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Differential properties of H-functions with applications to optically thick planetary atmospheres and spherical clouds

J.W. Hovenier, C.V.M. van der Mee★, and D. de Heer

Department of Physics and Astronomy, Free University, De Boelelaan 1081, NL-1081 HV Amsterdam, The Netherlands

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Summary. Integral expressions are derived for the first two derivatives of H-functions with respect to the directional variable. Assuming isotropic scattering these expressions are used to develop algorithms for numerical computation. Some results are presented in figures and tables with five decimal accuracy. The precise nature of the angular distribution of light emerging from a homogeneous semi-infinite plane-parallel planetary atmosphere is established. Several functions occurring in the asymptotic theory of isotropic scattering in spherical clouds are numerically evaluated. To clarify their nature when the albedo of single scattering is close to zero or one, expansions for H-functions and related functions are derived.

Key words: radiative transfer – atmospheres – light scattering – semi-infinite atmospheres

1. Introduction

The so-called Chandrasekhar H-functions (Chandrasekhar, 1950) play a prominent role in radiative transfer theory. In the context of semi-infinite homogeneous plane-parallel atmospheres with isotropic scattering the solutions of two basic problems can be expressed in H-functions. These are (i) the Milne problem, also referred to as the problem with a constant net flux, and (ii) the problem of diffuse reflection. The first problem is primarily considered for stellar atmospheres, the second one for planetary atmospheres. For details we refer to Chandrasekhar (1950), Busbridge (1960), Ambarzumian (1960), Kourganoff (1952), Sobolev (1963, 1975), Ivanov (1973), Van de Hulst (1980) and references cited in these books. H-functions are also used to model the reflection of light by particulate surface material of celestial bodies such as planets, moons and asteroids (See e.g. Hapke, 1981, 1984, 1986; Lumme and Bowell, 1981, and Simonelli and Veverka, 1986). Finally, H-functions are employed in studies of radiation transport in spherical clouds (see e.g. Van de Hulst, 1987, 1988).

A multitude of authors has contributed to creating an impressive body of knowledge on properties of H-functions. Restricting ourselves to isotropic scattering we mention the following aspects of H-functions (cf. the books mentioned above):

(i) their physical meaning (see also Van de Hulst, 1948);
(ii) tables with numerical values (see also Bosma and De Rooij, 1983; Domke, 1988);
(iii) integral equations, integral properties (in particular, moment relations) and integral representations;
(iv) expansions in the directional variable as well as in the albedo of single scattering.

Until now little attention has been paid in the literature to the derivatives of H-functions (and related functions) with respect to the directional variable, which we shall call μ. Yet it is known, for instance, that for isotropic scattering the first such derivative generally has a singularity at $\mu = 0$. The theory of light scattering in optically large spheres developed by Van de Hulst (1988) has called for a better understanding and an increased knowledge of these derivatives, since the first two derivatives of the H-function with respect to μ play an important role in this theory. Thus the main purposes of this paper are (i) to establish certain differential properties of H-functions and related functions, in particular for isotropic scattering; (ii) to provide some computational procedures along with tables of numerical results, and (iii) to present some applications to plane-parallel atmospheres and spherical clouds.

2. The H-function and its first two derivatives

H-functions depend on a directional variable μ $(0 \le \mu \le 1)$ and the albedo of single scattering a $(0 \le a \le 1)$. Not writing the latter dependence explicitly we have

$$\frac{1}{H(\mu)} = \left[1 - 2\int_0^1 \Psi(x)\,dx\right]^{1/2} + \int_0^1 \frac{x\Psi(x)}{\mu + x} H(x)\,dx \tag{1}$$

where $\Psi(x)$ is the so-called characteristic function depending on the type of scattering considered (see e.g. Chandrasekhar, 1950, Sect. 38; Busbridge, 1960; Sobolev, 1975, Sect. 5.1). It is a real and continuous function of x on $[0,1]$ satisfying

$$2\int_0^1 \Psi(x)\,dx \le 1\,. \tag{2}$$

Send offprint requests to: J.W. Hovenier

★ Permanent address: Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, PA 15260, USA

