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Abstract Kinetic Equations Relevant to Supercritical Media

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The abstract Hilbert space equation $(Tf)'(x) = -(Af)(x)$, $x \in \mathbb{R}_+$, is studied with a partial range boundary condition $(Q_+ f)(0) = f_+ \in \operatorname{Ran} Q_+$. Here T is bounded, injective and self-adjoint, A is Fredholm and self-adjoint, with finite-dimensional negative part, and Q_+ is the orthogonal projection onto the maximal T-positive T-invariant subspace. This models half-space stationary transport problems in supercritical media. A complete existence and uniqueness theory is developed.

I. INTRODUCTION

Considerable effort in linear transport theory has gone into the study of various equations of the form

$$(T\psi)'(x) = -A\psi(x), \qquad 0 < x < \infty, \tag{1}$$

with partial range boundary conditions

$$(Q_+\psi)(0) = f_+, \qquad \|\psi(x)\| = O(1) \quad (x \to \infty) \tag{2}$$

for Q_+ an appropriate projection onto "ingoing fluxes." The specific examples studied represent a variety of transport phenomena, usually under

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steady-state conditions, including neutrons, electrons, rarefied gases, phonons, etc. In addition to Hangelbroek's pioneering investigation of the neutron transport equation with isotropic scattering [18], we note particularly Ball [1], Beals [2, 3], Greenberg [1, 16, 15], Hangelbroek [19], Lekkerkerker [25], van der Mee [16, 17, 28] and Zweifel [17, 35]. In each of these publications, T and A are either specific or abstract self-adjoint operators on a Hilbert space H, the null space of T is trivial, and A is Fredholm. Moreover, with the exception of the Ball–Greenberg study [1], in all of these the operator A was assumed positive, and in most cases with zero null space.[1] This restriction to positive A excludes all applications to super-critical (we borrow the terminology of neutron transport) media a priori.

In this article we drop the positivity restriction on A. More precisely, we treat the abstract Hilbert space equation (1), with T bounded injective and A Fredholm with finite negative part (both self-adjoint). The completion H_A of the domain of A with respect to the form

$$(x, y)_A = (Ax, y) \qquad (x, y \in D(A)) \tag{3}$$

now is an indefinite inner product space with finite-dimensional nonpositive part. If A is injective, then H_A will be nondegenerate (i.e., the set of vectors $x \in H_A$ with $(x, y)_A = 0$ for every $y \in H_A$ is trivial) and therefore H_A will be a Pontryagin space (cf. [4, Chap. IX]). We note that in this case the operator $A^{-1}T$ is self-adjoint with respect to (3). Except for finitely many eigenvalues off the real axis with finite algebraic multiplicity, the spectrum of $T^{-1}A$ is real and $T^{-1}A$ allows a spectral function, as do the usual self-adjoint operators (see [21, 22]). The major possible complication, in addition to the nonreal eigenvalues, arises from the finitely many real eigenvalues where the spectral function is unbounded. Throughout all of this paper we shall assume the absence of such "irregular critical points" and this assumption we shall call the T-regularity of A.

Unlike the Ball–Greenberg paper [1], which deals with a specific example from neutron transport, we shall analyze the abstract half-space problem (1)–(2), rather than the spectral structure of the operator $T^{-1}A$. Assuming A to be T-regular, we provide a complete existence and uniqueness theory for the abstract boundary value problem. In a natural way we arrive at a reduction of this problem into two more or less independent subproblems:

(i) a half-space problem of the form (1)–(2), where A is replaced by the strictly positive operator A_β that coincides with A on a subspace of H of finite co-dimension.

[1] Results for the problem treated in [1] were claimed independently by R. Hangelbroek (oral presentation, "Fourth National Conference on Transport Theory," 1975) but not published.

(ii) a finite-dimensional evolution equation on a subspace $Z(K)$, which admits an elementary solution.

For positive A a reduction of this type was exploited previously in [28, 17, 16]. If A is not T-regular, such reduction is impossible.

The first subproblem can be solved in two alternative ways, imposing different assumptions on T, A and the type of solution. Under general assumptions on A the method of Beals [2] leads to solutions in a suitable extension of the original Hilbert space H. It was first developed for bounded A in [2] and generalized to unbounded A in [3, 16, 17]. Under the assumption that $I - A$ is a compact operator taking its values in the range of $|T|^{\alpha}$ for some $0 < \alpha < 1$, the method of van der Mee [28] yields solutions in the original Hilbert space H (strong solutions) rather than in some extension of H (weak solutions). One-speed and symmetric multigroup neutron transport satisfy the hypotheses of [28].

The second subproblem turns out to be an analysis of certain maximal positive and negative subspaces of $Z(K)$ with respect to the indefinite inner product

$$[u, v] = (Tu, v). \tag{4}$$

A close connection will be demonstrated between the solvability of the half-space problem (1)–(2) and the sign characteristics (with respect to (4)) of the restriction of $T^{-1}A$ to $Z(K)$. This method of characterizing self-adjoint matrices (with respect to an indefinite inner product) is due to Gohberg, Lancaster, and Rodman [11–14]. After the full analysis of both subproblems one obtains the solution of the problem (1)–(2) as a corollary. It turns out that for invertible A the half-space problem (1)–(2) has a unique solution for every $f_+ \in \operatorname{Ran} Q_+$ if and only if A is positive. For specific choices of A both the uniqueness and the existence of solutions may break down, which does not occur for positive A (see [28, 17]).

The method sketched above is a generalization of previous work by Zweifel and the authors (cf. [17]; also [28]), which applies to bounded T and positive A. The modifications required for unbounded T (but positive unbounded A) were explained in [16], but throughout we shall consider bounded T only. A new aspect of the present generalization is the connection between existence and uniqueness theory and sign characteristics. (For positive A this aspect is not useful.) The result will be applied to one-speed and symmetric multigroup neutron transport.

Section II contains preliminaries and decompositions needed to accomplish the reduction of the half-space problem to the above subproblems. Both of these subproblems are solved in Section III. The fourth section contains solvability criteria in terms of sign characteristics. Sections V and VI are devoted to applications, and a comparison with recent results is drawn in the last section.

II. Decompositions

Let T be a bounded self-adjoint operator on a complex Hilbert space H whose null space Ker T is trivial. Let A be a self-adjoint Fredholm operator on H with finite-dimensional negative part; i.e., the spectrum of A on $(-\infty, 0]$ consists of a finite number of eigenvalues of finite multiplicity. By the Fredholm assumption $K = T^{-1}A$ is densely defined. Since K is closed (which follows from the boundedness of T), the adjoint K^* is densely defined, while $K^* \supseteq AT^{-1}$. Because K is densely defined, the operator K^* is closed.

Given a linear operator S and a complex number λ, we define the λ root linear manifold $Z_\lambda(S)$ by

$$Z_\lambda(S) = \{x \in D(S) \mid \exists n \in \mathbb{N} : x \in D(S^n), (S-\lambda)^n x = 0\}.$$

If λ is not an eigenvalue of S, then, of course, $Z_\lambda(S) = \{0\}$.

Let $\lambda_1, ..., \lambda_k$ be the distinct non- (strictly) positive eigenvalues of A. Put

$$Z_{(0,\infty)}(A) = \left\{ \bigoplus_{j=1}^{k} Z_{\lambda_j}(A) \right\}^{\perp}.$$

Then this subspace has finite co-dimension in H, is A-invariant and on this subspace the operator A is positive, injective, and has closed range. Also

$$D(A) = D(A|_{Z_{(0,\infty)}(A)}) \oplus \left\{ \bigoplus_{j=1}^{k} Z_{\lambda_j}(A) \right\}.$$

Now consider the (indefinite) inner product

$$(x, y)_A = (Ax, y) \qquad (x, y \in D(A)). \tag{5}$$

Let us denote by H_A the direct sum of the k finite-dimensional root manifolds $Z_{\lambda_j}(A)$ $(j = 1, ..., k)$ and the completion Z_A of $D(A \mid Z_{(0,\infty)}(A))$ with respect to the inner product (5). (Note that (5) is positive definite on $D(A \mid Z_{(0,\infty)}(A))$). Then H_A can be written as the direct sum of $(\cdot, \cdot)_A$-orthogonal subspaces,

$$H_A = \left\{ \bigoplus_{\substack{j=1 \\ \lambda_j < 0}}^{k} Z_{\lambda_j}(A) \right\} \oplus Z_0(A) \oplus Z_A. \tag{6}$$

In this direct sum the first constituent subspace is strictly negative with respect to (5), the subspace $Z_0(A) = \operatorname{Ker} A$ is $(\cdot, \cdot)_A$-neutral (i.e., all vectors $x \in Z_0(A)$ satisfy $(x, x)_A = 0$) and Z_A is strictly positive. We remark that $H_A \subseteq H$.

It is straightforward to exploit the boundedness of T and the closed range of A to prove that $T^{-1}A$ is self-adjoint (and not just symmetric) with respect to the inner product (5) of H_A. First we suppose that $\operatorname{Ker} A = \{0\}$. Then H_A is a Pontryagin space with fundamental decomposition (6), where $Z_0(A) = \{0\}$ (cf. [4, Chap. IX]). Standard Pontryagin space theory implies the following properties of $T^{-1}A$:

(i) The nonreal spectrum of $T^{-1}A$ is symmetric with respect to the real line and consists of finitely many eigenvalues of finite algebraic multiplicity. If $\lambda \notin \mathbb{R}$ is such an eigenvalue, then the Jordan structure of $T^{-1}A$ at λ coincides with the Jordan structure of $T^{-1}A$ at $\bar{\lambda}$.

(ii) The sum of the algebraic multiplicities of the eigenvalues of $T^{-1}A$ in the open upper (or lower) half-plane does not exceed the dimension κ of a maximal negative subspace of H_A.

(iii) The behavior of the real spectrum of $T^{-1}A$ can be described by a spectral function E from the real line into the bounded projections on H_A, which is monotonically nondecreasing except at finitely many so-called *critical points.*

(iv) The length of a Jordan chain at a real eigenvalue of $T^{-1}A$ does not exceed $2\kappa + 1$.

(v) All critical points are eigenvalues of $T^{-1}A$. They can be divided into the *regular critical points*, where the spectral function is bounded and has a jump discontinuity (in the strong operator topology), and the *irregular critical points*, where the spectral function is unbounded.

(vi) A real eigenvalue λ of $T^{-1}A$ is an irregular critical point if and only if

$$Z_\lambda(T^{-1}A) \cap \{x \in H_A \mid (x, y)_A = 0 \text{ for all } y \in Z_\lambda(T^{-1}A)\} \neq \{0\}.$$

The spectral function has been analyzed in [21, 22]; the elementary properties that do not involve the spectral function can be found in [4].

