Conditions for runaway phenomena in the kinetic theory of particle swarms

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The velocity distribution of a spatially uniform diluted guest population of charged particles moving within a host medium under the influence of a D. C. electric field is studied. A simplified one-dimensional Boltzmann model of the Kač type is adopted. Necessary conditions and sufficient conditions are established for the existence, uniqueness, and attractivity of a stationary non-negative distribution corresponding to a specified concentration level. Conditions for the onset of the runaway process are established.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

In this paper we are concerned with some mathematical aspects of the behavior of a population of charged particles under the influence of a spatially uniform D. C. (i.e., time independent) electric field. Problems of this type appear in a number of distinct scientific areas, e.g., in the theory of swarms of charged particles in a neutral background gas,[1,2] in the study of "runaway" electrons in fully ionized plasmas,[3–5] in the calculation of D. C. conductivity in biological membranes,[6] and in semiconductor theory. In many of these cases the charged particles of interest are electrons; however, in some instances ions or positive vacancies are considered as well.

Let us consider a spatially uniform dilute population of charged particles that are initially at thermal equilibrium with a neutral environment. Suppose that for times $t \geqslant 0$ a uniform D. C. electric field is applied to the system. The charged particles are accelerated by the electric field but return some of the acquired kinetic energy to the host medium via some interaction process (collisions). The heating of the host medium is assumed to be negligible enough for the temperature of the background host medium to remain approximately time independent. Further, we assume the existence of a balance between ionization of host particles and recombination of charged particles, so that the total number of charged particles appears to be invariant.

Two main physical situations may occur: (i) the collision process is sufficiently effective to force the velocity distribution of the charged particles towards a steady state nonzero profile, which is usually a heavily distorted Maxwellian at a temperature exceeding the reference temperature of the background gas, or (ii) the collision process is not effective in removing kinetic energy from the population of charged guest particles, so that no relaxation of the distribution function towards a nonzero steady state distribution occurs. On the contrary, a "travelling wave in velocity space" is generated (the so-called runaway case). In case (i) the contribution to D. C. conductivity of the guest particles is the ratio of the magnitude of the current due to their motion (in steady state conditions) to the intensity of the D. C. field; in the runaway case (ii) one does not have a (finite) D. C. conductivity, since the speed of the charged particles increases indefinitely. In the Appendix we present two simple model problems, based on the BGK approximation, to illustrate the two kinds of behavior. A more sophisticated model problem has been presented by Corngold and Rollins.[7]

The physical aspects of the picture sketched above have been well understood for a number of years.[2,3] For instance, it is recognized that the key ingredient in determining whether a given process will involve "relaxation" [case (i)] or "runaway" [case (ii)] is the dependence of the collision frequency $\nu(v)$ upon the speed v of the charged particles for large values of v. Indeed, if $\nu(v)$ drops towards zero too rapidly as $v \to \infty$, the collision process can be shown[2] to be unable to slow down the most energetic charged particles. As a consequence, these particles "runaway."

In spite of this body of existing knowledge, we feel that the mathematical aspects of the runaway process—as opposed to the strictly phenomenological physical ones—still require some study. For one thing, the approximations adopted in the literature are often so drastic[4] as to make one wonder about the reliability of the results (beyond, maybe, an order of magnitude level of precision). On the other hand, at a more fundamental level, even the physical outline given above is open to some criticism. In fact, one could consider intermediate cases [besides the cases (i) and (ii) given above]. For instance, one could construct an ad hoc model according to which the charged particle distribution function relaxes towards an asymptotic profile whose first (or second) velocity moment is unbounded; then, the drift velocity (or the temperature) would diverge even under case (i) conditions. Conversely, under case (ii) conditions one could envisage, as an alternative to the travelling wave in velocity space, a distribution function which relaxes (uniformly with respect to velocity) towards zero as time grows; under such conditions the velocity moments may or may not converge to finite values as $t \to \infty$. Thus there are cases in which the distinction between the runaway and the "absence of runaway" situation becomes blurred.

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Other instances of confusion can be encountered. For instance, one author has erroneously presented estimates of the D. C. conductivity even in cases when the steady state distribution fails to exist (see the cases $p < -1$ in Ref. 6).

These questions have motivated the present introductory study on some mathematical aspects of the behavior of a collection of charged particles under the influence of an electric field. At this point we would like to present some additional remarks. First of all, we recall that—as observed by Corngold and Rollins[7]—much of the literature in the field deals with the problem of a steady state population of charged particles generated by a time independent source of cold particles. It should be noted that the two cases mentioned above for the sourceless problem—namely, case (i) of no runaways and case (ii) where runaways are present—correspond to the impossibility or the existence of a steady state population, respectively, when the source is present. The sourceless point of view taken in this paper has been described above. Another question concerns the choice of the mathematical model to employ in the description of the collective dynamics of the population of charged particles. In this preliminary study we assume, somewhat artificially, that charged particles move on a straight line parallel to the electric field (cf. the celebrated Kač model[8] of the Boltzmann equation); moreover, we usually assume that the collision process is described by a collision integral; the differential counterpart has been studied by Corngold and Rollins.[7] Finally we would like to mention that one of the problems we face is that of deciding upon the mathematical environment to adopt. On the one hand, we may introduce an L_1 space of distribution functions with at all times a finite total number of charged particles. On the other hand, we may adopt an L_1 space of distribution functions with at all times a finite number of collisions between the charged particles and the host medium. In part for reasons of mathematical convenience, we have made the former choice, especially as the general solution of the time dependent problem will turn out to have both a finite total number of charged particles and a finite number of guest-host interactions at all times. For the steady state problem we will be in the same rather fortunate situation, provided we assume the charged particle cross section $\nu(v) \geqslant 0$ to be nonintegrable with respect to velocity [in the sense that $\int_{-\infty}^{+\infty} \nu(v)dv = +\infty$]. On the other hand, if the cross section is integrable with respect to velocity [i.e., if $\int_{-\infty}^{+\infty} \nu(v)dv < +\infty$], we will have the rather unphysical situation of a "steady state" with finite total number of collisions but a nonzero particle density for infinite speed. We will make our assumptions more precise below.

Thus let us consider the simplified linear Boltzmann equation

$$\frac{\partial f}{\partial t}(v,t) + a\frac{\partial f}{\partial v}(v,t) + \nu(v)f(v,t) = \int_{-\infty}^{+\infty} k(v,v')\nu(v')f(v',t)dv'. \tag{1.1}$$

This equation describes the electron distribution $f(v,t)$ in a weakly ionized host medium as a function of the velocity $v \in (-\infty, +\infty)$ and time $t \geqslant 0$. The electrostatic acceleration a is assumed constant and positive. Recombination and ionization effects are assumed to balance each other. The expressions $\nu(v)$ and $k(v,v')$ denote the collision frequency (between an electron and the host medium) and the corresponding scattering kernel; accordingly, $k(v,v')dv$ is the probability that an electron entering the collision with velocity v' will come out of the collision with its velocity in the interval $(v, v+dv)$. We have

$$k(v,v') \geqslant 0, \tag{1.2}$$

$$\int_{-\infty}^{+\infty} k(v,v')dv \equiv 1. \tag{1.3}$$

The electron distribution $f(v,t)$ and the collision frequency $\nu(v)$ must, of course, be non-negative. By reciprocity symmetry, we also have

$$\nu(-v) = \nu(v), \tag{1.4}$$

$$k(-v,-v') = k(v,v'). \tag{1.5}$$

In connection with Eq. (1.1), we will study two mathematical problems. In the first place we will prove the unique solvability of the time-dependent evolution equation (1.1) under the initial condition

$$f(v,0) = f_0(v) \tag{1.6}$$

in a suitable functional setting, as well as the non-negativity of the solution for a non-negative initial condition, and establish the appropriate semigroup properties and bounds on the solution. In the second place we will establish necessary and sufficient conditions for the existence of a (unique) non-negative solution of the corresponding stationary equation

$$a\frac{\partial f}{\partial v}(v) + \nu(v)f(v) = \int_{-\infty}^{+\infty} k(v,v')\nu(v')f(v')dv'. \tag{1.7a}$$

We add the plausible requirements of a finite electron concentration and a finite collision rate (per unit volume); namely, we require

$$\int_{-\infty}^{+\infty} f(v)dv < +\infty, \tag{1.7b}$$

$$\int_{-\infty}^{+\infty} \nu(v)f(v)dv < +\infty. \tag{1.7c}$$

An additional plausible requirement is that in velocity space there should be no electrons entering or leaking out from the system. Since the acceleration a takes the role of "velocity" in velocity space, we require

$$\lim_{v\to -\infty} af(v) = \lim_{v\to +\infty} af(v) = 0. \tag{1.7d}$$

Hence

$$f(-\infty) = f(+\infty) = 0. \tag{1.7e}$$

Along with it we will establish under which conditions the stationary solution can be obtained from the solution of the time-dependent problem at $t \to \infty$.