If we take the derivative with respect to μ on both sides of Eq. (1) we find

$$\frac{dH(\mu)}{d\mu} \equiv H'(\mu) = H(\mu)^2 \int_0^1 \frac{x\Psi(x)H(x)}{(\mu+x)^2}\,dx\,. \tag{3}$$

Similarly we find for the second derivative

$$\frac{d^2H(\mu)}{d\mu^2} \equiv H''(\mu) = 2H(\mu)H'(\mu)\int_0^1 \frac{x\Psi(x)H(x)}{(\mu+x)^2}\,dx - 2H(\mu)^2\int_0^1 \frac{x\Psi(x)H(x)}{(\mu+x)^3}\,dx \tag{4}$$

which, on using Eq. (3), yields

$$H''(\mu) = \frac{2[H'(\mu)]^2}{H(\mu)} - 2H(\mu)^2\int_0^1 \frac{x\Psi(x)H(x)}{(\mu+x)^3}\,dx\,. \tag{5}$$

From hereon we consider isotropic scattering. In this case

$$\Psi(x) = \frac{a}{2} \tag{6}$$

and Eqs. (1), (3) and (5) take the form

$$\frac{1}{H(\mu)} = (1-a)^{1/2} + \frac{a}{2}\int_0^1 \frac{xH(x)}{\mu+x}\,dx \tag{7}$$

$$H'(\mu) = \frac{a}{2}H(\mu)^2\int_0^1 \frac{xH(x)}{(\mu+x)^2}\,dx \tag{8}$$

$$H''(\mu) = \frac{2[H'(\mu)]^2}{H(\mu)} - aH(\mu)^2\int_0^1 \frac{xH(x)}{(\mu+x)^3}\,dx\,. \tag{9}$$

The moments of the H-function are defined by

$$\alpha_i = \int_0^1 \mu^i H(\mu)\,d\mu \tag{10}$$

where $i = 0, 1, 2, \ldots$

If $a = 0$ Eqs. (7)–(9) show that $H(\mu) \equiv 1$ and thus $H'(\mu)$ and $H''(\mu)$ vanish for every value of μ. Henceforth, unless stated otherwise, we shall assume $a > 0$. For every a the H-function is monotonically increasing in μ from $H(0) = 1$ to $H(1)$ (See Fig. 1). Close to $\mu = 0$ the slope is large and Eq. (8) shows that $H'(\mu)$ tends to $+\infty$ as $\mu \to 0$, which becomes apparent by adding and subtracting $(\mu + x)^{-1}$ under the integral sign on the right-hand side of Eq. (8). The result is

$$H'(\mu) = \frac{a}{2}H(\mu)^2\left[\int_0^1 \frac{xH(x)-\mu-x}{(\mu+x)^2}\,dx + \ln\left(1+\frac{1}{\mu}\right)\right] \tag{11}$$

which yields in particular [cf. Van de Hulst, 1980, Sect. 8.3]

$$\lim_{\mu\to 0}\left[\frac{H(\mu)-1}{\mu} - \frac{a}{2}\ln\left(1+\frac{1}{\mu}\right)\right] = \frac{a}{2}\int_0^1 \frac{H(x)-1}{x}\,dx = a\ln H(1)\,. \tag{12}$$

From Eqs. (8) and (9) we easily derive the two identities

$$\lim_{\mu\to 0}\mu H'(\mu) = \frac{a}{2}\lim_{\mu\to 0}H(\mu)^2\int_0^1 \frac{\mu x H(x)}{(\mu+x)^2}\,dx = 0 \tag{13}$$

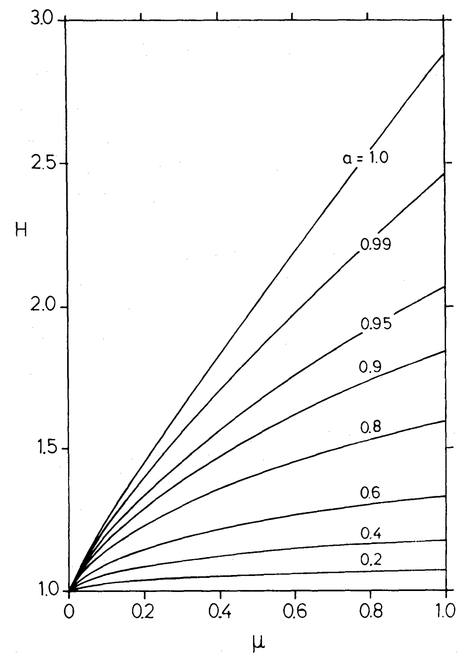

Fig. 1. Dependence of the H-function for isotopic scattering on the angular variable, μ, for various values of the albedo of single scattering, a