If $\operatorname{Ker} A$ is nontrivial, Eq. (4) still represents a fundamental decomposition of H_A, but now $Z_0(A) \neq \{0\}$ and H_A is not a Pontryagin space. It is no longer possible to formulate a spectral theorem for $T^{-1}A$ directly. Let us call the operator A *T-regular at the point* $\lambda = 0$ if the zero root linear manifold $Z_0(T^{-1}A)$ has finite dimension and

$$Z_0(T^{-1}A) \cap \{x \in H_A \mid (x, y)_A = 0 \text{ for all } y \in Z_0(T^{-1}A)\} = \{0\}. \tag{7}$$

From Eq. (7) and a simple dimension argument one finds that

$$Z_0(T^{-1}A) \oplus Z_0(AT^{-1})^\perp = H, \tag{8a}$$

$$Z_0(AT^{-1}) \oplus Z_0(T^{-1}A)^\perp = H. \tag{8b}$$

Using that $Z_0(T^{-1}A) \subseteq D(A)$, one also finds

$$Z_0(T^{-1}A) \oplus \overline{Z_0(AT^{-1})^{\perp} \cap D(A)}^{(H_A)} = H_A, \tag{8c}$$

where the closure is taken with respect to the topology of H_A. Because the neutral part $\operatorname{Ker} A = Z_0(A)$ of the fundamental decomposition (6) of H_A is contained in $Z_0(T^{-1}A)$, the closure of $Z_0(AT^{-1})^{\perp} \cap D(A)$ in H_A is a Pontryagin space with respect to $(\cdot, \cdot)_A$, in which $T^{-1}A$ is self-adjoint. The restriction of $T^{-1}A$ to this complement of $Z_0(T^{-1}A)$ in H_A now has properties (i)–(vi). We call A *T-regular* if A is T-regular at the point $\lambda = 0$ and if the restriction of $T^{-1}A$ to the closure of $Z_0(AT^{-1})^{\perp} \cap D(A)$ in H_A does not have irregular critical points.

In case A is positive, all zero Jordan chains of $T^{-1}A$ have length one or two (see [17, Lemma 1]) and therefore $Z_0(T^{-1}A)$ has finite dimension. Further, A is T-regular at $\lambda = 0$. To see this, take $x \in Z_0(T^{-1}A)$ such that $(x, y)_A = 0$ for all $y \in Z_0(T^{-1}A)$. Then the positivity of A entails $x \in \operatorname{Ker} A$. Also $x \in Z_0(AT^{-1})^{\perp} \subseteq (\operatorname{Ker} K^*)^{\perp} = \overline{\operatorname{Ran} K}$, where $K = T^{-1}A$. Using that T is bounded and A has closed range, one finds $x = T^{-1}Ay$ for some $y \in Z_0(T^{-1}A)$. Now $(Ay, y) = (Tx, y) = (x, Ty) = 0$ and therefore $Tx = Ay = 0$ and $x = 0$, which proves the T-regularity at $\lambda = 0$. Notice that on the closure of $Z_0(AT^{-1})^{\perp} \cap D(A)$ in H_A the operator $T^{-1}A$ is self-adjoint in the positive definite inner product $(\cdot, \cdot)_A$ and hence does not have irregular critical points. We conclude that A always is T-regular if A is positive. For this reason the condition of T-regularity does not appear in [17].

PROPOSITION 1. *Let A be T-regular. Then there exists a $T^{-1}A$-invariant subspace $Z(K)$ of H of finite dimension with the following properties:*

(a)
$$Z(K) \oplus (T[Z(K)])^{\perp} = H; \tag{9a}$$
$$T[Z(K)] \oplus Z(K)^{\perp} = H. \tag{9b}$$

(b) *The subspace $Z(K)$ is contained in the domain of A and*

$$Z(K) \oplus (\{T[Z(K)]\}^{\perp} \cap H_A) = H_A. \tag{9c}$$

(c) *The subspace $\{T[Z(K)]\}^{\perp} \cap H_A$ is $T^{-1}A$-invariant and strictly positive with respect to $(\cdot, \cdot)_A$.*

Proof. We assume that A is T-regular. Put

$$Z = \bigoplus_{\lambda} Z_\lambda(K),$$

where the sum is taken over all nonreal eigenvalues, all nonzero critical points and the zero eigenvalue of $K = T^{-1}A$. Let κ denote the dimension of a maximal (strictly) negative subspace of H_A^1, which is (temporarily) defined as the closure in H_A of $Z_0(AT^{-1})^\perp \cap D(A)$. Then the restriction of $T^{-1}A$ to H_A^1 is a self-adjoint operator on the Pontryagin space H_A^1 (with respect to $(\cdot,\cdot)_A$), which has properties (i)–(vi) and does not have irregular critical points. Let $\bar{\kappa}$ denote the sum of the algebraic multiplicities of the eigenvalues of $T^{-1}A$ in the open upper (or lower) half-plane. Because there are no irregular critical points, the subspaces $Z_\lambda(K)$, with λ running through all nonzero critical points, are Pontryagin spaces with dimension of maximal negative subspace $\kappa(\lambda)$, which satisfy

$$\sum_\lambda \kappa(\lambda) = \kappa - \bar{\kappa}$$

(cf. [21, 22]). For every critical point λ we choose a maximal negative $T^{-1}A$-invariant subspace M_λ of $Z_\lambda(K)$. Because of Pontryagin's theorem (see [30]; also [4]) such a subspace indeed exists. Now we extend M_λ to the $T^{-1}A$-invariant subspace N_λ of $Z_\lambda(K)$ spanned by the maximal Jordan chains of $T^{-1}A$ corresponding to the eigenvalue λ which have at least one vector in M_λ. Property (iv) now implies that

$$\dim N_\lambda \leqslant (2\kappa+1)\dim M_\lambda = (2\kappa+1)\,\kappa(\lambda).$$

Next we put

$$Z(K) = \left\{\bigoplus_\zeta Z_\zeta(K)\right\} \oplus \left\{\bigoplus_\lambda N_\lambda\right\} \oplus Z_0(K), \tag{10}$$

where ζ runs through the nonreal eigenvalues and λ through the nonzero critical points of $T^{-1}A$. In this way we find

$$\begin{aligned}\dim Z(K) &= 2\bar{\kappa} + \sum_\lambda \dim N_\lambda + \dim Z_0(K)\\ &\leqslant 2\bar{\kappa} + (2\kappa+1)(\kappa-\bar{\kappa}) + \dim Z_0(K) < \infty.\end{aligned}$$

Obviously, $Z(K)$ is $T^{-1}A$-invariant and contained in $D(A)$.

Next we exploit the computational rule

$$(M_1 \oplus M_2)^\perp = M_1^\perp \cap M_2^\perp$$

for closed subspaces and the T-regularity of A (namely, Eq. (7) and the converse of property (vi)) and obtain

$$Z(K) \cap (T[Z(K)])^\perp = \{0\}.$$

Simple dimension arguments imply the decompositions (9a)–(9c).

Finally, we notice that (i) the constituent subspaces in the decomposition

(10) are $(\cdot,\cdot)_A$-orthogonal, (ii) the subspace $\{\oplus_\zeta Z_\zeta(K)\} \oplus \{\oplus_\lambda N_\lambda\}$ is a Pontryagin space with maximal negative subspace of dimension $\bar{\kappa} + \sum_\lambda \kappa(\lambda) = \kappa$, and (iii) $A[Z(K)] \subseteq T[Z(K)]$, and therefore $Z(K)$ and $(T[Z(K)])^\perp \cap H_A$ are $(\cdot,\cdot)_A$-orthogonal. Because

$$(T[Z(K)])^\perp \cap H_A \subseteq \overline{Z_0(AT^{-1})^\perp \cap D(A)}^{(H_A)}$$

and the right-hand side represents a Pontryagin space with maximal negative subspace of dimension κ, the subspace $T[Z(K)]^\perp \cap H_A$ necessarily is strictly positive with respect to $(\cdot,\cdot)_A$. ∎

The construction of $Z(K)$ given above is by no means unique. This is due to the non-uniqueness of the maximal $(\cdot,\cdot)_A$-negative $T^{-1}A$-invariant subspace M_λ of $Z_\lambda(K)$. However, the dimension $\kappa(\lambda)$ of M_λ is independent of the specific choice of M_λ.

If decompositions of the type (9a)–(9c) existed for given operators T and A, then $T^{-1}A$ would be the direct sum of a self-adjoint Hilbert space operator and an operator on a finite-dimensional space, which excludes the occurrence of irregular critical points (see [21, 22]). Thus the T-regularity of A cannot be dropped in this proposition. As shown by Jonas and Langer (see [23]), it is possible to construct T and A in such a way that $\operatorname{Ker} A = \{0\}$ and $I - A$ has rank one, but A is not T-regular.

We now generalize a construction from [17, 28] that will facilitate the reduction of the half-space problem (1)–(2) to one with strictly positive A.

PROPOSITION 2. *Let A be T-regular, and let β be an invertible operator on $Z(K)$ that is positive with respect to the indefinite inner product $[x, y] = (Tx, y)$. Denote by P the projection of H onto $(T[Z(K)])^\perp$ along $Z(K)$. Then*

$$A_\beta = T\beta^{-1}(I-P) + AP \tag{11a}$$

is strictly positive with respect to the original inner product $(\cdot,\cdot)$ of H, while

$$A_\beta^{-1}T = \beta \oplus (T^{-1}A\,|_{(T[Z(K)])^\perp}). \tag{11b}$$

Moreover, if $I - A$ is a compact operator with $\operatorname{Ran}(I-A) \subseteq \operatorname{Ran} |T|^\alpha$ for some $0 < \alpha < 1$, then $I - A_\beta$ also is a compact operator such that $\operatorname{Ran}(I - A_\beta) \subseteq \operatorname{Ran} |T|^\alpha$.

Proof. Equation (11b) is immediate from Eq. (11a). Because P is the projection of H onto $(T[Z(K)])^\perp$ along $Z(K)$ and $A[Z(K)] \subseteq (T[Z(K)])$, one finds

$$(A_\beta x, x) = [\beta^{-1}(I-P)x, (I-P)x] + (APx, Px), \qquad x \in H.$$

As β is positive with respect to $[\cdot, \cdot]$ and $(T[Z(K)])^{\perp}$ is strictly positive with respect to $(\cdot, \cdot)_A$, the operator A_β is strictly positive.

Let $I-A$ be compact and let $\operatorname{Ran}(I-A) \subseteq \operatorname{Ran}|T|^\alpha$ for some $0<\alpha<1$. Then $I-A_\beta$ is a compact operator. To prove that $\operatorname{Ran}(I-A_\beta) \subseteq \operatorname{Ran}|T|^\alpha$, it suffices to show that $Z_\lambda(K) \subseteq \operatorname{Ran}|T|^\alpha$ for every eigenvalue λ of $T^{-1}A$. First suppose $Ax=\lambda Tx$; then $x=(I-A)x+\lambda Tx \in \operatorname{Ran}|T|^\alpha$, which proves that $\operatorname{Ker}(T^{-1}A-\lambda) \subseteq \operatorname{Ran}|T|^\alpha$. Assuming $\operatorname{Ker}(T^{-1}A-\lambda)^n \subseteq \operatorname{Ran}|T|^\alpha$, we suppose $(T^{-1}A-\lambda)^{n+1}y=0$. Then for $z \in \operatorname{Ker}(T^{-1}A-\lambda)^n$ one has $(A-\lambda T)y=Tz$, and therefore

$$y=(I-A)y+\lambda Ty+Tz \in \operatorname{Ran}|T|^\alpha.$$

Hence, $\operatorname{Ker}(T^{-1}A-\lambda)^{n+1} \subseteq \operatorname{Ran}|T|^\alpha$. Thus we have shown that $Z_\lambda(K) \subseteq \operatorname{Ran}|T|^\alpha$. ∎

We remark that $D(A_\beta)=D(A)$ and $A_\beta^{-1}T$ is a bounded self-adjoint operator on H_A with respect to the (positive definite) inner product

$$(x, y)_{A_\beta}=(A_\beta x, y) \qquad (x, y \in D(A)). \tag{12}$$

Since the operators A_β and A coincide on the subspace of H of finite codimension $(T[Z(K)])^{\perp}$, all inner products $(\cdot, \cdot)_{A_\beta}$ are equivalent on $D(A)$. Hence, all these inner products make H_A into the completion of $D(A)$.