In this paper we will investigate both the stationary and the time-dependent problem, as well as the decay to equilibrium of the solution in the time-dependent case. The time-dependent problem was already solved in Sec. XIII.4 of Ref. 9 as an application of the theory of time-dependent kinetic equations of Beals and Protopopescu (see Ref. 10; also Ref. 9, Chap. XI and Sec. XII.1–2). Here we shall give a direct proof based on semigroup considerations, which does not rely on this theory. In part we shall recover a well-known result. Note that, if the collision frequency is unbounded, the initial-value problem cannot be treated directly within the framework of Refs. 9 and 10; however, our proof will extend to this case. In fact, we will develop one of the few theories of kinetic equations where the usual cutoff leads to an unbounded collision frequency and an unbounded gain part of the collision operator. Different theories of this type were recently developed, for linearized Maxwell–Boltzmann equations by Arlotti[11] and for Fokker–Planck type equations by Cosner *et al.*[12]

Prior to developing the proper functional formulation of the problem, we make a number of assumptions on a, $\nu(v)$, and $k(v,v')$. Concerning a and ν we have:

Assumption (i): the acceleration a is a fixed positive constant;

Assumption (ii): the collision frequency $\nu(v)$ is a Lebesgue measurable, non-negative, and even function of v on $(-\infty,+\infty)$, which is almost everywhere nonzero and Lebesgue integrable on every bounded Lebesgue measurable set.

It is more complicated to formulate proper conditions on $k(v,v')$. On the one hand, we shall consider measurable functions $k(v,v')$ on $\Re^2$ satisfying (1.2), (1.3), and (1.5); on the other hand, we would like to include $k(v,v')=\delta(v-\alpha v')$ in our description. For this reason we shall consider the Banach spaces $L_1(\Re,dv)$ and $L_1(\Re,\nu\,dv)$ with the norms

$$\|f\|_1=\int_{-\infty}^{+\infty}|f(v)|dv,$$

$$\|f\|_\nu=\int_{-\infty}^{+\infty}\nu(v)|f(v)|dv,$$

and postulate the following assumption.

Assumption (iii): The operator K which is formally represented as

$$(Kf)(v)=\int_{-\infty}^{+\infty}k(v,v')\nu(v')f(v')dv'$$

is a positive linear operator $K: L_1(\Re,\nu\,dv)\to L_1(\Re,dv)$ satisfying

$$\|Kf\|_1=\|f\|_\nu,\quad \text{if } f\in L_1(\Re,\nu\,dv) \text{ and } f\geqslant 0, \tag{1.8a}$$

as well as the reciprocity principle

$$(Kf)(v)=(Kg)(-v),$$
$$\text{if } f(v)=g(-v) \text{ and } f\in L_1(\Re,\nu\,dv). \tag{1.8b}$$

If we define the (distributional) derivative f' of a function f in $L_1(\Re,dv)$ by

$$\int_{-\infty}^{+\infty}f'(v)g(v)dv=-\int_{-\infty}^{+\infty}f(v)g'(v)dv,$$

for every $g\in C_c^1(\Re)$, where $C_c^1(\Re)$ is the set of continuously differentiable complex functions on $\Re$ of compact support, *by a solution of the stationary equation* (1.7a) *we mean a function* φ *satisfying*

$$\varphi'(v)=-(1/a)\nu(v)\varphi(v)+(1/a)(K\varphi)(v),\quad v\in\Re, \tag{1.9a}$$

$$\varphi\in L_1(\Re,\nu dv). \tag{1.9b}$$

Since such a solution obviously has its derivative in $L_1(\Re,dv)$, each solution of the stationary problem will be absolutely continuous on $[-b,b]$ for all $b>0$. We have seen above that a physically acceptable solution ought to be non-negative and to obey requirements (1.7b), (1.7c), (1.7d), and (1.7e). Accordingly, among the possible solutions of problem (1.9) we shall be mostly interested in those non-negative solutions φ which also belong to $L_1(\Re,dv)$ and satisfy $\varphi(-\infty)=\varphi(+\infty)=0$. The following theorem gives a necessary condition for the existence of a non-negative solution of (1.9) in $L_1(\Re,dv)$.

Theorem 1: Let a, ν, and K satisfy the assumptions (i), (ii), and (iii) stated above. Then a necessary condition for problem (1.9) to admit a nontrivial non-negative solution $\varphi\in L_1(\Re,dv)$ is that

$$\int_{-\infty}^{+\infty}\nu(v)dv=+\infty.$$

Proof: Let φ be a nontrivial non-negative solution of problem (1.9). Then φ is continuous on $\Re$ and there exists $v_0\in\Re$ such that $\varphi(v_0)>0$. However, since K is positive, Eq. (1.9a) yields

$$\begin{aligned}\varphi(v)&=\exp\left\{-\frac{1}{a}\int_{v_0}^{v}\nu(v'')dv''\right\}\varphi(v_0)\\&\quad+\frac{1}{a}\int_{v_0}^{v}\exp\left\{-\frac{1}{a}\int_{v'}^{v}\nu(v'')dv''\right\}(K\varphi)(v')dv'\\&\geqslant\exp\left\{-\frac{1}{a}\int_{v_0}^{v}\nu(v'')dv''\right\}\varphi(v_0),\end{aligned}$$

so that

$$\liminf_{v\to+\infty}\varphi(v)\geqslant\exp\left\{-\frac{1}{a}\int_{v_0}^{\infty}\nu(v'')dv''\right\}\varphi(v_0).$$

Then $\varphi\in L_1(\Re,dv)$ only if $\int_{v_0}^{\infty}\nu(v'')dv''=+\infty$, i.e., only if $\int_{-\infty}^{+\infty}\nu(v'')dv''=+\infty$. □

II. THE STATIONARY PROBLEM

In this section we shall discuss the stationary problem (1.9). Throughout this section, with the exception of the final part, we shall also make the additional assumption

$$\int_{-\infty}^{+\infty}\nu(v)dv=+\infty, \tag{2.1}$$

whose motivation is given by Theorem 1 above. Note that, as a consequence of assumption (ii), Eq. (2.1) characterizes the frequency behavior of $\nu(v)$ as $v\to\infty$. As observed above, (2.1) is equivalent to

$$\int_{\alpha}^{+\infty}\nu(v)dv=+\infty,\quad \text{for some } \alpha\in\Re.$$

Our first step is to convert the integrodifferential equation (1.9) into an (equivalent) integral equation. To this purpose, we define the following operator:

$$L: L_1(\Re,dv)\to L_1(\Re,\nu\,dv),$$

$$(Lf)(v)=\int_{-\infty}^{v}\frac{1}{a}\exp\left\{-\frac{1}{a}\int_{v'}^{v}\nu(v'')dv''\right\}f(v')dv'.$$

On integrating Lf with respect to the measure $\nu(v)dv$ and changing the order of integration we obtain

$$\|Lf\|_\nu = \int_{-\infty}^{+\infty}\left(1 - \lim_{v\to+\infty} \exp\left\{-\frac{1}{a}\int_{v'}^{v} \nu(v'')dv''\right\}\right)f(v')dv', \quad f\geqslant 0, \tag{2.2}$$

which implies that L is a positive contraction from $L_1(\Re,dv)$ to $L_1(\Re,\nu\, dv)$ and hence LK is a positive contraction on $L_1(\Re,\nu\, dv)$. Under our assumptions, if $\nu(v)$ obeys (2.1), then $\|Lf\|_\nu = \|f\|_1$ for all non-negative $f\in L_1(\Re,dv)$.

Theorem 2: If (2.1) holds, then every solution of the integrodifferential equation (1.9a) in $L_1(\Re,\nu\, dv)$ is a solution of the linear stationary problem

$$\varphi = LK\varphi, \quad \varphi\in L_1(\Re,\nu\, dv), \tag{2.3}$$

and conversely. Moreover, for every solution φ of the two equivalent problems we have $\varphi(-\infty) = \varphi(+\infty) = 0$.