$$\lim_{\mu\to 0}\mu^2 H''(\mu) = \lim_{\mu\to 0}\left[\frac{2[\mu H'(\mu)]^2}{H(\mu)} - aH(\mu)^2\int_0^1 \frac{\mu^2 x H(x)}{(\mu+x)^3}\,dx\right] = 0\,. \tag{14}$$

Here we have used the principle of dominated convergence (See e.g. Rudin, 1964, Theorem 10.32) and the fact that the integrands in Eqs. (13) and (14) are dominated by $H(x)$. In fact, we may generalize Eq. (13) to find that $\mu^\alpha H'(\mu)$ vanishes as $\mu \to 0$ for every $\alpha > 0$. Using this observation we easily prove the identity

$$\lim_{\mu\to 0}\mu H''(\mu) = \lim_{\mu\to 0}\left[\frac{2[\mu^{1/2}H'(\mu)]^2}{H(\mu)} - aH(\mu)^2\int_0^1 \frac{\mu x^2}{(\mu+x)^3}\frac{H(x)-1}{x}\,dx - a\mu H(\mu)^2\int_0^1 \frac{x}{(\mu+x)^3}\,dx\right] = -\frac{a}{2} \tag{15}$$

since the second integrand is dominated by $[H(x)-1]/x$. Hence $H''(\mu)$ tends to $-\infty$ as μ tends to zero. Furthermore, it is clear from the shape of $H(\mu)$ that $H'(\mu)$ is monotonically decreasing and thus $H''(\mu) < 0$ in the entire range of μ.

Equations (8) and (9) show that two integrations over the directional variable suffice for the numerical evaluation of $H'(\mu)$ and $H''(\mu)$ provided $H(\mu)$ is known. Several methods for the numerical computation of $H(\mu)$ by iteration have been discussed by Bosma and De Rooij (1983). We used their second method which is an iterative procedure for solving Eq. (7) starting the iterations with $H(\mu) \equiv 1$ and dividing each iterate by its value at $\mu = 0$ before starting the next iteration step. Therefore the basic job is to evaluate integrals of the type

$$\int_0^1 \frac{xH(x)}{(\mu+x)^j}\,dx \tag{16}$$

with $j = 1, 2, 3$. For this purpose we used Gaussian quadrature either with N_G division points in the entire interval $(0, 1)$ or with N_{G_1} points in $(0, 0.1)$ and N_{G_2} points in $(0.1, 1)$ with $N_{G_1} \geq N_{G_2}$. In our calculations we used N_G, N_{G_1}, $N_{G_2} \in \{8, 16, 32, 64, 128, 256\}$. Values of $H(\mu)$ in non-Gaussian division points were obtained by one additional integration after computing $H(\mu)$ in the Gaussian division points (cf. Eq. (7)). A similar procedure was used for $H'(\mu)$ and $H''(\mu)$.

Some results of our calculations are shown in Tables 1a–1c. We expect the numbers to be accurate within one unit of the last decimal given. To assess the accuracy of our numerical results we have not only varied N_G, N_{G_1} and N_{G_2} but also performed a number of checks in some of which alternative approaches were involved. For example:

(i) $H(\mu)$ was checked by comparison to numerical data in the literature (See Van de Hulst, 1980, and Bosma and De Rooij, 1983, and references cited by these authors).