Now let us define the projection Q_+ appearing in Eq. (2). Because T is self-adjoint with zero null space, the $(\cdot, \cdot)$-orthogonal projection of H onto the maximal $(\cdot, \cdot)$-positive/negative invariant subspace of T can be defined uniquely. We shall denote this projection by $Q_\pm$. We notice that $Q_\pm$ are complementary projections.

Let us construct analogous projections for $A_\beta^{-1}T$. Since $A_\beta^{-1}T$ is self-adjoint with zero null space (with respect to (12)), the $(\cdot, \cdot)_{A_\beta}$-orthogonal projections $P_\pm$ of H_A onto the maximal $(\cdot, \cdot)_{A_\beta}$-positive/negative invariant subspace of $A_\beta^{-1}T$ are complementary. The projection P of H onto $(T[Z(K)])^{\perp}$ along $Z(K)$ also maps H_A along $Z(K)$ $(\subseteq H_A)$ onto the closed subspace $(T[Z(K)])^{\perp} \cap H_A$ of H_A (cf. Eq. (9c)). Since the decompositions (9a) and (9c) both reduce $A_\beta^{-1}T$ and the restriction of $A_\beta^{-1}T$ to $(T[Z(K)])^{\perp}$ does not depend on β (cf. Eq. (11b)), the operators $PP_\pm$ on H_A are disjoint projections that do not depend on β. In fact, PP_+, PP_-, and $I-P$ form a set of three disjoint and complementary projections on H_A.

PROPOSITION 3. *The subspace*

$$M_\pm=[\operatorname{Ran} PP_\mp \oplus \operatorname{Ran} Q_\pm] \cap Z(K)$$

is a maximal strictly positive/negative subspace of $Z(K)$ *with respect to the indefinite inner product* $[x, y] = (Tx, y)$ *and*

$$(M_\pm)^\perp = T[M_\mp] \oplus Z(K)^\perp.$$

Proof. Take $x \in [\operatorname{Ran} PP_- \oplus \operatorname{Ran} Q_+] \cap Z(K)$. Then there exist $y \in \operatorname{Ran} PP_- \subseteq \operatorname{Ran} P_-$ and $z \in \operatorname{Ran} Q_+$ such that $x = y + z$. Then

$$0 \leqslant (Tz, z) = (Tx, x) + (Ty, y) - (Tx, y) - (Ty, x).$$

As $Tx \in T[Z(K)]$ and $y \in (T[Z(K)])^\perp$, one has $(Tx, y) = (x, Ty) = 0$. Moreover, $y \in \operatorname{Ran} P_-$ implies

$$(Ty, y) = (A_\beta^{-1} Ty, y)_{A_\beta} \leqslant 0, \tag{13}$$

and therefore $(Tx, x) \geqslant 0$. If (Tx, x) were zero, both (Tz, z) and (Ty, y) would vanish, which implies $y = z = 0$ and $x = 0$. We then conclude that $[\operatorname{Ran} PP_- \oplus \operatorname{Ran} Q_+] \cap Z(K)$ is strictly positive. Similarly one shows that $[\operatorname{Ran} PP_+ \oplus \operatorname{Ran} Q_-] \cap Z(K)$ is strictly negative.

We now calculate that

$$(\operatorname{Ran} PP_\mp)^\perp = \overline{T[\operatorname{Ran} PP_\pm]} \oplus T[Z(K)], \tag{14}$$

$$(\operatorname{Ran} Q_\pm)^\perp = \operatorname{Ran} Q_\mp = \overline{T[\operatorname{Ran} Q_\mp]}. \tag{15}$$

Therefore,

$$(M_\pm)^\perp = [(\operatorname{Ran} PP_\mp)^\perp \cap (\operatorname{Ran} Q_\pm)^\perp] \oplus Z(K)^\perp = T[M_\mp] \oplus Z(K)^\perp.$$

Hence,

$$(M_+ \oplus M_-)^\perp = (M_+)^\perp \cap (M_-)^\perp = Z(K)^\perp.$$

This implies

$$M_+ \oplus M_- = Z(K), \tag{16}$$

and thus $M_\pm$ is a maximal $[\cdot, \cdot]$-positive/negative subspace of $Z(K)$. ∎

Because $(M_\pm)^\perp = T[M_\mp] \oplus Z(K)^\perp$, the decomposition (16) is a fundamental decomposition of $Z(K)$ with respect to $[\cdot, \cdot]$. With respect to this inner product $Z(K)$ is nondegenerate.

III. Half-Space Solvability

The *(right) half-space problem* is the partial range boundary value problem (1)–(2) where f_+ is a given vector in Ran Q_+. The usual method of solution consists of the determination of an albedo operator (first introduced in [24]) satisfying $Ef_+ = \psi(0)$, after which Eqs. (1)–(2) can be solved with the help of standard semigroup theory. To define E on the whole space (rather than Ran Q_+) one also considers the *left half-space problem*

$$(T\psi)'(x) = -A\psi(x), \qquad -\infty < x < 0, \tag{17a}$$

$$Q_-\psi(0) = f_-, \qquad \|\psi(x)\| = O(1) \quad (x \to -\infty), \tag{17b}$$

where $f_- \in \operatorname{Ran} Q_-$, and defines $Ef_- = \psi(0)$. Modified right/left half-space problems are defined by requiring the solution $\psi(x)$ to vanish for $x \to \pm\infty$:

$$\lim_{x \to \pm\infty} \|\psi(x)\| = 0. \tag{18}$$

Albedo operators (or related concepts) offer interesting perspectives if the operator A is positive (cf., e.g., [24, 18, 2, 28, 3, 17]), because in this case half-space problems are usually uniquely solvable. As we shall see, for nonpositive A the unique solvability is in fact lost so that albedo operators lose their usefulness. Instead we shall describe the measures of non-uniqueness and noncompleteness for the solution of these problems. Put

$$\mathbb{K}_\pm = \bigoplus_\lambda Z_\lambda(K) \bigoplus_\zeta \operatorname{Ker}(K-\zeta), \qquad \mathbb{L}_\pm = \bigoplus_\lambda Z_\lambda(K) \bigoplus_\zeta (K-\zeta)[Z_\zeta(K)],$$

where λ runs through all eigenvalues of $T^{-1}A\,|_{Z(K)}$ in the open right/left half-plane and ζ through all eigenvalues of this operator on the imaginary axis. In order that the solution $\psi(x)$ of problem (1)–(2) (resp. (17a)–(17b)) be bounded at $+\infty$ (resp. $-\infty$), the initial value $\psi(0)$ has to belong to $\operatorname{Ran} PP_\pm \oplus \mathbb{K}_\pm$. In order that the solution of Eqs. (1)–(2) (resp. (17a)–(17b)) vanish at $+\infty$ (resp. $-\infty$), one has to choose $\psi(0)$ in the subspace $\operatorname{Ran} PP_\pm \oplus \mathbb{K}^0_\pm$, where

$$\mathbb{K}^0_\pm = \bigoplus_\lambda Z_\lambda(K)$$

and λ runs through the eigenvalues of $T^{-1}A\,|_{Z(K)}$ in the open right/left half-plane. For later use we derive

$$(\mathbb{K}_\pm)^\perp = T[\mathbb{L}_\mp] \oplus Z(K)^\perp, \tag{19a}$$

$$(\mathbb{K}^0_\pm)^\perp = T[\mathbb{L}^0_\mp] \oplus Z(K)^\perp, \tag{19b}$$

where

$$\mathbb{L}_{\pm}^{0} = \bigoplus_{\zeta} Z_{\zeta}(K)$$

and ζ runs through all eigenvalues of $T^{-1}A\,|_{Z(K)}$ in the closed right/left half-plane. Now let us reduce the right half-space problem (1)–(2) to two subproblems. Let us apply the projection P of H onto $(T[Z(K)])^{\perp}$ along $Z(K)$ to both sides of Eq. (1). We then find the following two subproblems:

$$\text{(i)} \qquad (T\psi_1)'(x) = -A\psi_1(x), \qquad 0 < x < \infty, \tag{20}$$

which must give a solution $\psi_1(x)$ in $(T[Z(K)])^{\perp}$;

$$\text{(ii)} \qquad (T\psi_0)'(x) = -A\psi_0(x), \qquad 0 < x < \infty, \tag{21}$$

which is an evolution equation for $\psi_0(x)$ on the finite-dimensional space $Z(K)$.

The full solution reads

$$\psi(x) = \psi_1(x) + \psi_0(x), \qquad 0 \leqslant x < \infty.$$

Equation (21) admits an elementary solution of the form

$$\psi_0(x) = e^{-xT^{-1}A}\psi_0(0), \qquad 0 \leqslant x < \infty,$$

where $\psi_0(0) \in \mathbb{K}_+$ if $\psi(x)$ must be bounded, and $\psi_0(0) \in \mathbb{K}_+^0$ if $\psi(x)$ must vanish at infinity. Though a correct specification of the first subproblem will follow, in principle standard semigroup theory yields

$$\psi_1(x) = e^{-xT^{-1}A}\psi_1(0), \qquad 0 \leqslant x < \infty,$$

where $\psi_1(0) \in \operatorname{Ran} PP_+$. Now let us add to Eq. (20) the dummy equation

$$(T\phi_0)'(x) = -A_\beta \phi_0(x), \qquad 0 < x < \infty, \tag{22}$$

on $Z(K)$, where A_β is given by Eq. (11a) for some $[\cdot, \cdot]$-positive β. The solutions $\phi_0(x)$ of Eq. (22) are easy to compute but do not concern us as, indeed, we shall project out this part of the solution. The relevant observation is that Eqs. (20) and (22) together can be written in the form

$$(T\phi)'(x) = -A_\beta \phi(x), \qquad 0 < x < \infty, \tag{23}$$

where A_β is strictly positive with respect to $(\cdot, \cdot)$ and $\phi = \psi_1 + \phi_0$. Our main task is the following: (i) to find the general solution of Eq. (23) that is bounded (resp. vanishes) at $+\infty$ and to project it onto $(T[Z(K)])^{\perp}$ along

$Z(K)$, (ii) to disregard the part of $\phi(x)$ obtained from projection onto $Z(K)$ along $(T[Z(K)])^{\perp}$, (iii) to solve Eq. (21) (which we did) and to write $\psi = P\phi + \psi_0$, and (iv) to fit the boundary condition $Q_+ \psi(0) = f_+$ if possible.