Proof: Let φ be a solution of problem (1.9). Setting

$$H(v) = \exp\left\{\frac{1}{a}\int_{v_0}^{v} \nu(v')dv'\right\},$$

where v_0 is some real number, we obtain from (1.9),

$$(H\varphi)'(v) = (1/a)H(v)(K\varphi)(v), \quad v\in\Re,$$

which in turn implies

$$H(v)\varphi(v) = \varphi(v_0) + \frac{1}{a}\int_{v_0}^{v} H(v')(K\varphi)(v')dv'.$$

Here we observe that the integral on the right-hand side is finite, because $H(v)$ is bounded on every interval of the type $(-\infty,A]$ with $A < +\infty$. As a result we find

$$\varphi(v) = \exp\left\{-\frac{1}{a}\int_{v_0}^{v} \nu(v')dv'\right\}\varphi(v_0) + \frac{1}{a}\int_{v_0}^{v}\exp\left\{-\frac{1}{a}\int_{v'}^{v}\nu(v'')dv''\right\}(K\varphi)(v')dv', \tag{2.4}$$

where $v\in\Re$. We now note that φ is of bounded variation on $\Re$, due to the fact that $\varphi'\in L_1(\Re,dv)$. This obviously implies the boundedness of φ on $\Re$. Letting v_0 tend to $-\infty$ and taking account of (2.1) in combination with the boundedness of φ, we must have $\varphi(-\infty) = 0$ by dominated convergence. Similarly, if $v_0 \to +\infty$, we get $\varphi(+\infty) = 0$. Thus any solution φ of problem (1.9) obeys $\varphi = LK\varphi$, with $\varphi(\pm\infty) = 0$.

Conversely, directly from the explicit form of $LK\varphi$, every solution of Eq. (2.3) in $L_1(\Re,\nu\, dv)$ is absolutely continuous on $[-b,b]$ for all $b > 0$ and of bounded variation on $(-\infty, +\infty)$. Moreover,

$$\frac{d}{dv}(LK\varphi)(v) = \frac{1}{a}(K\varphi)(v) - \frac{1}{a}\nu(v)(LK\varphi)(v)$$

and the right-hand side belongs to $L_1(\Re,dv)$; hence the solution φ of Eq. (2.3) satisfies Eq. (1.9a). □

Theorem 3: If condition (2.1) is satisfied, then the set of all φ satisfying problem (2.3) is at most one dimensional and, when nontrivial, contains a nontrivial non-negative function.

Proof: Let us suppose that Eq. (2.3) admits solutions. Then every such solution is non-negative, apart from a constant factor. Indeed, if $\varphi = LK\varphi$ for some $\varphi\in L_1(\Re,\nu\, dv)$, one first chooses φ real. We then have

$$|\varphi| = |LK\varphi| \leqslant LK|\varphi|,$$

while

$$\begin{aligned}\int_{-\infty}^{+\infty} \nu(v)\{(LK|\varphi|)(v) - |\varphi(v)|\}dv &= \|LK|\varphi|\|_\nu - \|\varphi\|_\nu \\ &= \|K|\varphi|\|_1 - \|\varphi\|_\nu \\ &= \|\varphi\|_\nu - \|\varphi\|_\nu = 0.\end{aligned}$$

Hence $|\varphi| = LK|\varphi|$ and the existence of a nontrivial solution of Eq. (2.3) in $L_1(\Re,\nu\, dv)$ implies the existence of a nontrivial non-negative solution of Eqs. (1.9) in $L_1(\Re,\nu\, dv)$. In fact, if the solution φ is real, then $\|LK\{|\varphi| \pm \varphi\}\|_\nu = 0$ and $LK\{|\varphi| \pm \varphi\} = \{|\varphi| \pm \varphi\}$ imply that φ does not change sign.

Finally, if φ is a nontrivial non-negative solution of Eq. (2.3) in $L_1(\Re,\nu\, dv)$ and $\varphi(v) = 0$ for some $v\in\Re$, then $(LK\varphi)(v) = 0$ yields $(K\varphi)(v') \equiv 0$ for $v' < v$ and hence $\varphi(v'') = (LK\varphi)(v'') = 0$ for all $v'' \leqslant v$. Putting $v_0 = \sup\{v\in\Re: \varphi(v'') = 0 \text{ for all } v'' \leqslant v\}$ we find $\varphi(v) > 0$ for all $v > v_0$, since otherwise $\varphi(v)$ would vanish for some $v > v_0$. Thus if φ_1 and φ_2 are two different non-negative solutions of Eq. (2.3) of unit norm in $L_1(\Re,\nu\, dv)$, then $(\varphi_1 - \varphi_2)$ will be a nontrivial real solution of Eq. (2.3), which must have constant sign. Since both φ_1 and φ_2 have unit norm in $L_1(\Re,\nu\, dv)$, we obtain $\|\varphi_1 - \varphi_2\|_\nu = |\|\varphi_1\|_\nu - \|\varphi_2\|_\nu|$, which is a contradiction. Hence the solution space of Eq. (2.3) is at most one dimensional. □

Let φ be a nontrivial non-negative stationary solution in $L_1(\Re,dv)$. Then, apparently, either $\varphi(v) > 0$ for all $v\in\Re$ or $\varphi(v) = 0$ for $v\leqslant v_0$ and $\varphi(v) > 0$ for $v > v_0$ where v_0 is some real constant. In the latter case we have $(K\varphi)(v) \equiv 0$ for all $v < v_0$, as a result of the equation $\varphi = LK\varphi$. Since, by assumption, $\nu(v)$ does not vanish on a set of positive measure, we must then have $k(v,v') \equiv 0$ for all $v < v_0$ and $v' > v_0$.

Theorem 4: If condition (2.1) is satisfied and if, in addition, LK is a weakly compact operator on $L_1(\Re,\nu\, dv)$, then the stationary problem has a unique non-negative solution in $L_1(\Re,\nu\, dv)$ of unit norm.

Proof: If condition (2.1) is satisfied and f is non-negative, we have $\|LKf\|_\nu = \|f\|_\nu$. Consequently, the spectral radius of LK, $\mathrm{spr}(LK)$, is one. Moreover, $(LK)^2$ is compact as an operator on $L_1(\Re,\nu\, dv)$, because the square of a weakly compact operator in L_1 is compact. Then the compactness of $(LK)^2$ in combination with $\mathrm{spr}(LK) = 1$ implies the existence of at least one non-negative $\varphi\in L_1(\Re,\nu\, dv)$ of unit norm such that Eq. (2.3) holds true (see Ref. 13, Chap. 2). By the previous theorem this solution is unique. □

Corollary 5: Let condition (2.1) be satisfied. If the operator

$$(Bf)(v) = \int_{-\infty}^{+\infty} k(v,v')f(v')dv'$$

is weakly compact on $L_1(\Re,dv)$ then the problem (2.3) has a unique non-negative solution of unit norm.

Proof: If the above operator B is weakly compact on $L_1(\Re,dv)$, then $LK = LB\nu$ is weakly compact on $L_1(\Re,\nu\, dv)$. Here we observe that ν is a bounded operator from $L_1(\Re,\nu\, dv)$ into $L_1(\Re,dv)$. The result follows directly from the previous theorem. □

We now consider some simple models for the collision term. They satisfy the assumptions we formulated to ensure the existence of stationary solutions.

