL CASE

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ABSTRACT

New characteristic methods for the solution of the x,y geometry discrete ordinates neutron transport equation have recently been introduced. Five polynomials, without any continuity requirement are defined on each mesh cell. A polynomial of order k is used to approximate the angular flux inside the cell, while polynomial approximations of order ℓ are used along the cell edges.

Error bounds for a pure absorber calculation by this $Ck\ell$ characteristic method are given here for ℓ lower or equal to 1 and for the simplest case. In this case, that we shall call the exceptional case, a uniform spatial mesh grid with rectangles of length Δx and height Δy is used, and the angular quadrature directions $\omega = (\mu,\nu)$ verify the condition $\left|\frac{\mu\Delta y}{\nu\Delta x}\right| = 1$ (the characteristic lines are the diagonals of the cells).

It is proved that, in a discrete L^2 norm, the $Ck\ell$ method has a convergence rate equal to Min $(k+2, 2\ell+1)$ for regular data and solution, and equal to Min $(k+1, \frac{\ell}{2}+1)$ for more realistic situations. We also provide some numerical results that show that the asymptotic values of the computed convergence rates are identical to the theoretical ones.

INTRODUCTION

The first formulation of a characteristic method which preserves flux spatial moments and satisfies a balance equation is due to Lathrop [1]. But Lathrop's step characteristic scheme, which

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