Let us analyze the operator differential equation (23). If $I - A$ is a compact operator such that $\operatorname{Ran}(I - A) \subseteq \operatorname{Ran} |T|^{\alpha}$ for some $0 < \alpha < 1$, then $I - A_\beta$ has the same property (see Proposition 2) and $(T, I - A_\beta)$ is a positive definite admissible pair on H in the sense of [28]. Then to every $f_+ \in \operatorname{Ran} Q_+$ there exists a unique vector function ϕ: $(0, \infty) \to H$ such that ϕ can be continuously extended to $[0, \infty)$, $T\phi$ is strongly differentiable, ϕ satisfies Eq. (23) and

$$Q_+ \phi(0) = f_+, \qquad \|\phi(x)\| = O(1) \quad (x \to +\infty);$$

this solution $\phi(x)$ vanishes for $x \to +\infty$ (see [28, Sect. IV.3]). In more general cases, however, the problem of solving Eq. (23) within the original Hilbert space H still is open. For our general abstract analysis we will employ a different solution method, which was initiated in [2] (where A and thus A_β is bounded) and further developed in [3, 17] (where A and A_β are unbounded). This method yields solutions in some extension space of H. Let us construct this extension space first. By H_T we denote the completion of H with respect to the inner product

$$(x, y)_T = (|T|\, x, y) \qquad (x, y \in H), \tag{24}$$

and by H_{K_β} the completion of H_A with respect to the inner product

$$(x, y)_{K_\beta} = (|A_\beta^{-1} T|\, x, y)_{A_\beta} \qquad (x, y \in H_A), \tag{25}$$

where $|A_\beta^{-1} T| = A_\beta^{-1} T(P_+ - P_-)$. Both inner products are positive definite. As $A_\beta^{-1} T$ does not depend on β on the closed invariant subspace $(T[Z(K)])^{\perp} \cap H_A$ of finite co-dimension in H_A, for all $[\cdot, \cdot]$-positive β the inner products (25) are equivalent on H_A and therefore we may suppress β in H_{K_β} and write H_K. It can be proved (see [17]; also [3]) that there exists a unique (but β-dependent) albedo operator $E: H_T \to H_K$, which is bounded and injective, such that

$$\phi(0) = Ef_+, \qquad \phi(x) = e^{-xT^{-1}A_\beta} Ef_+, \qquad 0 \leqslant x < \infty.$$

This operator E also acts as a bounded operator from H_T into H_T. The range of E is dense in both H_K and H_T, and its kernel is zero. The albedo operator E also generates the solution of the left half-space analogue of Eq. (20). We obtain for the solution of Eq. (20) the expression

$$\psi_1(x) = e^{-xT^{-1}A} PEf_+, \qquad 0 < x < \infty,$$

where, indeed, P extends to a bounded projection on H_K (with kernel $Z(K)$ and range the closure of $(T[Z(K)])^{\sim}$ in H_K).

Let us now consider the full right half-space problem (1)–(2), requiring solutions that either are bounded or vanish at $+\infty$. The unique solvability of the related right half-space problem (23) (with similar type of boundary conditions) implies

$$[\operatorname{Ran} PP_+ \cap \operatorname{Ran} Q_-] \subseteq \operatorname{Ran} P_+ \cap \operatorname{Ran} Q_- = \{0\} \quad \text{(uniqueness)}, \tag{26a}$$

$$\overline{(\operatorname{Ran} PP_+ \oplus Z(K)) + \operatorname{Ran} Q_-} \supseteq \overline{\operatorname{Ran} P_+ + \operatorname{Ran} Q_-} = H \quad \text{(existence)}, \tag{26b}$$

where the closure is taken in H. If $I-A$ is compact and $\operatorname{Ran}(I-A) \subseteq \operatorname{Ran} |T|^{\alpha}$ for some $0 < \alpha < 1$, in Eq. (26b) no closures are needed. We first formulate two theorems.

THEOREM 1. *Let A be T-regular. Then the right/left half-space problem* (1)–(2) (*resp.* (17a)–(17b)) *has at most one solution that is bounded at* $+\infty$ (*resp.* $-\infty$) *for every* $f_+ \in \operatorname{Ran} Q_+$ (*resp.* $f_- \in \operatorname{Ran} Q_-$) *if and only if* $\mathbb{K}_{\pm}$ *is positive/negative with respect to* $[\cdot, \cdot]$. *Problem* (1)–(2) (*resp.* (17a)–(17b)) *has at least one solution that is bounded at* $+\infty$ (*resp.* $-\infty$) *if and only if* $\mathbb{L}_{\mp}$ *is negative/positive with respect to* $[\cdot, \cdot]$.

The *measure of non-uniqueness* $\delta^{\pm}$ for the solution of the right/left half-space problem coincides with the dimension of a maximal strictly negative/positive subspace of $\mathbb{K}_{\pm}$. The *measure of noncompleteness* $\gamma^{\pm}$, being the co-dimension in $\operatorname{Ran} Q_{\pm}$ of the closure of the subspace $f_{\pm} \in \operatorname{Ran} Q_{\pm}$ for which the right/left half-space problem (1)–(2) (resp. (17a)–(17b)) has at least one bounded solution, coincides with the dimension of a maximal strictly positive/negative subspace of $\mathbb{L}_{\mp}$. These properties will be proved together with Theorem 1.

THEOREM 2. *Let A be T-regular. Then the right/left half-space problem* (1)–(2) (*resp.* (17a)–(17b)) *has at most one solution that vanishes at* $+\infty$ (*resp.* $-\infty$) *for every* $f_+ \in \operatorname{Ran} Q_+$ (*resp.* $f_- \in \operatorname{Ran} Q_-$) *if and only if* $\mathbb{K}^0_{\pm}$ *is positive/negative with respect to* $[\cdot, \cdot]$. *Problem* (1)–(2) (*resp.* (17a)–(17b)) *has at least one solution that vanishes at* $+\infty$ (*resp.* $-\infty$), *if and only if* $\mathbb{L}^0_{\mp}$ *is negative/positive with respect to* $[\cdot, \cdot]$.

The *measure of non-uniqueness* $\delta_0^{\pm}$ for the solution of the right/left half-space problem that vanishes at $\pm\infty$, equals the dimension of a maximal strictly negative/positive subspace of $\mathbb{K}^0_{\pm}$. The corresponding *measure of noncompleteness* $\gamma_0^{\pm}$ coincides with the dimension of a maximal strictly positive/negative subspace of $\mathbb{L}^0_\mp$. These properties may be proved together with Theorem 2.

We only prove the first theorem and the properties of $\delta^\pm$ and $\gamma^\pm$ below its statement. The proof of Theorem 2 and accompanying properties is analogous. First a lemma is needed.

LEMMA 1. *Put*

$$M_\pm = [\operatorname{Ran} PP_\mp \oplus \operatorname{Ran} Q_\pm] \cap Z(K). \tag{27}$$

Then $\mathbb{K}_+ \cap M_\pm$ *(resp.* $\mathbb{K}^0_+ \cap M_\pm$*) is a maximal strictly positive/negative subspace of* $\mathbb{K}_+$ *(resp.* $\mathbb{K}^0_+$*) with respect to the inner product* $[x, y] = (Tx, y)$. *Moreover,*

$$(\mathbb{K}_+ \cap M_+) \oplus (\mathbb{K}_+ \cap M_-) = \mathbb{K}_+, \qquad (\mathbb{K}^0_+ \cap M_+) \oplus (\mathbb{K}^0_+ \cap M_-) = \mathbb{K}^0_+, \tag{28}$$

$$(\mathbb{L}_- + M_+) \cap (\mathbb{L}_- + M_-) = \mathbb{L}_-, \qquad (\mathbb{L}^0_- + M_+) \cap (\mathbb{L}^0_- + M_-) = \mathbb{L}^0_-. \tag{29}$$

Proof. Consider the operator $T^{-1}A\,|_{Z(K)}$. Its spectrum consists of the different real eigenvalues $\lambda_1, \ldots, \lambda_k$ and the conjugate pairs of different nonreal eigenvalues $\mu_1, \bar{\mu}_1, \ldots, \mu_l, \bar{\mu}_l$, all of which have finite algebraic multiplicity. Clearly, the subspaces

$$Z_{\lambda_1}(T^{-1}A), \ldots, Z_{\lambda_k}(T^{-1}A),$$
$$Z_{\mu_1}(T^{-1}A) \oplus Z_{\bar{\mu}_1}(T^{-1}A), \ldots, Z_{\mu_l}(T^{-1}A) \oplus Z_{\bar{\mu}_l}(T^{-1}A)$$

are $[\cdot, \cdot]$-orthogonal. Because $M_\pm$ is a maximal strictly positive/negative subspace of $Z(K)$ with respect to $[\cdot, \cdot]$, its intersection with $Z_{\lambda_j}(T^{-1}A)$ (resp. $Z_{\mu_j}(T^{-1}A) \oplus Z_{\bar{\mu}_j}(T^{-1}A)$) must be a maximal strictly positive/negative subspace of $Z_{\lambda_j}(T^{-1}A)$ (resp. $Z_{\mu_j}(T^{-1}A) \oplus Z_{\bar{\mu}_j}(T^{-1}A)$) with respect to $[\cdot, \cdot]$. Hence, $\mathbb{K}^0_+ \cap M_+$ is a maximal strictly $[\cdot, \cdot]$-positive/-negative subspace of $\mathbb{K}^0_+$. In the same way as in [17] we may prove that $\operatorname{Ker} A \cap M_\pm$ is a maximal strictly $[\cdot, \cdot]$-positive/-negative subspace of $\operatorname{Ker} A$. Thus from the definition of $\mathbb{K}_+$ we find that $\mathbb{K}_+ \cap M_\pm$ is a maximal strictly $[\cdot, \cdot]$-positive/-negative subspace of $\mathbb{K}_+$.

Equations (28) are immediate from the contents of the previous paragraph. Using Eqs. (19b) and Proposition 3 one easily finds Eqs. (29). ∎

Proof of Theorem 1. Let us first consider the homogeneous (i.e., $f_+ = 0$) right half-space problem (1)–(2). Then $\psi(0) \in \operatorname{Ran} PP_+ \oplus \mathbb{K}_+$ and $Q_+\psi(0) = 0$ imply

$$\psi(0) \in \operatorname{Ran} Q_- \cap [\operatorname{Ran} PP_+ \oplus \mathbb{K}_+].$$

The measure of non-uniqueness δ^+ equals the dimension of the subspace $[\operatorname{Ran} PP_+ \oplus \mathbb{K}] \cap \operatorname{Ran} Q_-$, which, because of Eq. (26a), can be written as

$$\delta^+ = \dim(\mathbb{K}_+ \cap M_-),$$

where M_- is given by Eq. (27). Lemma 1 now implies that δ^+ equals the dimension of a maximal strictly negative subspace of $\mathbb{K}_+$.

Next consider the measure of noncompleteness γ^+, which is the co-dimension in $\operatorname{Ran} Q_+$ of the closure of the subspace of $f_+ \in \operatorname{Ran} Q_+$ for which Eqs. (1) and (2) have at least one solution. One finds

$$\gamma^+ = \operatorname{codim}\{\overline{[\operatorname{Ran} PP_+ \oplus \mathbb{K}_+] + \operatorname{Ran} Q_-}\}.$$

Using Eq. (26b) and the property $\dim(M/N) = \dim(N^\perp/M^\perp)$ for closed subspaces M and N, one gets

$$\gamma^+ = \dim \frac{(\operatorname{Ran} PP_+)^\perp \cap \mathbb{K}_+^\perp \cap (\operatorname{Ran} Q_-)^\perp}{(\operatorname{Ran} PP_+)^\perp \cap Z(K)^\perp \cap (\operatorname{Ran} Q_-)^\perp}.$$

With the help of Eqs. (14) and (15) one arrives at the equality

$$\gamma^+ = \dim \frac{T\{[\operatorname{Ran} PP_- \oplus \mathbb{L}_-] \cap \operatorname{Ran} Q_+\}}{T\{\operatorname{Ran} PP_- \cap \operatorname{Ran} Q_+\}}.$$

Now substituting the analog of Eq. (26a) gives

$$\begin{aligned}\gamma^+ &= \dim T\{[\operatorname{Ran} PP_- \oplus \mathbb{L}_-] \cap \operatorname{Ran} Q_+\} \\ &= \dim\{[\operatorname{Ran} PP_- \oplus \mathbb{L}_-] \cap \operatorname{Ran} Q_+\} = \dim(\mathbb{L}_- \cap M_+),\end{aligned}$$

where M_+ is given by Eq. (27). Following the method of proof of Lemma 1 one has at once that $\mathbb{L}_- \cap M_+$ is a maximal strictly positive subspace of $\mathbb{L}_-$, which establishes the theorem. ∎