Example 1 [*The Bhatnagar–Gross–Krook (BGK) model*]*:* The idea behind this model is the assumption that the average effect of collisions is to provide a "source" which is proportional to the deviation of the distribution function $f(v)$ from a Maxwellian $f_m(v)$. Thus the collision term is assumed to take the following form:

$$\int_{-\infty}^{+\infty} k(v,v')\nu(v')f(v')dv' = \nu(v)F_m(v)\int_{-\infty}^{+\infty}\nu(v')f(v')dv', \tag{2.5}$$

where

$$F_m(v) = \frac{f_m(v)}{\int_{-\infty}^{+\infty}\nu(v')f_m(v')dv'}.$$

In this case, the operator K defined by (2.5) is a compact operator from $L_1(\Re,\nu\, dv)$ to $L_1(\Re,dv)$. Hence, if condition (2.1) is satisfied, the results of Theorem 4 hold true. In the Appendix we will give a more elaborate account of the BGK model. For a discussion of the reliability of the BGK model in the transport theory of charged particles see, for instance, the paper of Corngold and Rollins.[14] □

Example 2: Consider a particular class of integral kernels $k(v,v')$, which is a finite linear combination of functions separated in the variables v and v'. A kernel of this type is said to be degenerate and can be written in the form

$$k(v,v') = \nu(v)f_m(v)f_m(v')\sum_{i=1}^{\infty}\alpha_i\psi_i(v)\varphi_i(v'), \tag{2.6}$$

where ψ_i and φ_i are given functions and the α_i are suitable positive accommodation coefficients. In the literature of the kinetic theory of gases this model is known as the generalized BGK model and is obtained by generalizing the linearized BGK model. If we suppose the functions φ_i and ψ_i to be essentially bounded, the operator K defined by (2.6) and (1.8) is a compact operator from $L_1(\Re,\nu\, dv)$ into $L_1(\Re,dv)$. □

Example 3: Consider the integral kernel defined by

$$k(v,v') = \begin{cases} 1/2r, & v\in[-r,r], \quad v'\in\Re, \\ 0, & \text{otherwise.}\end{cases}$$

Then the integral operator B defined by the above kernel has the property that for every $\epsilon>0$ there exists $\delta>0$ such that

$$\int_{-\infty}^{+\infty}|(Bf)(v+h)-(Bf)(v)|dv<\epsilon$$

for every f belonging to a bounded subset of $L_1(\Re,dv)$ and every h with $|h|<\delta$. Moreover, there exists a subset $[-r,r]\subset G\subset\Re$ such that

$$\int_{\Re\setminus\overline{G}}|(Kf)(v)|dv<\epsilon,$$

is trivially satisfied for every $\epsilon>0$. From Theorem 2.1 of Ref. 15 it follows that B is compact on $L_1(\Re,dv)$. By virtue of Corollary 5, we have a stationary solution if $\int_{-\infty}^{+\infty}\nu(v')dv' = +\infty$. More generally, we may replace $k(v,v')$ with a bounded continuous non-negative function with support on $[-r,r]\times\Re$. A sufficient condition for compactness will then be the existence, for every $\epsilon>0$, of a number $\delta>0$ such that

$$\int_{-r}^{+r}|k(v+h,v')-k(v,v')|dv<\epsilon$$

for $|h|<\delta$, uniformly in v' on $\Re$. □

In the remaining part of this section, we consider the case when the behavior of the collision frequency $\nu(v)$ at infinity is such that condition (2.1) is not satisfied. In other words, from now on in this section, we replace (2.1) by the alternative assumption

$$\int_{-\infty}^{+\infty}\nu(v)dv<+\infty. \tag{2.7}$$

If condition (2.7) is satisfied, then $\|Lf\|_\nu \leqslant (1-\delta)\|f\|_1$ for all non-negative $f\in L_1(\Re,dv)$ where

$$\delta = \exp\left\{-\frac{1}{a}\int_{-\infty}^{+\infty}\nu(v')dv'\right\}>0.$$

In this case dominated convergence applied to Eq. (2.4) yields the existence of the continuous limits $\varphi(\pm\infty)$, whence the integrodifferential equation (1.9) can be put in the equivalent form

$$\varphi(v)-(LK\varphi)(v) = \exp\left\{-\frac{1}{a}\int_{-\infty}^{v}\nu(v')dv'\right\}\varphi(-\infty). \tag{2.8}$$

As a result we find that $\varphi(\pm\infty)$ are finite, while an easy integration of Eq. (1.9) over $\Re$ yields $\varphi(-\infty)=\varphi(+\infty)$. Now the integral equation to be investigated is Eq. (2.8). Equation (2.8) is uniquely solvable in $L_1(\Re,\nu\, dv)$, which is easily seen from the norm estimate

$$\|LK\varphi\|_\nu \leqslant (1-\delta)\|K\varphi\|_1 = (1-\delta)\|\varphi\|_\nu,$$

where $\delta\in(0,1)$. We summarize the result as follows.

Theorem 6: If condition (2.7) is satisfied, then the stationary problem (2.8) has a unique non-negative solution φ in $L_1(\Re,\nu\, dv)$ of unit norm with $\varphi(-\infty)=\varphi(+\infty)>0$.

Remark: Note that the solution of (2.8) under assumption (2.7), which is referred to in Theorem 6, is physically irrelevant, since it corresponds to an infinite population level.

It is possible to give necessary and sufficient conditions for the existence of a stationary solution in $L_1(\Re,\nu\, dv)$ in terms of the spectral properties of LK. The stationary solution will be unique apart from a normalization factor. If condition (2.1) holds true, the necessary and sufficient condition is that 1 is an eigenvalue of LK. The corresponding eigenfunction will then be non-negative. In particular, if $(LK)^n$ is compact for some $n\in\mathbb{N}$, there will be a stationary solution. On the other hand, if condition (2.7) is satisfied, there always exists a unique non-negative stationary (unphysical) solution in $L_1(\Re,\nu\, dv)$ of unit norm, because $\mathrm{spr}(LK)<1$; its values at $\pm\infty$ are equal and positive.

Instead of Eqs. (2.3) and (2.8) on $L_1(\Re,\nu\, dv)$, we may also study the equivalent equations

$$\psi - KL\psi = 0, \tag{2.9}$$

$$\psi - KL\psi = \varphi(-\infty)K\omega, \tag{2.10}$$

on $L_1(\Re,dv)$, where

$$\omega(v) = \exp\left\{-\frac{1}{a}\int_{-\infty}^{v} \nu(v'')dv''\right\}.$$

In fact, if φ is a (non-negative) solution of Eq. (2.3) in $L_1(\Re,\nu\,dv)$, then $K\varphi$ is a (non-negative) solution of Eq. (2.9) in $L_1(\Re,dv)$; conversely, if ψ is a (non-negative) solution of Eq. (2.9) in $L_1(\Re,dv)$, then $L\psi$ is a (non-negative) solution of Eq. (2.3) in $L_1(\Re,\nu\,dv)$. Moreover, in this manner nontrivial solutions of Eq. (2.3) in $L_1(\Re,\nu\,dv)$ correspond to nontrivial solutions of Eq. (2.9) in $L_1(\Re,dv)$. A similar connection exists between solutions of Eq. (2.8) in $L_1(\Re,\nu\,dv)$ and solutions of Eq. (2.10) in $L_1(\Re,dv)$, but now $K\varphi$ is a solution of Eq. (2.10) if φ is a solution of Eq. (2.8), while $\varphi(-\infty)\omega + L\psi$ is a solution of Eq. (2.8) if ψ is a solution of Eq. (2.10). However, since in general the operator K does not map absolutely continuous functions of $L_1(\Re,\nu\,dv)$ into continuous functions, the solutions of Eqs. (2.9) and (2.10) need not be continuous. On the other hand, if K (or the above operator B) has finite rank, it is much easier to solve Eqs. (2.9) and (2.10) than to solve Eqs. (2.3) and (2.8). Finally, it should be observed that the nonzero spectra and eigenvalue spectra of LK and KL coincide.

III. THE TIME-DEPENDENT PROBLEM

In order to study the time-dependent problem, we shall analyze the operator

$$(Tf)(v) = -a\frac{\partial f}{\partial v} - \nu(v)f(v) + (Kf)(v)$$

on the intersection $\mathscr{M}$ of $L_1(\Re,dv)$, $L_1(\Re,\nu\,dv)$ and the set of functions which are absolutely continuous on $[-b,b]$ for all $b>0$, are of bounded variation and vanish at $-\infty$. We shall prove an extension of T to be the generator of a strongly continuous semigroup on $L_1(\Re,dv)$ using the Hille–Phillips theorem. For this purpose we solve the equation

$$(\lambda - T)f = g \tag{3.1}$$

for $f\in\mathscr{M}$, where g is an arbitrary function in $L_1(\Re,dv)$ and $\lambda > 0$. We obtain

$$f = L_\lambda Kf + L_\lambda g, \tag{3.2}$$

where

$$(L_\lambda f)(v) = \frac{1}{a}\int_{-\infty}^{v} \exp\left\{-\frac{1}{a}\int_{v'}^{v}[\nu(v'') + \lambda]dv''\right\} f(v')dv'.$$

The derivation of Eq. (3.2) is the same as the derivation of (2.3) with $\nu(v)$ replaced by $\nu(v)+\lambda$, since for $\lambda > 0$ the integral $\int_{-\infty}^{+\infty}\{\nu(v)+\lambda\}dv$ is infinite. As a result we obtain

$$\|L_\lambda f\|_\nu + \lambda\,\|L_\lambda f\|_1 = \|f\|_1, \quad f\geqslant 0. \tag{3.3}$$

Here we have replaced $\nu(v)$ by $\nu(v)+\lambda$ in the identity $\|Lf\|_\nu = \|f\|_1$ for $f\geqslant 0$. This is allowed, since $L_\lambda f$ coincides with Lf on replacing $\nu(v)+\lambda$. Consequently,

$$\|L_\lambda Kf\|_\nu + \lambda\,\|L_\lambda Kf\|_1 = \|f\|_\nu, \quad f\geqslant 0, \tag{3.4}$$

whence L_λ maps $L_1(\Re,dv)$ and $L_\lambda K$ maps $L_1(\Re,\nu\,dv)$ into the intersection of $L_1(\Re,dv)$ and $L_1(\Re,\nu\,dv)$. Hence for every $\lambda>0$ and $g\in L_1(\Re,dv)$ the solutions f of Eq. (3.2) belong to this intersection.

Theorem 7: For every $\lambda>0$ and $g\in L_1(\Re,dv)$ there exists a unique solution $T_\lambda g$ of Eq. (3.2), which belongs to $L_1(\Re,dv)$. Then T_λ is the resolvent of a strongly continuous positive contraction semigroup $\{S(t)\}_{t\geqslant 0}$ on $L_1(\Re,dv)$ whose generator G is a closed extension of T. Moreover, the semigroup $\{S(t)\}_{t\geqslant 0}$ satisfies

$$\|S(t)f\|_1 = \|f\|_1, \quad f\geqslant 0, \tag{3.5}$$

if and only if G is the (minimal) closure of T.

Proof: Put

$$T_\lambda g = \sum_{n=0}^{\infty}(L_\lambda K)^n L_\lambda g, \quad g\in L_1(\Re,dv).$$

Then for $g\geqslant 0$ in $L_1(\Re,dv)$ Eq. (3.4) implies

$$\lambda\,\|(L_\lambda K)^n L_\lambda g\|_1 = \|(L_\lambda K)^{n-1}L_\lambda g\|_\nu - \|(L_\lambda K)^n L_\lambda g\|_\nu, \quad n = 1,2,3,\ldots,$$

and therefore for $g\geqslant 0$

$$\|T_\lambda g\|_1 = \sum_{n=0}^{\infty}\|(L_\lambda K)^n L_\lambda g\|_1 = \|L_\lambda g\|_1 + \frac{1}{\lambda}\|L_\lambda g\|_\nu - \frac{1}{\lambda}\beta_\lambda(L_\lambda g) = \frac{1}{\lambda}\,\|g\|_1 - \frac{1}{\lambda}\beta_\lambda(L_\lambda g) \leqslant \frac{1}{\lambda}\|g\|_1.$$

Here

$$\beta_\lambda(f) = \lim_{n\to\infty}\|(L_\lambda K)^n f\|_\nu, \quad 0\leqslant f\in L_1(\Re,\nu\,dv),$$

extends uniquely to a positive linear functional of $L_1(\Re,\nu\,dv)$. Thus there exists a non-negative function $\varphi_\lambda\in L_\infty(\Re,\nu\,dv)$ such that

$$\beta_\lambda(f) = \int_{-\infty}^{+\infty}\nu(v)f(v)\varphi_\lambda(v)dv, \quad f\in L_1(\Re,\nu\,dv).$$

Since obviously $\beta_\lambda(L_\lambda Kf) = \beta_\lambda(f)$ for all $f\in L_1(\Re,\nu\,dv)$, we have $(L_\lambda K)^*\varphi_\lambda = \varphi_\lambda$, where the adjoint is defined on $L_\infty(\Re,\nu\,dv)$.

To prove that 1 is not an eigenvalue of $L_\lambda K$, suppose $c(\lambda)$ is an eigenvalue of $L_\lambda K$. If φ is a corresponding eigenfunction, then

$$|c(\lambda)|\{\|\varphi\|_\nu + \lambda\,\|\varphi\|_1\} = \|L_\lambda K\varphi\|_\nu + \lambda\,\|L_\lambda K\varphi\|_1 \leqslant \|L_\lambda K|\varphi|\|_\nu + \lambda\,\|L_\lambda K|\varphi|\|_1 = \||\varphi|\|_\nu = \|\varphi\|_\nu,$$

which implies that $|c(\lambda)|\leqslant 1$. Moreover, $|c(\lambda)|\neq 1$, since otherwise $\|\varphi\|_1 = 0$ and thus $\varphi = 0$. Thus $f = T_\lambda g$ is the unique solution of Eq. (3.2) in $L_1(\Re,dv)$.

Clearly, T_λ is the resolvent of a bounded strongly continuous positive contraction semigroup on $L_1(\Re,dv)$, i.e., $T_\lambda = (\lambda - G)^{-1}$, where G is the generator and $D(G) = \operatorname{Ran} T_\lambda$. Moreover,

$$T_\lambda g = \int_0^\infty e^{-\lambda t}S(t)g\,dt, \quad g\geqslant 0 \text{ in } L_1(\Re,dv),$$

while

$$\|T_\lambda g\|_1 + (1/\lambda)\beta_\lambda(L_\lambda g) = (1/\lambda)\|g\|_1, \quad g\geqslant 0 \text{ in } L_1(\Re,dv).$$

Thus if $1\notin\sigma_r(L_\lambda K)$, the residual spectrum of $L_\lambda K$, and hence $1\notin\sigma_p((L_\lambda K)^*)$, we have $\beta_\lambda(L_\lambda g) = 0$ and therefore

$$\|T_\lambda g\|_1 = (1/\lambda)\|g\|_1, \quad g\geqslant 0 \text{ in } L_1(\Re,dv).$$

Hence

$$\|S(t)g\|_1 = \|g\|_1, \quad g\geqslant 0 \text{ in } L_1(\Re,dv).$$

Conversely, if the last two equations are true, $\beta_\lambda(L_\lambda g) = 0$ for all $g\geqslant 0$ in $L_1(\Re,dv)$. Since $\{L_\lambda g: g\notin L_1(\Re,dv)\}$ is dense in $L_1(\Re,\nu\, dv)$, we get $\varphi_\lambda = 0$ and hence $1\notin\sigma_r(L_\lambda K)$.

If Eq. (3.5) is true and $1\notin\sigma_r(L_\lambda K)$, then $\mathbb{1} - L_\lambda K$ maps $L_1(\Re,\nu\, dv)$ into a dense subspace of $L_1(\Re,\nu\, dv)$ and Eq. (3.2) can be solved in $\mathscr{M}$ for all $g\in\mathscr{D}$, where $\mathscr{D}$ is a suitable dense subset of $L_1(\Re,dv)$. For every $g\in L_1(\Re,dv)$ and $\epsilon>0$ we then choose $g_0\in\mathscr{D}$ such that $\|g-g_0\|_1<\epsilon\lambda/(\lambda+1)$ so that

$$\|T_\lambda[g-g_0]\|_1<\epsilon/(\lambda+1).$$

Therefore $f = T_\lambda g\in D(\lambda - G)$ can be approximated by some $f_0 = T_\lambda g_0\in\mathscr{M}$ such that

$$\|f-f_0\|_1 + \|(\lambda - G)f - (\lambda - T)f_0\|_1$$
$$= \|f-f_0\|_1 + \|g-g_0\|_1<\epsilon.$$

Consequently, $(\lambda - G)$ is the closure of $(\lambda - T)$. On the other hand, if $(\lambda - G)$ is not the closure of $(\lambda - T)$, there exist $g\in L_1(\Re,dv)$ and $\epsilon>0$ such that

$$\|T_\lambda g - f\|_1 + \|g - (\lambda - T)f\|_1\geqslant\epsilon, \quad f\in\mathscr{M}.$$

Hence if $f = \Sigma_{n=0}^{N}(L_\lambda K)^n L_\lambda g$, we have

$$\|T_\lambda(KL_\lambda)^{N+1}g\|_1 + \|(L_\lambda K)^{N+1}g\|_1\geqslant\epsilon.$$

By choosing N large enough, the first term can be made arbitrarily small, because the series defining $T_\lambda g$ converges absolutely in $L_1(\Re,dv)$. But then

$$\beta_\lambda(g) = \lim_{N\to\infty}\|(L_\lambda K)^{N+1}g\|_1\geqslant\epsilon,$$

whence $1\in\sigma_r(L_\lambda K)$. □

Remarks: (1) If there exists a nonzero stationary solution φ, then $\varphi\geqslant 0$ apart from a constant factor, $S(t)\varphi\equiv\varphi$ and $\lambda T_\lambda\varphi\equiv\varphi$. But then $\beta_\lambda(L_\lambda\varphi) = 0$ and hence $\beta_\lambda = 0$, which implies that $1\notin\sigma_r(L_\lambda K)$ for all $\lambda>0$. Consequently, the existence of a nonzero stationary solution implies (3.5).