IV. Sign Characteristics and Solvability

In this section the measures of non-uniqueness $\delta^\pm$ and $\delta_0^\pm$ and noncompleteness $\gamma^\pm$ and $\gamma_0^\pm$ are related to the sign characteristics of the self-adjoint "matrix" $T^{-1}A|_{Z(K)}$ with respect to the indefinite inner product $[\cdot, \cdot]$. The sign characteristics were developed in [11, 12] and a clear exposition of them can be found in [13, 14]. Let $J_1, \ldots, J_a$ be the Jordan blocks of $T^{-1}A|_{Z(K)}$ that correspond to real eigenvalues. Let $J_{a+1}, J_{a+2}, \ldots, J_{a+2b}$ be the Jordan blocks corresponding to the nonreal eigenvalues ordered in such a way that for $k = 1, \ldots, b$ the corresponding entries of J_{a+2k-1} and J_{a+2k} are complex conjugates. For $i = 1,\dots, a+2b$ let P_i be the square matrix of the same size as J_i with trailing diagonal entries $+1$ and elements 0 off the trailing diagonal. Define the block diagonal matrix

$$\mathbb{P} = \operatorname{diag}\left(\varepsilon_1 P_1,\dots, \varepsilon_a P_a, \begin{bmatrix} \varnothing & P_{a+1} \\ P_{a+2} & \varnothing \end{bmatrix},\dots, \begin{bmatrix} \varnothing & P_{a+2b-1} \\ P_{a+2b} & \varnothing \end{bmatrix}\right),$$

where $\varepsilon_1,\dots, \varepsilon_a$ are signs ± 1 which will be specified shortly. Then there is a similarity transformation $S\colon Z(K) \to \mathbb{C}^N$, $N = \dim Z(K)$, and there are certain signs $\varepsilon_1,\dots, \varepsilon_a$ such that

$$S(T^{-1}A\,|_{Z(K)})\, S^{-1} = \operatorname{diag}(J_1,\dots, J_{a+2b}), \tag{30a}$$

$$[x,y] = \langle \mathbb{P}Sx, Sy\rangle \qquad (x, y \in Z(K)), \tag{30b}$$

where $\langle \cdot, \cdot\rangle$ is the usual inner product of $\mathbb{C}^N$. We call $(J, \mathbb{P})$, where $J = \operatorname{diag}(J_1,\dots, J_{a+2b})$, the canonical form of $T^{-1}A\,|_{Z(K)}$ with respect to $[\cdot,\cdot]$. The signs $\varepsilon_1,\dots, \varepsilon_a$ do not depend on the specific choice of S and are called the sign characteristics (or just signs) of $T^{-1}A\,|_{Z(K)}$ with respect to $[\cdot,\cdot]$.

Let us reformulate the construction of sign characteristics. Instead of a similarity $S\colon Z(K) \to \mathbb{C}^N$ we employ a special Jordan basis of $T^{-1}A\,|_{Z(K)}$ which we obtain from the canonical basis of $\mathbb{C}^N$ consisting of unit vectors by applying S^{-1}. Conversely, S is fully specified by this basis. First notice that the invariant subspaces of $T^{-1}A\,|_{Z(K)}$ that correspond to the different real eigenvalue Jordan blocks, to a real and a nonreal eigenvalue Jordan block, or to different conjugate pairs of nonreal eigenvalue Jordan blocks, are $[\cdot,\cdot]$-orthogonal. To each real eigenvalue λ corresponding to a Jordan block of size n one chooses a Jordan chain $z_0, z_1,\dots, z_{n-1}$ such that

$$[z_j, z_k] = \pm\delta_{j,n-1-k},$$

where ± 1 is the corresponding sign. For each pair of conjugate nonreal eigenvalue Jordan blocks one chooses Jordan chains $x_0, x_1,\dots, x_{n-1}$ and $y_0, y_1,\dots, y_{n-1}$ such that

$$[x_i, x_j] = [y_i, y_j] = 0 \qquad (i, j = 0, 1,\dots, n-1),$$
$$[x_j, y_k] = [y_k, x_j] = \delta_{j,n-1-k}.$$

All these Jordan chains together yield a *canonical Jordan basis* of $T^{-1}A\,|_{Z(K)}$ with respect to $[\cdot,\cdot]$.

THEOREM 3. *Let $2m_1,\dots, 2m_r$ be the even sizes and $2n_1+1,\dots, 2n_s+1$ the odd sizes of the Jordan blocks of $T^{-1}A\,|_{Z(K)}$ for the positive/negative eigenvalues, and let $\varepsilon_1,\dots, \varepsilon_s$ be the signs of these Jordan blocks of odd size. Then the measure of non-uniqueness $\delta^{\pm}$ for the solutions of the right/left* *half-space problem that are bounded at* $\pm\infty$, *is the sum of the following three contributions*:

(i) *The sum of the algebraic multiplicities of the eigenvalues of* $T^{-1}A$ *in the quarter plane* $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > 0,\ \operatorname{Re}\lambda \gtrless 0\}$, *and the geometric multiplicity of the positive imaginary eigenvalues.*

(ii) *The number of negative/positive signs of zero Jordan blocks of* $T^{-1}A$ *of order one.*

(iii) $(\sum_{i=1}^{r} m_i) + (\sum_{j=1}^{s} n_j) + \sum_{\varepsilon_j = \mp 1} 1$.

The measure of non-uniqueness $\delta_0^{\pm}$ *for the solutions of the right/left half-space problem that vanish at* $\pm\infty$ *is the sum of the following two contributions*:

(i) *The sum of the algebraic multiplicities of the eigenvalues of* $T^{-1}A$ *in the quarter plane* $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > 0,\ \operatorname{Re}\lambda \gtrless 0\}$.

(ii) $(\sum_{i=1}^{r} m_i) + (\sum_{j=1}^{s} n_j) + \sum_{\varepsilon_j = \mp 1} 1$.

Proof. Let us first choose, as before, a canonical Jordan basis of $T^{-1}A\,|_{Z(K)}$ with respect to $[\cdot,\cdot]$. By $\operatorname{sp} M$ we denote the linear span of a set M. If the Jordan chain $z_0, z_1, ..., z_{n-1}$ corresponds to a real eigenvalue Jordan block, then

$$\begin{aligned}
\operatorname{sp}\{z_0, z_1, ..., z_{n-1}\} &= \operatorname{sp}\{z_0, z_{n-1}\} \oplus \operatorname{sp}\{z_1, z_{n-2}\} \\
&\qquad \oplus \cdots \oplus \operatorname{sp}\{z_{(1/2)n-1}, z_{(1/2)n}\} \quad (n \text{ even}), \\
&= \operatorname{sp}\{z_0, z_{n-1}\} \oplus \operatorname{sp}\{z_1, z_{n-2}\} \\
&= \oplus \cdots \oplus \operatorname{sp}\{z_{(1/2)n-(3/2)}, z_{(1/2)n+(1/2)}\} \\
&\qquad \oplus \operatorname{sp}\{z_{(1/2)n-(1/2)}\} \quad (n \text{ odd}),
\end{aligned} \tag{31a}$$

is a $[\cdot,\cdot]$-orthogonal decomposition. For $k = 0, 1, ...,$ integer $\frac{1}{2}(n-2)$ we have the further $[\cdot,\cdot]$-orthogonal decomposition

$$\operatorname{sp}\{z_k, z_{n-1-k}\} = \operatorname{sp}\{z_k + z_{n-1-k}\} \oplus \operatorname{sp}\{z_k - z_{n-1-k}\}, \tag{31b}$$

where

$$[z_k \pm z_{n-1-k}, z_k \pm z_{n-1-k}] = \pm 2[z_0, z_{n-1}].$$

Thus one of the subspaces in the decomposition (31b) is strictly positive and the other strictly negative. Furthermore, if n is odd, we have

$$[z_{(1/2)(n-1)}, z_{(1/2)(n-1)}] = [z_0, z_{n-1}],$$

which has the same sign as the corresponding Jordan block.

If the Jordan chains $x_0, x_1, \dots, x_{n-1}$ and $y_0, y_1, \dots, y_{n-1}$ correspond to conjugate nonreal eigenvalue Jordan blocks, then

$$\begin{aligned}
&\operatorname{sp}\{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\} \\
&\quad = \operatorname{sp}\{x_0 + y_{n-1}, x_0 - y_{n-1}\} \\
&\qquad \oplus \cdots \oplus \operatorname{sp}\{x_{n-2} + y_1, x_{n-2} - y_1\} \oplus \operatorname{sp}\{x_{n-1} + y_0, x_{n-1} - y_0\}
\end{aligned} \tag{32}$$

is a $[\cdot, \cdot]$-orthogonal decomposition into two-dimensional subspaces, for which one of the two $[\cdot, \cdot]$-orthogonal basis vectors is strictly positive and the other one strictly negative.

To calculate the dimension of a maximal strictly negative/positive subspace of $\mathbb{K}_{\pm}$, which according to the uniqueness part of Theorem 1 characterizes $\delta^{\pm}$, one adds the following contributions:

(a) all conjugate pairs of nonreal eigenvalue Jordan blocks of $T^{-1}A\,|_{Z(K)}$ in the open right/left half-plane. As shown by (32), the total contribution is the sum of the algebraic multiplicities of the eigenvalues in the quarter plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \gtrless 0,\ \operatorname{Im} \lambda \gtrless 0\}$. Because the spectrum of $T^{-1}A$ is symmetric with respect to the real line, it is also the same as the sum of the algebraic multiplicities of the eigenvalues in the quarter plane $\{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda > 0, \operatorname{Re} \lambda \gtrless 0\}$.

(b) all conjugate pairs of Jordan blocks for nonzero eigenvalues in the upper/lower imaginary line. As (32) shows, this contribution is obtained by adding the geometric multiplicities of the nonzero eigenvalues on the upper/lower (and thus upper) imaginary line.

(c) the zero Jordan blocks of order one. This contribution is the sum of the negative/positive signs of the zero Jordan blocks of order one.

(d) the Jordan blocks of even size corresponding to positive/negative eigenvalues. Their contribution is one half of the size of each such block.

(e) the Jordan blocks of odd size corresponding to positive/negative eigenvalues. If such a block has size $2n+1$ with sign ε, the contribution is n if $\varepsilon = \pm 1$, and $n+1$ if $\varepsilon = \mp 1$.