(2) Another case when (3.5) is true occurs if $\nu(v)$ is essentially bounded. In that case $\|g\|_\nu\leqslant\|\nu\|_\infty\|g\|_1$ for all $g\in L_1(\Re,dv)\subseteq L_1(\Re,\nu\, dv)$ Thus if $\{\alpha_m\}_{m=1}^\infty$ is an increasing sequence in (0,1) with limit 1, then $f_m = (\mathbb{1} - \alpha_m L_\lambda K)^{-1}L_\lambda g\in D(T)$, increases with m if $g\geqslant 0$ and satisfies

$$(\lambda - T)f_m = g - (1-\alpha_m)Kf_m.$$

As a result, $\|f-f_m\|_1\to 0$ as $m\to\infty$. On the other hand,

$$\|g - (\lambda - T)f_m\|_1$$
$$= (1-\alpha_m)\|Kf_m\|_1\leqslant(1-\alpha_m)\|f_m\|_\nu$$
$$\leqslant(1-\alpha_m)\|\nu\|_\infty\|f_m\|_1,$$

which vanishes as $m\to\infty$. Consequently, $f\in D(G)$ and $(\lambda - G)f = g$. Moreover, $G = \overline{T}$.

(3) If $\nu(v)$ is integrable, then $\|L_\lambda K\|\leqslant\|LK\|<1$ on $L_1(\Re,\nu\, dv)$. Then $1\in\sigma_r(L_\lambda K)$ and Eq. (3.5) must be satisfied. This is also the case if KL_λ is weakly compact on $L_1(\Re,dv)$ [or $L_\lambda K$ is weakly compact on $L_1(\Re,\nu\, dv)$]. The reason is that power compact operators do not have a residual spectrum.

(4) In general, T is not a closed operator and hence G is a proper extension of T. This is, for instance, the case if $k(v,v') = \delta(v-v')$ and $\nu(v)$ is not essentially bounded. In this case $(Tf)(v) = -a(\partial f/\partial v)$ defined on $\mathscr{M}$ while $\mathscr{M}$ does not coincide with the (generally) larger domain of the generator of the semigroup $\{S_{0,\nu=0}(t)\}_{t\geqslant 0}$ on $L_1(\Re,dv)$ defined by $(S_{0,\nu=0}(t)g)(v) = g(v-at)$. However, if $\mathrm{spr}(L_\lambda K)<1$ [which occurs, for instance, if KL_λ is weakly compact on $L_1(\Re,dv)$ or if $\nu(v)$ is integrable], we can easily prove that T_λ maps $L_1(\Re,dv)$ into $\mathscr{M}$ and therefore that $G = T$.

Suppose there is a nontrivial non-negative stationary solution φ in $L_1(\Re,dv)$ Then, as known, either $\varphi(v)>0$ for all $v\in\Re$ or $\varphi(v) = 0$ for $v\leqslant v_0$ and $\varphi(v)>0$ for $v>v_0$. In order to derive some properties of the semigroup $\{S(t)\}_{t\geqslant 0}$ in the latter case, we consider the free streaming semigroup $\{S_0(t)\}_{t\geqslant 0}$ on $L_1(\Re,dv)$ generated by the operator $T_0 = -(a(d/dv) + \nu(v))$ on the domain $D(T_0) = \mathscr{M}$. Then $\{S_0(t)\}_{t\geqslant 0}$ is a contraction semigroup whose generator satisfies

$$(\lambda - T_0)^{-1}g = L_\lambda g = \int_0^\infty e^{-\lambda t}S_0(t)g\, dt, \quad \mathrm{Re}\ \lambda>0. \tag{3.6}$$

It is possible to write down $S_0(t)$ in closed form. In fact,

$$(S_0(t)g)(v) = M(t,v)g(v-at), \tag{3.7}$$

where

$$M(t,v) = \exp\left\{-\frac{1}{a}\int_{v-at}^{v}\nu(v')dv'\right\}. \tag{3.8}$$

Hence $\|S_0(t)\| = \mathrm{ess\ sup}\ \{M(t,v): v\in\Re\}$, so that the type $\omega_0(S_0)$ of the semigroup $\{S_0(t)\}_{t\geqslant 0}$ is given by

$$\omega_0(S_0) = \lim_{t\to\infty}(1/t)\log\operatorname*{ess\,sup}_{v\in\Re}M(t,v). \tag{3.9}$$

Since in an L_1 space the type and the spectral bound of a positive semigroup coincide (see Ref. 16), we may extend (3.6) to all $\mathrm{Re}\,\lambda>\omega_0(S_0)$. Writing $(R_\alpha g)(v) = \exp\{i\alpha v/a\}g(v)$ we have for $\lambda = \sigma + i\tau$ with $\sigma,\tau\in\Re$

$$L_\lambda = R_\tau^{-1}L_\sigma R_\tau,$$

while $\|L_\sigma g\|_1$ increases monotonically as σ decreases from $+\infty$ to $-\infty$ for all non-negative $g\in L_1(\Re,dv)$. Hence

$$\sigma(T_0) = \{\lambda\in\mathbb{C}: \mathrm{Re}\,\lambda\leqslant\omega_0(S_0)\}$$

whenever $\omega_0(S_0)>-\infty$, while $\sigma(T_0) = \varnothing$ whenever $\omega_0(S_0) = -\infty$. We now observe that

$$[(\lambda - G)^{-1} - (\lambda - T_0)^{-1}]g$$
$$= [T_\lambda - L_\lambda]g = \sum_{n=1}^\infty(L_\lambda K)^n L_\lambda g$$

implies that, for all $\mathrm{Re}\,\lambda>0$, $[T_\lambda - L_\lambda]g = 0$ for all g of support within $[v_0,\infty)$. Since

$$[T_\lambda - L_\lambda]g = \int_0^\infty e^{-\lambda t}[S(t) - S_0(t)]g\,dt, \quad \operatorname{Re}\lambda > 0,$$

we find that $[S(t) - S_0(t)]g = 0$ for all g of support within $[v_0, \infty)$. For later use we also mention that

$$\lim_{t\to\infty} \|S_0(t)g\|_1 = 0, \quad g \in L_1(\mathfrak{R}, dv)$$

as a consequence of (3.7), (3.8) and the nonintegrability of $\nu(v)$.

We call $\{S(t)\}_{t>0}$ *mean ergodic*[17,18] if for every $g\in L_1(\mathfrak{R},dv)$ there exists a vector $Pg \in L_1(\mathfrak{R},dv)$ such that

$$\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t S(t')g\,dt' - Pg \right\|_1 = 0. \qquad (3.10)$$

It then follows that P is the (bounded) projection of $L_1(\mathfrak{R},dv)$ onto the fixed space

$$\mathscr{F} = \{g \in L_1(\mathfrak{R},dv) : S(t)g = g \text{ for all } t \geqslant 0\}$$

of the semigroup $\{S(t)\}_{t>0}$ along the space

$$\mathfrak{g} = \overline{\operatorname{span}\{[\mathbb{1} - S(t)]g : t > 0, g \in L_1(\mathfrak{R},dv)\}}.$$

Theorem 8: Suppose there is a nontrivial stationary solution φ in $L_1(\mathfrak{R},dv)$. Then the semigroup $\{S(t)\}_{t>0}$ is mean ergodic and the limit Pg is a one-dimensional projection of the form

$$(Pg)(v) = \alpha(g)\varphi(v), \quad v \in \mathfrak{R}, \qquad (3.11)$$

where

$$\alpha(g) = \int_{-\infty}^{\infty} \psi(v')g(v')dv'$$

for some non-negative function $\psi \in L_\infty(\mathfrak{R},dv)$ with $\|\psi\|_\infty < +\infty$ and $\int_{-\infty}^{+\infty} \psi(v')\varphi(v')dv' = 1$.

Proof: First, if G is the generator of $\{S(t)\}_{t>0}$, then $\mathscr{F} = \{\varphi \in L_1(\mathfrak{R},\nu\,dv) : G\varphi = 0\}$. Thus, if $G = T$, then $\mathscr{F}$ coincides with the set of stationary solutions in $L_1(\mathfrak{R},dv)$. Now recall that φ is continuous and suppose that φ does not have (finite) zeros. Then $0 \leqslant S(t)\varphi = \varphi$ for all $t \geqslant 0$, and the mean ergodicity of $\{S(t)\}_{t>0}$ is immediate from Ref. 18 (Corollary 1 of Theorem V 8.4).

Next, suppose φ has a finite zero. Then there exists $v_0 \in \mathfrak{R}$ such that $\varphi(v) \equiv 0$ on $(-\infty, v_0]$ and $\varphi(v) > 0$ on (v_0, ∞). Then $(K\varphi)(v) \equiv 0$ on $(-\infty, v_0]$, and therefore $(Ku)(v) \equiv 0$ on $(-\infty, v_0]$ and for all characteristic functions u of compact support within (v_0, ∞). Since every nonnegative function in $L_1(\mathfrak{R},\nu\,dv)$ of support within $[v_0, \infty)$ is the monotone limit of a sequence of finite linear combinations of characteristic functions of compact support within (v_0, ∞), we have $(Ku)(v) \equiv 0$ on $(-\infty, v_0)$ for all $u \in L_1(\mathfrak{R},\nu\,dv)$ of support within $[v_0, \infty)$. Thus K leaves invariant the closed invariant ideal in $L_1(\mathfrak{R},\nu\,dv)$ of functions with support in $[v_0, \infty)$. Then, by the second paragraph following the proof of Theorem 7, this must also be the case for $S(t)$. We may now restrict $S(t)$ to $L_1([v_0, \infty), dv)$ and apply the same corollary in Ref. 18 to get the ergodicity of the reduced semigroup. From the ergodicity of the reduced semigroup and the special form of φ we immediately have the ergodicity of the full semigroup $\{S(t)\}_{t>0}$.

Finally, as the stationary problem has at most one linearly independent solution in $L_1(\mathfrak{R},\nu\,dv)$, we easily obtain the specific form (3.11) of the projection P. □

IV. DECAY TO EQUILIBRIUM

In this section we shall prove that under certain quite natural conditions the solution of the time-dependent problem converges in the norm of $L_1(\mathfrak{R},dv)$ to a solution of the stationary problem. Obviously, one of these conditions is that there exists a nontrivial stationary solution in $L_1(\mathfrak{R},dv)$. The other condition is that the generator G of the time evolution semigroup $\{S(t)\}_{t>0}$ of Eq. (1.1) does not have purely imaginary eigenvalues. Of course, the second condition is suggested by the fact that if $i\alpha$ is a purely imaginary eigenvalue of the generator G and g is a corresponding eigenfunction, then the solution of Eq. (1.1) with initial condition g has the form

$$S(t)g = e^{i\alpha t}g$$

and therefore does not converge at $t\to\infty$.

More specifically, if the generator G of the semigroup $\{S(t)\}_{t>0}$ has purely imaginary eigenvalues, then under certain conditions one may prove that every solution $S(t)g$ of the time-dependent problem converges in the strong topology of $L_1(\mathfrak{R},dv)$ to a periodic function (cf. Ref. 19, Theorem C IV 2.14). Indeed, suppose that $\lambda = 0$ is an isolated eigenvalue of G and that $i\alpha \in \sigma_p(T)$ for some nonzero real α. Let us also suppose that the distribution kernel $k(v,v')$ does not vanish on a set of positive measure, so that the semigroup $L_1(\mathfrak{R},dv)$ is irreducible (i.e., does not have nontrivial closed invariant ideals). Then the spectrum of G on the imaginary line consists of a sequence $\{in\alpha\}_{n=-\infty}^{\infty}$ of simple eigenvalues. On denoting by Q a suitable projection of $L_1(\mathfrak{R},dv)$ onto the closed linear span of the corresponding eigenfunctions, we obtain

$$\lim_{t\to\infty} \|S(t)g - e^{i\gamma t}Qg\|_1 = 0$$

for some period $\gamma > 0$.

Assume now that there are no purely imaginary eigenvalues and that a nontrivial stationary solution exists.

In order to establish the decay to equilibrium we shall apply the 0-2 law for positive semigroups in L_1 spaces (see Ref. 19, Theorem C IV 2.6 plus corollary), which may be formulated as follows. Let $\{S(t)\}_{t>0}$ be a positive semigroup on the Banach space $L_1(E,\Sigma,\mu)$ and let $e(\mu)$ be a non-negative function in the kernel of its generator which does not vanish on a set of positive μ measure. Then for every $\tau > 0$ there exists a partition of E into two μ-measurable subsets $E_{0\tau}$ and $E_{2\tau}$ with the following properties:

(1) For every $t > 0$ the closed ideals of all functions in $L_1(E,\Sigma,\mu)$ having their support on $E_{0\tau}$ and $E_{2\tau}$, respectively, are invariant under $S(t)$.

(2) $|S(t) - S(t+\tau)|e_{0\tau} \downarrow 0$ as $t\to\infty$.

(3) $|S(t) - S(t+\tau)|e_{2\tau} = 2e_{2\tau}$ for all $t \geqslant 0$.

Here $e_{0\tau} = e\chi_{0\tau}$ and $e_{2\tau} = e\chi_{2\tau}$, where $\chi_{0\tau}$ and $\chi_{2\tau}$ denote the characteristic functions of $E_{0\tau}$ and $E_{2\tau}$, respectively.

Moreover, if the point spectrum $\sigma_p(G)$ of the generator G of $\{S(t)\}_{t>0}$ satisfies $\sigma_p(G) \cap \{\operatorname{Re}\lambda = 0\} = \{0\}$, then

$S(t)g$ converges in $L_1(E,\Sigma,\mu)$ strongly as $t\to\infty$ for all $g\in L_1(E,\Sigma,\mu)$ that vanish on $E_{2\tau}$.

Theorem 9: Suppose there is a nontrivial stationary solution φ in $L_1(\mathfrak{R},dv)$, while G does not have purely imaginary eigenvalues. Then

$$\lim_{t\to\infty}\|S(t)g-Pg\|_1=0,\quad g\in L_1(\mathfrak{R},dv), \tag{4.1}$$

where P is the projection given by (3.11).

Proof: As a result of the previous section we may write

$$T_\lambda g=\sum_{n=0}^{\infty}(L_\lambda K)^n L_\lambda g,\quad \text{Re }\lambda>0.$$

From this equality we easily derive that the closed invariant ideals of T_λ in $L_1(\mathfrak{R},dv)$ are ideals of all functions in $L_1(\mathfrak{R},dv)$ that have their support in $[v_0,\infty)$ for some $v_0\in\mathfrak{R}$. This in turn implies that for all $\tau\geqslant 0$ one of the sets $E_{0\tau}$ and $E_{2\tau}$ in the 0-2 law has zero measure.

First suppose $E_{2\tau}=\mathfrak{R}$. Then $e_{0\tau}=0$ and $e_{2\tau}=\varphi$ and hence

$$|S(t)-S(t+\tau)|\varphi=2\varphi=\{S(t)+S(t+\tau)\}\varphi,\quad t\geqslant 0,$$

which is impossible. Indeed, there is a sequence of functions $g_n\in L_1(\mathfrak{R},dv)$ with $|g_n|\leqslant\varphi$ such that for every $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that for $n\geqslant n_0$ and for $t\geqslant 0$

$$(\{S(t)+S(t+\tau)\}\varphi)(v)\leqslant(\{S(t)-S(t+\tau)\}g_n)(v)+\epsilon,\quad v\in\mathfrak{R}.$$

Writing $g_n^{\pm}=\sup(\pm g_n,0)$ we have

$$(S(t)[\varphi-g_n^+])(v)+(S(t)g_n^-)(v)+(S(t+\tau)[\varphi-g_n^-])(v)+(S(t+\tau)g_n^+)(v)\leqslant\epsilon.$$