All contributions are added to find the measure of non-uniqueness $\delta^{\pm}$. The measure of non-uniqueness $\delta_0^{\pm}$ is found similarly, with the help of the uniqueness part of Theorem 2. ∎

THEOREM 4. *Let $2m_1, \dots, 2m_r$ be the even sizes and $2n_1 + 1, \dots, 2n_s + 1$ the odd sizes of the Jordan blocks of $T^{-1}A\,|_{Z(K)}$ for the negative/positive eigenvalues, and let $2\bar{m}_1, \dots, 2\bar{m}_{\bar{r}}$ be the even sizes and $2\bar{n}_1 + 1, \dots, 2\bar{n}_{\bar{s}} + 1$ the odd sizes of the zero Jordan blocks of $T^{-1}A$. Denote the signs of the Jordan* *blocks of odd size for the negative/positive eigenvalues by $\varepsilon_1,\dots,\varepsilon_s$, and those for the blocks of odd size $\geqslant 3$ for the zero eigenvalue by $\bar\varepsilon_1,\dots,\bar\varepsilon_{\bar s}$. Then the measure of noncompleteness $\gamma^\pm$ for the solutions of the right/left half-space problem that are bounded at $\pm\infty$ is given by the sum of the following three contributions*:

(i) $(\sum_{i=1}^r m_i)+(\sum_{j=1}^s n_j)+\sum_{\varepsilon_j=\pm 1} 1$;

(ii) $(\sum_{i=1}^{\bar r}(\bar m_i-1))+(\sum_{j=1}^{\bar s}(\bar n_j-1))+\sum_{\bar\varepsilon_j=\pm 1} 1$;

(iii) *the sum of the algebraic multiplicities of the eigenvalues of $T^{-1}A$ in the quarter plane $\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda>0,\ \operatorname{Re}\lambda\lesseqqgtr 0\}$ minus the sum of the geometric multiplicities of the positive imaginary eigenvalues.*

The measure of noncompleteness $\gamma_0^\pm$ for the solutions of the right/left half-space problem that vanish at $\pm\infty$ is the sum of the following three contributions:

(i) $(\sum_{i=1}^r m_i)+(\sum_{i=1}^s n_j)+\sum_{\varepsilon_j=\pm 1} 1$;

(ii) $(\sum_{i=1}^{\bar r}\bar m_i)+(\sum_{j=1}^{\bar s}\bar n_j)+\sum_{\bar\varepsilon_j=\pm 1} 1$;

(iii) *the sum of the algebraic multiplicities of the eigenvalue of $T^{-1}A$ in the quarter plane $\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda>0,\ \operatorname{Re}\lambda\lesseqqgtr 0\}$.*

Proof. According to the existence part of Theorem 1, $\gamma^\pm$ equals the dimension of a maximal strictly positive/negative subspace of $\mathbb{L}_\mp$. To compute this dimension we make use of a canonical Jordan basis of $T^{-1}A|_{Z(K)}$ with respect to $[\cdot,\cdot]$ and repeat the argument of the proof of Theorem 3. Now one has to consider the following contributions in order to calculate $\gamma^\pm$:

(a) all conjugate pairs of Jordan blocks corresponding to the nonreal eigenvalues in the open left/right half-plane. Their contribution to $\gamma^\pm$ is the sum of the algebraic multiplicities of the eigenvalues of $T^{-1}A$ in the quarter plane $\{\lambda\in\mathbb{C}\mid\operatorname{Im}\lambda>0,\ \operatorname{Re}\lambda\lesseqqgtr 0\}$ with an appropriate subtraction for purely imaginary eigenvalues;

(b) the Jordan blocks of $T^{-1}A|_{Z(K)}$ of even size corresponding to the negative/positive eigenvalues giving the contribution $\sum_{i=1}^r m_i$;

(c) the Jordan blocks of $T^{-1}A|_{Z(K)}$ of odd size corresponding to the negative/positive eigenvalues giving the contribution $\sum_{j=1}^s n_j+\sum_{\varepsilon_j=\pm 1} 1$;

(d) the zero Jordan blocks of $T^{-1}A$.

The latter blocks can be treated for each block separately. Consider a Jordan chain $x_0,\ x_1,\dots,x_{n-1}$ of some canonical Jordan basis of $T^{-1}A|_{Z(K)}$ corresponding to some zero Jordan block, and put

$$M=T^{-1}A\ \mathrm{sp}\{x_0,x_1,\dots,x_{n-1}\}=\mathrm{sp}\{x_0,x_1,\dots,x_{n-2}\}.$$

Then

$$M = \mathrm{sp}\{x_0\} \oplus \mathrm{sp}\{x_1, x_{n-2}\} \oplus \cdots \oplus \mathrm{sp}\{x_{(1/2)n-1}, x_{(1/2)n}\} \qquad (n \text{ even}),$$

$$M = \{0\} \qquad (n = 1),$$

$$M = \mathrm{sp}\{x_0\} \oplus \mathrm{sp}\{x_1, x_{n-2}\} \oplus \cdots \oplus \mathrm{sp}\{x_{(1/2)n-(3/2)}, x_{(1/2)n+(1/2)}\}$$
$$\oplus\ \mathrm{sp}\{x_{(1/2)n-(1/2)}\} \qquad (n \geqslant 3 \text{ odd}).$$

By counting the dimension of a maximal strictly positive/negative subspace of M one finds $\frac{1}{2}(n-2)$ for n even, zero for $n = 1$, $\frac{1}{2}(n-3)+1$ for $n \geqslant 3$ odd and sign ± 1, and $\frac{1}{2}(n-3)$ for $n \geqslant 3$ odd and sign ∓ 1.

The proposition can be found by adding all contributions, as far as $\gamma^{\pm}$ is concerned. The measure of noncompleteness $\gamma_0^{\pm}$ is computed similarly using the existence part of Theorem 2. ∎

COROLLARY 1. *Let A be positive. Then*

(a) $\gamma^{\pm} = 0$ *and* $\delta^{\pm}$ *is the number of negative/positive signs for zero Jordan blocks of* $T^{-1}A$ *of order one.*

(b) $\delta_0^{\pm} = 0$ *and* $\gamma_0^{\pm}$ *is the sum of the number of positive/negative signs for zero Jordan blocks of* $T^{-1}A$ *of order one and the number of zero Jordan blocks of order two.*

Proof. For A positive one may take $Z(K) = Z_0(K)$. Using that all zero Jordan chains have length one or two, one derives the corollary directly from Theorems 3 and 4. ∎

Part (a) of Corollary 1 appears as the main result of [17], although in [17] sign characteristics were avoided. Under an extra symmetry condition and assuming $I - A$ compact with range contained in $\operatorname{Ran} |T|^{\alpha}$ for some $0 < \alpha < 1$, Part (a) was obtained earlier in [28].

In order that the right/left half-space problem, with solutions required to be bounded, has a unique solution for every $f_{\pm} \in \operatorname{Ran} Q_{\pm}$, it is necessary and sufficient that $\{T[Z_0(K)]\}^{\perp} \cap H_A$ is strictly positive with respect to $(\cdot, \cdot)_A$ and all zero Jordan blocks of $T^{-1}A$ have order $\leqslant 3$, where the first order blocks have positive/negative signs and the third order blocks have negative/positive signs. In order that the right/left half-space problem, with solutions required to vanish at $\pm\infty$, has a unique solution for every $f_{\pm} \in \operatorname{Ran} Q_{\pm}$, it is necessary and sufficient that $\{T[Z_0(K)]\}^{\perp} \cap H_A$ is strictly positive with respect to $(\cdot, \cdot)_A$ and all zero Jordan blocks of $T^{-1}A$ have order one with negative/positive sign. Note that if $\operatorname{Ker} A$ is trivial, then existence and uniqueness of the solutions of the right/left half-space problem (irrespective of whether bounded solutions are required, or solutions which vanish at $\pm\infty$) occur if and only if A is (strictly) positive.

V. One-Speed Neutron Transport Equation

In the one-speed approximation and after Fourier decomposition the neutron transport equation for the mth Fourier component $f^m(x,\mu)$, $m = 0, 1, 2, \ldots$, of the angular density is written as

$$\mu \frac{\partial f^m}{\partial x}(x,\mu) + f^m(x,\mu) = \int_{-1}^{1} g^m(\mu,\mu') f^m(x,\mu')\, d\mu', \tag{33}$$

where $\mu \in [-1, 1]$ is the cosine of the angle describing the direction of propagation and x is a position coordinate in units of neutron mean free path. (For the physical background we refer to [6, 7] and for the details on the Fourier decomposition to [27].) The scattering kernel has the form

$$g^m(\mu,\mu') = \frac{c}{4\pi} \int_0^{2\pi} p(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos\alpha)\cos m\alpha\, d\alpha, \tag{34}$$

where the redistribution function p is nonnegative, belongs to $L_1[-1, 1]$ and satisfies the normalization $\int_{-1}^{1} p(t)\, dt = 2$, while $c \geqslant 0$ denotes the number of secondaries per collision. It is customary to expand $p(t)$ into a series of Legendre polynomials by putting

$$f_l = \tfrac{1}{2} \int_{-1}^{1} p(t)\, P_l(t)\, dt \qquad (l = 0, 1, 2, \ldots),$$

where $P_l(t) = (2^l l!)^{-1}(d/dt)^l (t^2-1)^l$ is the usual Legendre polynomial of degree l. In this way one may write

$$g^m(\mu,\mu') = \frac{1}{2} c \sum_{l=m}^{\infty} f_l(2l+1) \frac{(l-m)!}{(l+m)!} P_l^m(\mu)\, P_l^m(\mu'), \tag{35}$$

where

$$P_l^m(\mu) = (1-\mu^2)^{(1/2)m} \left(\frac{d}{d\mu}\right)^m P_l(\mu)$$

is an associated Legendre function.

On computing the Fourier components one writes

$$f^m(x,\mu) = (2\pi)^{-1} \int_0^{2\pi} f(x,\mu,\phi)\cos m\phi\, d\phi,$$

where $f(x,\mu,\phi)$ is the full (azimuth-dependent) angular density (cf. [27]). By physical necessity one must require that $f(x,\mu,\phi) \geqslant 0$, which imposes additional constraints on the solution of Eq. (33). Similar conditions must be imposed on the functions appearing in the boundary conditions below. However, we choose not to deal with such extra constraints and for this reason we actually investigate the existence and uniqueness of real (but often nonphysical) solutions of Eq. (33).

We study Eq. (33) with scattering kernel (34) (or (35)) on the Hilbert space $H = L_2[-1, 1]$ and define the operators T, A, Q_+, and Q_- on this space by

$$(Th)(\mu) = \mu h(\mu), \qquad (Ah)(\mu) = h(\mu) - \int_{-1}^{1} g^m(\mu, \mu')\, h(\mu')\, d\mu',$$

$$(Q_\pm h)(\mu) = h(\mu) \quad (\mu \gtrless 0), \qquad (Q_\pm h)(\mu) = 0 \quad (\mu \lesseqgtr 0).$$

For the present study it is quite sufficient only to assume that $p(t)$ is a real-valued function in $L_1[-1, 1]$, $c \in \mathbb{R}$, but still $\int_{-1}^{1} p(t)\, dt = 2$. Then T and A are bounded self-adjoint operators, T has zero null space and $Q_\pm$ is the orthogonal projection onto the maximal positive/negative T-invariant subspace. According to a result of Vladimirov [34, Appendix XII.8] the operator $I - A$ is compact, the associated Legendre functions $P_l^m(\mu)$ $(l \geqslant m)$ form a complete and orthogonal set of eigenfunctions of A, while

$$AP_l^m = (1 - cf_l)\, P_l^m, \qquad l \geqslant m. \tag{36}$$

Hence, $\sigma(A) = \{1 - cf_l \mid l \geqslant m\} \cup \{1\}$. As shown by van der Mee [28, Theorem VI.1.1], one has

$$\operatorname{Ran}(I - A) \subseteq \operatorname{Ran} |T|^\alpha$$

for all $0 < \alpha < (r-1)/2r$, whenever $p \in L_r[-1, 1]$ with $r > 1$. We thus conclude that T and A satisfy the general hypotheses of previous sections, except possibly the T-regularity of A. Because $I - A$ is compact and T is bounded, the operator A is T-regular at $\lambda = 0$.

Equation (33) on the half-line $x \in (0, \infty)$ is usually endowed with the boundary conditions

$$\lim_{x \downarrow 0} \int_0^1 |f(x, \mu) - f_+(\mu)|^2\, d\mu = 0, \qquad \int_{-1}^{1} |f(x, \mu)|^2\, d\mu = O(1) \quad (x \to \infty)$$

(corresponding to Eqs. (1) and (2)), or with the alternative set of conditions

$$\lim_{x \downarrow 0} \int_0^1 |f(x, \mu) - f_+(\mu)|^2\, d\mu = 0, \qquad \lim_{x \to \infty} \int_{-1}^{1} |f(x, \mu)|^2\, d\mu = 0$$

(corresponding to Eqs. (1)–(18)).