If $g_n^{\pm}=0$, then $|g_n|=g_n^{\mp}\leqslant\varphi$ yields $\varphi(v)\leqslant\epsilon$ for all $v\in\mathfrak{R}$, which is a contradiction for sufficiently small ϵ. Consequently, $E_{0\tau}=\mathfrak{R}$ for all $\tau>0$, which implies that for every $g\in L_1(\mathfrak{R},dv)$

$$\lim_{t\to\infty}\|S(t)g-Qg\|_1=0$$

for some vector Qg (cf. Ref. 19 Corollary to Theorem C IV 2.6). But then the inequality

$$\|S(\tau)-Qg\|_1\leqslant\|S(\tau)-\mathbf{1}\|\,\|S(t)g-Qg\|_1+\|[S(t+\tau)-S(t)]g\|_1$$

implies $Q=P$, which completes the proof. □

Remarks: (1) If the generator $G\neq\overline{T}$, then there are no nontrivial stationary solutions (see Remark 1, after Theorem 7). On the other hand, if $G=T$, then T_λ is bounded as an operator from $L_1(\mathfrak{R},dv)$ into $L_1(\mathfrak{R},\nu\,dv)$. Then

$$T_\lambda=L_\lambda+L_\lambda KT_\lambda,\quad \lambda>0, \tag{4.2}$$

implies the Duhamel formula

$$S(t)=S_0(t)+\int_0^t S_0(t-\tau)KS(\tau)d\tau. \tag{4.3}$$

Now, if φ is an eigenfunction to the imaginary eigenvalue $i\lambda$ of T and hence $S(t)\varphi=e^{i\lambda t}\varphi$ and $S(t)|\varphi|=|\varphi|$ (see Nagel,[19] Corollary 2.3 on p. 297), we find, following an argument by Arlotti,[20]

$$e^{i\lambda t}\varphi=S_0(t)\varphi+\int_0^t e^{i\lambda\tau}S_0(t-\tau)K\varphi\,d\tau$$

and

$$|\varphi|=S_0(t)|\varphi|+\int_0^t S_0(t-\tau)K|\varphi|d\tau.$$

A simple comparison of the L_1 norms yields $\varphi\geqslant 0$ and $\lambda=0$. Hence if $G=T$ [which occurs if $\nu(v)$ is integrable or if $L_\lambda K$ is weakly compact on $L_1(\mathfrak{R},\nu\,dv)$], G does not have purely imaginary eigenvalues. Finally, if ν is essentially bounded, then $\|Kg\|_1\leqslant\|\nu\|_\infty\|g\|_1$ implies Eqs. (4.2) and (4.3). We may then repeat the above reasoning and conclude that $G=\overline{T}$ does not have purely imaginary eigenvalues.

(2) If we consider the case $k(v,v')=\delta(v-v')$ where $[S(t)g](v)=g(v-at)$ and hence $G=\overline{T}$, we see that $\varphi(v)=\exp\{-i\lambda v/a\}$ seemingly is an eigenfunction of $S(t)$ corresponding to the eigenvalue $e^{i\lambda t}$. However, $\varphi\notin L_1(\mathfrak{R},dv)$, though $\varphi\in L_1(\mathfrak{R},\nu\,dv)$ if ν is integrable.

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APPENDIX: AN ILLUSTRATIVE EXAMPLE

The object of this Appendix is to illustrate—with the help of two simple model problems—the typical patterns of behavior that one may expect from a population of charged particles moving through a host medium under the influence of a D.C. electric field. We shall consider two distinct versions of a simplified one-dimensional BGK model. A parallel and more sophisticated treatment has been proposed by Corngold and Rollins[7] who have adapted a one-dimensional Fokker–Planck model.

One simplified model is represented by the kinetic equation

$$\frac{\partial f}{\partial t}(v,t)+a\frac{\partial f}{\partial v}(v,t)=\nu(v)\{c(t)f_m(v)-f(v,t)\},\quad v\in\mathfrak{R},\quad t\geqslant 0, \tag{A1}$$

where

$$c(t)=\frac{\int_{-\infty}^{\infty}\nu(v)f(v,t)dv}{\int_{-\infty}^{\infty}\nu(v)f_m(v)dv}$$

is a normalization parameter, and $f_m(v)=\sqrt{\beta/\pi}\times\exp(-\beta v^2)$ is the normalized Maxwellian with

$$\langle v^2\rangle=\int_{-\infty}^{\infty}v^2 f_m(v)dv=(2\beta)^{-1}.$$

We consider the two following cases:

(i) $\nu(v) \equiv \nu_0 > 0$,

$$\text{(ii)}\ \ \nu(v) = \begin{cases} \nu_0, & -w \leqslant v \leqslant w, \\ 0, & |v| > w, \end{cases} \tag{A2}$$

where w and ν_0 are positive constants. Note that

$$\int_{-\infty}^{\infty} \nu(v) dv = +\infty$$

in case (i) whereas

$$0 < \int_{-\infty}^{\infty} \nu(v) dv = 2\nu_0 w < +\infty$$

in case (ii). Moreover, note that in case (ii) assumption (ii) is violated. We shall study the typical problem of the time evolution of a swarm of guest particles following the switching-on of the acceleration field at time $t = 0$, with $f(v,0) = f_m(v)$.

It is easy to establish the following results.

Case (i): Here $f(v,t)$ relaxes towards a steady profile. In fact, one can show that

$$f(v,t) = f_\infty(v) + \frac{a}{\nu_0} \exp(-r_0 t) \frac{d}{dv} f_\infty(v - at),$$

where

$$f_\infty(v) = (\nu_0/2a) \exp\{-\beta(v^2 - \lambda^2)\} \operatorname{erfc}(\lambda\sqrt{\beta}),$$

$$\text{with } \lambda = -v + (\nu_0/2a\beta)$$

and $f(v,t) \to f_\infty(v)$, as $t \to +\infty$.

Now we introduce the normalized velocity moments $\mu_k(t)$ which are defined by

$$\mu_k(t) = \int_{-\infty}^{\infty} v^k f(v,t) dv \left(\int_{-\infty}^{\infty} f(v,t) dv \right)^{-1}, \quad k = 0,1,2,\dots.$$

Then it is easy to show that

$$\begin{aligned} \langle v \rangle (t) &= \mu_1(t) \\ &= (a/\nu_0)\{1 - \exp(-\nu_0 t)\} \sim a/\nu_0, \quad \text{as } t \to +\infty. \end{aligned}$$

Further

$$\begin{aligned} \tfrac{1}{2}\langle v^2 \rangle (t) &= \mu_2(t) \\ &= (1/4\beta) + (a/\nu_0)^2 \{1 - (1 + \nu_0 t) \\ &\quad \times \exp(-\nu_0 t)\} \sim (1/4\beta) + (a/\nu_0)^2, \quad \text{as } t \to +\infty. \end{aligned}$$

Accordingly, for thermal agitation (relative to average velocity) we have

$$\tfrac{1}{2}(\langle v^2 \rangle (t) - \langle v \rangle^2 (t)) \sim 1/4\beta + \tfrac{1}{2}(a/\nu_0)^2, \quad \text{as } t \to +\infty.$$

In this case there is no runaway process.

Case (ii): For problem (A1) subject to (A2) and to the initial condition $f(v,0) = f_m(v)$, it is obvious that, within the quadrant $v \geqslant w$, $t \geqslant 0$, the solution $f(v,t)$ remains constant along the characteristics $v = \bar{v} + as$, $t = s$ $(s \geqslant 0)$. Accordingly, we can write

$$f(\bar{v} + at, t) = g_t(\bar{v}) = \begin{cases} f_m(\bar{v}), & \bar{v} \geqslant w \\ f(w, (w - \bar{v})/a), & w - at < \bar{v} < w, \end{cases}$$

where $t \geqslant 0$. Therefore,

$$f(\bar{v} + at, t) \sim g_\infty(\bar{v}), \text{ as } t \to +\infty, \quad \bar{v} \in \Re,$$

i.e., there is convergence towards a travelling wave. Note that this is not an explicit expression.

We can summarize the results as follows. Under case (i) conditions [for which $\int_{-\infty}^{\infty} \nu(v) dv = +\infty$] there are no runaways and the distribution function relaxes towards an asymptotic profile $f_\infty \in L_1(\Re, dv) \cap L_1(\Re, \nu\, dv)$ whose velocity moments are finite. Note that, in this case, if cold charged particles were fed continuously into the system, then the distribution function would not relax towards a steady state value. However, in both situations the velocity moments would relax towards finite values.

On the contrary, in case (ii) [for which $0 < \int_{-\infty}^{\infty} \nu(v) dv < +\infty$] $f(v,t)$ converges towards a "travelling wave" and all velocity moments diverge as $t \to +\infty$. Under steady feeding, the velocity moments would diverge as $t \to +\infty$, whereas $f(v,t)$ would converge to a steady profile belonging to $L_1(\Re, \nu\, dv)$; however, this profile would not belong to $L_1(\Re, dv)$.

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