LEMMA 2. *Corresponding to a nonzero eigenvalue of $T^{-1}A$ there exists precisely one independent maximal Jordan chain of eigenvectors and generalized eigenvectors.*

Proof. If A is invertible (i.e., $cf_l \neq 1$ for $l \geqslant m$), an induction argument shows that for $e(\mu) = (1-\mu^2)^{(1/2)m}$ the vector $(A^{-1}T)^n e$ is the product of $e(\mu)$ and a polynomial of degree n (cf. Eqs. (35) and (36)), which means that e is a cyclic vector of $A^{-1}T$. Thus for invertible A the lemma is obvious.

Therefore let us assume that A is not invertible. Clearly the solutions of Eq. (33) that are vector polynomials in x have the form

$$\psi(x) = e^{-xT^{-1}A}h = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} x^n (T^{-1}A)^n h, \qquad 0 < x < \infty,$$

where $(T^{-1}A)^N h = 0$ for some $N \in \mathbb{N}$. (Thus $h \in Z_0(T^{-1}A)$). It is not difficult to prove (e.g., by induction on the above N) that $Z_0(T^{-1}A)$ entirely consists of products of $e(\mu) = (1-\mu^2)^{(1/2)m}$ and polynomials, simply because the associated Legendre functions have this form. The subspace $Z_0(AT^{-1})^\perp$ is $T^{-1}A$-invariant and may be obtained by taking the closure in $L_2[-1, 1]$ of the set $\mathbb{P}^m = \{h(\mu)(1-\mu^2)^{m/2} \mid h \text{ polynomial}\}$ intersected by $Z_0(AT^{-1})^\perp$. There is a unique bounded operator S on $Z_0(AT^{-1})^\perp$ such that

$$ASk = Tk, \qquad k \in Z_0(AT^{-1})^\perp.$$

The set $\mathbb{P}^m \cap Z_0(AT^{-1})^\perp$ is S-invariant.

Let $\hat{e}$ denote the (unique) function in $\mathbb{P}^m \cap Z_0(AT^{-1})^\perp$ which is a polynomial with leading coefficient 1 and minimal degree multiplied by $(1-\mu^2)^{(1/2)m}$. We shall prove that $\hat{e}$ is a cyclic vector for S, i.e.,

$$\{h(S)\hat{e} \mid h \text{ polynomial}\} = \mathbb{P}^m \cap Z_0(AT^{-1})^\perp.$$

First we note that $h(S)f = 0$ for $f \in \mathbb{P}^m$ and h a nonzero polynomial implies $f = 0$. Indeed, otherwise one has $(S-\xi)g = 0$ for some $0 \neq g \in \mathbb{P}^m$ and therefore $(T-\xi)g = -\xi(I-A)g = \hat{g} \in \mathbb{P}^m$, where $\hat{g}(\mu)(1-\mu^2)^{-(1/2)m}$ is a polynomial of degree at most the degree of the polynomial $g(\mu)(1-\mu^2)^{-m/2}$ (cf. (35)). Thus

$$(\mu - \xi)\, g(\mu) = \hat{g}(\mu)$$

belongs to $\mathbb{P}^m$, but the left- and right-hand sides are products of $(1-\mu^2)^{(1/2)m}$ and polynomials of different degrees, which is a contradiction. Hence, we may conclude that $f = 0$.

If $q \in \mathbb{P}^m$, put $\deg q$ for the degree of the polynomial $q(\mu)(1-\mu^2)^{-m/2}$. For the degree of a polynomial h we write $\deg h$. Assume that for $n \geqslant 0$,

$$\{h(S)\hat{e} \mid h \text{ polynomial with degree } \leqslant n\}$$
$$= \{q \in \mathbb{P}^m \mid \deg q \leqslant n + \deg \hat{e}\}. \tag{37}$$

Certainly, $\deg h(S)\hat{e} \leqslant \deg h + \deg \hat{e}$ for $h \neq 0$. If for some $0 \neq h \in \mathbb{P}^0$ one would have $\deg h(S)\hat{e} < \deg h + \deg \hat{e}$, then there exists a polynomial $r \neq 0$ such that $r(S)\hat{e} = 0$, which implies the false statement $\hat{e} = 0$. Thus

$$\deg h(S)\hat{e} = \deg h + \deg \hat{e}, \qquad 0 \neq h \in \mathbb{P}^0.$$

Next let $q \in \mathbb{P}^m$ and $\deg q = n + 1 + \deg \hat{e}$. For a suitable constant $c \neq 0$ the function $q - cS^{n+1}\hat{e} \in \mathbb{P}^m$ has degree strictly less than $n + 1 + \deg \hat{e}$. Using the induction hypothesis (37) one finds a polynomial r with $\deg r \leqslant n$ such that

$$q - cS^{n+1}\hat{e} = r(S)\hat{e},$$

which implies Eq. (37) with n replaced by $n + 1$. Hence, $\hat{e}$ is a cyclic vector for S. ∎

If $f_l = 0$ for $l \geqslant L + 1$, the operator $I - A$ has finite rank. Then the determinant of the operator $(T-z)^{-1}(T-zA)$, $z \notin [-1, 1]$, is well defined and

$$\det(T-z)^{-1}(T-zA) = \Lambda^m(z), \qquad z \notin [-1, 1], \tag{38}$$

where $\Lambda^m(z)$ is the *dispersion function*

$$\Lambda^m(z) = 1 + z \int_{-1}^{1} \frac{\psi^m(\mu, \mu)}{\mu - z}\, d\mu$$

and the *characteristic binomial* is given by

$$\psi^m(v, \mu) = \frac{1}{2} c \sum_{l=m}^{L} f_l(2l+1) \frac{(l-m)!}{(l+m)!}\, g_l^m(v)(1-v^2)^{(1/2)m}\, P_l^m(\mu).$$

The polynomials $g_l^m(\xi)$ satisfy the recurrence relation

$$(2l+1)(1-cf_l)\,\xi g_l^m(\xi) = (l-m+1)\, g_{l+1}^m(\xi) + (l+m)\, g_{l-1}^m(\xi). \tag{39}$$

For $m = 0$ and $c < 1$ the relationship (38) between spectral properties of $T^{-1}A$ and the well-known dispersion function of transport theory was proved by Hangelbroek [19], but his proof carries over to general m and c. As a consequence of Lemma 2 and Eq. (38), if $f_l = 0$ for $l \geqslant L + 1$, the algebraic multiplicity of an eigenvalue $1/\lambda$ of $T^{-1}A$ coincides with the order of the zero of $\Lambda^m(z)=0$ at $z=\lambda$.

Let us make a few remarks concerning history. Feldman [8] observed that with respect to the basis $(P_l^m)_{m \geqslant l}$ of $L_2[-1, 1]$ the operator $A-\lambda T$ has a matrix representation of Jacobi type, and from this observation he derived Lemma 2 (stated for $c \geqslant 1$). Shultis and Hill [31] and Case [5] have exploited the recurrence relation (39) to prove the simplicity of the finite zeros of $\Lambda^m(z)=0$ for $c \leqslant 1$. The observation (made for $m=0$) that, if $cf_l \neq 1$ for $l \geqslant m$, the vector $e(\mu)=(1-\mu^2)^{m/2}$ is a cyclic vector of $A^{-1}T$, and its major consequences are due to Hangelbroek [19] (also [25]). We emphasize that the finite zeros of $\Lambda^m(z)=0$ (and thus the nonzero eigenvalues of $T^{-1}A$) may have multiple order and be situated off the real and imaginary axes (see [29]).

In order to develop full and half range theory one must prove the absence of irregular critical points of $T^{-1}A$, and for this purpose it suffices to show the absence of eigenvalues embedded in the continuous spectrum. In view of (38), this is intimately related to the nonexistence, for $v \in [-1, 1]$, of zero limiting values of $\Lambda(v \pm i\varepsilon)$ as $\varepsilon \to 0$. Except for the case $v=\pm 1$ and $m \geqslant 1$, where zero limiting values may occur, this was shown by Garcia and Siewert [10] (for $m=0$ also [9]). Partial results were derived by Hangelbroek [19] ($m=0$, $c<1$, $-1<v<1$) and Lekkerkerker [25] ($m=0$, $c<1$, $v=\pm 1$). We shall give a new and concise proof of these results and at the same time establish the T-regularity of A.

LEMMA 3. *If $a_l=0$ for $l \geqslant L+1$ and $m=0, 1$, the operator A is T-regular.*

Proof. Let us search for nonzero solutions of Eq. (33) of the form

$$f^m(x, \mu)=e^{-x/v}\phi(v, \mu), \tag{40}$$

where $v \in [-1, 1]$. By substitution in Eq. (33) one gets

$$(v-\mu)\,\phi(v, \mu)=\frac{1}{2}cv \sum_{l=m}^{L} f_l(2l+1)\frac{(l-m)!}{(l+m)!}P_l^m(\mu) \times \int_{-1}^{1} \phi(v, \hat{\mu})\,P_l^m(\hat{\mu})\,d\hat{\mu}. \tag{41}$$

Using the three term recurrence relation for the associated Legendre functions one sees that the functions $\int_{-1}^{1} \phi(v, \hat{\mu})\,P_l^m(\hat{\mu})\,d\hat{\mu}$ $(l \geqslant m)$ satisfy the recurrence relation (39), and thus, on proper normalization, one gets

$$\int_{-1}^{1} \phi(v, \hat{\mu})\,P_l^m(\hat{\mu})\,d\hat{\mu}=g_l^m(v). \tag{42}$$

As a consequence we have

$$(v-\mu)\,\phi(v,\mu)=v\psi^m(v,\mu)\overset{\text{def}}{=}v\tilde{\psi}^m(v,\mu)(1-\mu^2)^{(1/2)m}. \tag{43}$$

For either $v\in(-1,1)$ or $m=0$, 1 the condition $\phi(v,\cdot)\in L_2[-1,1]$ implies $\tilde{\psi}^m(v,v)=0$. This in turn implies $\phi(v,\cdot)\in\mathbb{P}^m$, which contradicts the proof of Lemma 2. Hence, for either $v\in(-1,1)$ or $m=0$, 1 there cannot be any eigenvalues of S on $[-1,1]$. ∎

For $v=\pm 1$ and $m\geqslant 1$ the operator S cannot have an eigenvalue, in spite of the fact that $\Lambda^m(\pm 1)$ may vanish (see [10]). The full Hangelbroek correspondence (38) between eigenvalues of S and zeros of the dispersion function does not carry over to this case. In fact, for this case $g_L^m(1)=g_{L+1}^m(1)=g_{L+2}^m(1)=\cdots$ (see [10]), but this number is nonzero, and therefore the function $\phi(1,\mu)$ does not belong to $L_2[-1,1]$ (otherwise $\phi(1,\cdot)\in\mathbb{P}^m$). Finally, if v is a finite zero of the dispersion function outside $[-1,1]$, one may use (43a) and $\phi(v,\cdot)\notin\mathbb{P}^m$ to derive $\psi^m(v,v)\neq 0$. In this way we recover another result of Garcia and Siewert [10] (for $m=0$ also [9]), which for $m=0$ and $c<1$ was announced in [19].

If $a_l=0$ for $l\geqslant L+1$, the T-regularity of A enables us to apply the abstract theory of the previous section. This is also possible for more general cases where A still is T-regular. We note that in the most general case of anisotropic one-speed neutron transport, the T-regularity of A is an open problem.

VI. Symmetric Multigroup Transport Equation

The symmetric multigroup approximation in neutron transport with isotropic scattering leads to the coupled set of N equations

$$\mu\frac{\partial f_i}{\partial x}(x,\mu)+\sigma_i f_i(x,\mu)$$
$$=\frac{1}{2}\sum_{j=1}^{N}C_{ij}\int_{-1}^{1}f_j(x,\mu')\,d\mu' \qquad (i=1,2,\ldots,N), \tag{44}$$

where $\mu\in[-1,1]$, $f_i(x,\mu)$ is the angular density of neutrons with speed $1/\sigma_i$ (in units of the largest speed) and $\mathbf{C}=(C_{ij})_{i,j=1}^N$ is a real symmetric matrix. We take $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_N=1$, denote by $\boldsymbol{\Sigma}$ the diagonal matrix with diagonal entries $\sigma_1,\ldots,\sigma_N$, but refrain from the condition $C_{ij}\geqslant 0$ $(1\leqslant i, j\leqslant N)$ required for physical reasons. We also disregard the physical necessity that $f_i(x,\mu)$ be nonnegative $(i=1,2,\ldots,N)$.

We study Eq. (44) in the Hilbert space H of N-vectors $\mathbf{h} = (h_i)_{i=1}^N$ of functions $h_i \in L_2[-1, 1]$. This space we endow with the inner product

$$(\mathbf{h}, \mathbf{k}) = \sum_{i=1}^{N} \sigma_i \int_{-1}^{1} h_i(\mu)\, \overline{k_i(\mu)}\, d\mu, \qquad \mathbf{h} = (h_i)_{i=1}^N, \qquad \mathbf{k} = (k_i)_{i=1}^N.$$

On this space we define the operators T, A, and $Q_\pm$ by

$$(T\mathbf{h})_i(\mu) = \sigma_i^{-1} \mu h_i(\mu),$$

$$(A\mathbf{h})_i(\mu) = h_i(\mu) - \frac{1}{2} \sigma_i^{-1} \sum_{j=1}^{N} C_{ij} \int_{-1}^{1} h_j(\mu')\, d\mu',$$

$$(Q_\pm \mathbf{h})_i(\mu) = h_i(\mu) \quad (\mu \gtrless 0), \qquad (Q_\pm \mathbf{h})_i(\mu) = 0 \quad (\mu \lessgtr 0),$$

where $\mathbf{h} = (h_i)_{i=1}^N$. Then T and A are self-adjoint, $\operatorname{Ker} T = \{0\}$, $Q_\pm$ is the orthogonal projection onto the maximal positive/negative T-invariant subspace of H, $I - A$ has finite rank and satisfies the condition

$$\operatorname{Ran}(I - A) \subseteq \operatorname{Ran} |T|^\alpha, \qquad 0 < \alpha < \tfrac{1}{2}.$$

The operator A is (strictly) positive if and only if

$$(\sigma_i \delta_{ij} - C_{ij})_{i,j=1}^N$$

is (strictly) positive, and invertible if and only if the above matrix is invertible.

LEMMA 4. *The operator A is T-regular.*

Proof. Because T is bounded and $I - A$ compact, the operator A is T-regular at $\lambda = 0$. Therefore it suffices to find out if $T^{-1}A$ has any eigenvalues within its continuous spectrum $(-\infty, -1] \cup [1, \infty)$ (which coincides with $\sigma(T^{-1})$).

Let $0 \neq \mathbf{h} = (h_i)_{i=1}^N \in H$ be an eigenfunction of $T^{-1}A$ at the eigenvalue $1/\lambda$, where $0 \neq \lambda \in [-1, 1]$. We thus assume $\lambda A\mathbf{h} = T\mathbf{h}$, which may be written as

$$\begin{aligned}(\lambda\sigma_i - \mu)\, h_i(\mu) \\ = \frac{1}{2} \lambda \sum_{j=1}^{N} C_{ij} \int_{-1}^{1} h_j(\mu')\, d\mu', \qquad 1 \leqslant i \leqslant N, \quad -1 \leqslant \mu \leqslant 1. \qquad (45)\end{aligned}$$

Now observe that there exists $1 \leqslant i \leqslant N$ (e.g., $i = N$) such that $-1 \leqslant \lambda\sigma_i \leqslant 1$. Also observe that Eq. (45) implies that all functions h_i are continuous on

$[-1, 1]\backslash\{\lambda\sigma_1,..., \lambda\sigma_N\}$. Hence for all i for which $\lambda\sigma_i \in [-1, 1]$ the requirement $h_i \in L_2[-1, 1]$ necessarily entails $\frac{1}{2}\sum_{j=1}^{N} C_{ij} \int_{-1}^{1} h_j(\mu')\, d\mu' = 0$, which in turn implies $h_i(\mu) \equiv 0$. Thus $T^{-1}A$ cannot have any eigenvalue within the set $(-\infty, -\sigma_1] \cup [\sigma_1, \infty)$. Now consider $1/\lambda \notin (-\sigma_1, -1] \cup [1, \sigma_1)$, and let $\alpha = \min\{i \mid \lambda\sigma_i \notin [-1, 1]\}$. Then $h_i(\mu) \equiv 0$ for $\alpha < i \leqslant N$, while

$$h_i(\mu) = \frac{\lambda \sum_{j=1}^{\alpha} C_{ij} \int_{-1}^{1} h_j(\mu')\, d\mu'}{2(\lambda\sigma_i - \mu)}, \qquad 1 \leqslant i \leqslant \alpha.$$

Putting $\boldsymbol{\xi} = (\int_{-1}^{1} h_i(\mu)\, d\mu)_{i=1}^{\alpha}$, we find for

$$\boldsymbol{\Lambda}_\alpha(\lambda) = \left(\delta_{ij} - \frac{1}{2}\lambda \sum_{j=1}^{\alpha} C_{ij} \int_{-1}^{1} (\lambda\sigma_i - \mu)^{-1}\, d\mu\right)_{i,j=1}^{\alpha}$$

that

$$\boldsymbol{\Lambda}_\alpha(\lambda)\boldsymbol{\xi} = 0.$$

We thus conclude that λ must be an eigenvalue of a truncated symmetric multigroup problem, where the only groups to be considered are the ones with $1 \leqslant i \leqslant \alpha$. This truncated multigroup problem reads

$$\mu \frac{\partial g_i}{\partial x}(x, \mu) + \sigma_i\, g_i(x, \mu) = \frac{1}{2} \sum_{j=1}^{\alpha} C_{ij} \int_{-1}^{1} g_j(x, \mu')\, d\mu' \qquad (i = 1, 2,..., \alpha), \tag{46}$$

where $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_\alpha > 1/|\lambda|$. The eigenvalue λ of $T^{-1}A$ is a *discrete* eigenvalue of Eq. (46) and the eigenfunction has the form $\mathbf{h} = (h_i)_{i=1}^{N}$ with $h_{\alpha+1} = \cdots = h_N = 0$ and $(h_i)_{i=1}^{\alpha}$ an eigenfunction of Eq. (46) corresponding to λ. We now repeat the whole reasoning leading to these conclusions for the generalized eigenvectors of $T^{-1}A$ corresponding to λ and we argue by induction on the length of the Jordan chain. As a result we find that

$$Z_{1/\lambda}(T^{-1}A) \subseteq Z_{1/\lambda}(T_\alpha^{-1}A_\alpha) \oplus \{0\}, \tag{47}$$

where T_α and A_α are the analogues of T and A for Eq. (46), $Z_{1/\lambda}(T_\alpha^{-1}A_\alpha)$ is a subspace of the direct sum of α copies of $L_2[-1, 1]$ and $\{0\}$ is the trivial subspace of the direct sum of $N - \alpha$ copies of $L_2[-1, 1]$. Now let $(A\mathbf{h}, \mathbf{k}) = 0$ for some $\mathbf{h} \in Z_{1/\lambda}(T^{-1}A)$ and all $\mathbf{k} \in Z_{1/\lambda}(T^{-1}A)$, and let us prove $\mathbf{h} = 0$. Certainly, $h_i = k_i = 0$ for $i \geqslant \alpha + 1$. Put $\mathbf{h}_\alpha = (h_1,..., h_\alpha)$, $\mathbf{k}_\alpha = (k_1,..., k_\alpha)$ and $\boldsymbol{l}_\alpha = (l_1,..., l_\alpha)$, where $\boldsymbol{l} = T^{-1}A\mathbf{h}$ satisfies $l_i = 0$ for $i \geqslant \alpha + 1$. Then

$$0 = (A\mathbf{h}, \mathbf{k}) = (T\boldsymbol{l}, \mathbf{k}) = (T_\alpha \boldsymbol{l}_\alpha, \mathbf{k}_\alpha) = (A_\alpha \mathbf{h}_\alpha, \mathbf{k}_\alpha),$$

and A_α is self-adjoint on the direct sum of α copies of $L_2[-1, 1]$ weighted by $\sigma_1,..., \sigma_\alpha$. Because $1/\lambda$ is either an isolated eigenvalue or a regular point of $T_\alpha^{-1}A_\alpha$, property (vi) of Section II implies that $\mathbf{h}_\alpha = 0$, whence $\mathbf{h} = 0$. Again using property (vi) of Section II, we find that $1/\lambda$ cannot be an irregular critical point of the restriction of $T^{-1}A$ to $Z_0(AT^{-1})^\perp$, endowed with $(\cdot, \cdot)_A$. Hence A is T-regular. ∎

We conclude the discussion of this example with some references. Full-range results were derived for strictly positive $\mathbf{\Sigma} - \mathbf{C}$ by Greenberg [15] and Lekkerkerker [26], who both excluded eigenvalues of $T^{-1}A$ within the continuous spectrum. Half-range theory was discussed in [17] (also [28, Sect. VI.7]) for (nonstrictly) positive A.

VII. Discussion

We have provided a complete theory of existence and uniqueness for partial range boundary value problems of type (1)–(2), as well as the modified problem (1)–(2)–(18). The indices $\delta^\pm$, $\gamma^\pm$ and $\delta_0^\pm$, $\gamma_0^\pm$ provide a measure for the non-uniqueness and noncompleteness of the solutions of these problems. For physical problems, such non-unique or noncomplete situations are of considerable importance, as they often reflect the existence of conservation laws. See, for example, Section VI.5 of [28], in which the neutron current density is shown to be conserved for $c = 1$; also, strong evaporation problems [32, 33].

The results of Sections III and IV of this paper are in disagreement with those in the recent monograph of Kaper *et al.* [20] concerning indefinite collision operators. This monograph treats extensively the one-speed neutron transport equation outlined in Section V. The simplicity of the finite zeros of the dispersion function $\Lambda^m(z)$ and the nonexistence of zeros of $\Lambda^m(z)$ off the real and imaginary axes correspond to similar properties of the reciprocals of the eigenvalues of $T^{-1}A$ (cf. (38)). These properties are violated in general in the abstract problem, and appear also not to hold for the problem treated in the monograph. Indeed, recently it has been shown [29] that $\Lambda^m(z)$ has zeros off the real and imaginary axes for $m = 0$, $c = 9$, $f_1 = 0.4$, $f_2 = \frac{1}{9}$, and $f_l = 0$ ($l \geqslant 3$). By the continuous dependence of these zeros on the parameters c, f_1, and f_2, the same can be made true under the noncritically assumption $f_2 \neq \frac{1}{9}$. Adaptation of an argument in [29], using the symmetry and continuous dependence of the eigenvalues, can also be used to obtain a model with nonsimple finite eigenvalues. This mistake in the eigenvalue problem for $T^{-1}A$ has overwhelming consequences, and is the principal reason why the Jordan structure of $T^{-1}A$ and the decompositions in this paper are considerably more complicated than the (incorrect) decompositions in the monograph.

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