(ii) $H'(\mu)$ was also computed by numerical integration from

$$H'(\mu) = \frac{H(\mu)[H(\mu) - 1]}{\mu} - \frac{a\mu}{2} H(\mu)^2 \int_0^1 \frac{H(x)}{(\mu + x)^2} dx \tag{17}$$

which is found by differentiation from (cf. Chandrasekhar, 1950, Sect. 37)

$$H(\mu) = 1 + \frac{a\mu}{2} H(\mu) \int_0^1 \frac{H(x)}{\mu + x} dx. \tag{18}$$

Our numerical experiments suggest that employing Eq. (11) or Eq. (17) instead of Eq. (8) does not yield a generally better method for computing $H'(\mu)$. Another check on $H'(\mu)$ was obtained from the obvious relation

$$H(\mu) = 1 + \int_0^\mu H'(x) dx \tag{19}$$

by numerical integration of $H'(x)$ over the interval $[0, \mu]$.

(iii) $H''(\mu)$ was also calculated from

$$H''(\mu) = \frac{2[H'(\mu)]^2}{H(\mu)} + 2\frac{H'(\mu)}{\mu} - 2\frac{H(\mu)}{\mu^2}[H(\mu) - 1] + a\mu H(\mu)^2 \int_0^1 \frac{H(x)}{(\mu + x)^3} dx \tag{20}$$

which easily follows from Eq. (17). Employing Eq. (20) rather than Eq. (9) did not yield an overall improvement. Further we checked by numerical integration whether

$$\int_\mu^1 H''(x) dx = H'(1) - H'(\mu) \tag{21}$$

and

$$\int_0^1 \mu(1 - \mu)^2 H''(\mu) d\mu = 1 - 4\alpha_0 + 6\alpha_1. \tag{22}$$

Equation (22) is readily verified by partial integration and using Eq. (8). The last check has the advantage of being independent of H'.

Table 1a. The H-function and its first two derivatives with respect to μ for isotropic scattering and albedos of single scattering 0.20, 0.40 and 0.60. Three moments of the H-functions are given on the bottom line

	ALBEDO IS .20			ALBEDO IS .40			ALBEDO IS .60		
μ	H	H'	H''	H	H'	H''	H	H'	H''
0.000	1.00000	∞	$-\infty$	1.00000	∞	$-\infty$	1.00000	∞	$-\infty$
.005	1.00273	.44695	-19.70328	1.00565	.93329	-39.15043	1.00886	1.47681	-58.20027
.010	1.00478	.37898	-9.75201	1.00994	.79832	-19.35372	1.01568	1.27647	-28.68753
.015	1.00657	.33960	-6.44264	1.01373	.72019	-12.78698	1.02175	1.16074	-18.92984
.020	1.00819	.31192	-4.79101	1.01718	.66522	-9.51602	1.02734	1.07937	-14.08252
.025	1.00969	.29062	-3.80158	1.02040	.62290	-7.55956	1.03258	1.01674	-11.18990
.050	1.01605	.22587	-1.82894	1.03416	.49368	-3.66843	1.05532	.82506	-5.46214
.100	1.02562	.16450	-.85005	1.05536	.36948	-1.74066	1.09135	.63858	-2.64414
.200	1.03892	.10911	-.36930	1.08578	.25411	-.78559	1.14516	.45994	-1.24648
.300	1.04830	.08098	-.21524	1.10790	.19338	-.47228	1.18587	.36179	-.77898
.400	1.05546	.06350	-.14204	1.12517	.15455	-.31963	1.21860	.29672	-.54495
.500	1.06118	.05154	-.10062	1.13919	.12737	-.23120	1.24581	.24973	-.40550
.600	1.06588	.04287	-.07470	1.15088	.10727	-.17469	1.26892	.21405	-.31395
.700	1.06982	.03632	-.05737	1.16080	.09184	-.13619	1.28887	.18603	-.24999
.800	1.07319	.03122	-.04522	1.16935	.07966	-.10874	1.30631	.16348	-.20334
.900	1.07610	.02716	-.03638	1.17681	.06985	-.08848	1.32170	.14498	-.16819
1.000	1.07865	.02387	-.02977	1.18338	.06180	-.07313	1.33541	.12958	-.14105
	α_0=1.05573	α_1= .53315	α_2 = .35679	α_0=1.12702	α_1= .57621	α_2= .38747	α_0=1.22515	α_1= .63663	α_2= .43092

Table 1b. As Table 1a but for albedos 0.80, 0.90 and 0.95

	ALBEDO IS .80			ALBEDO IS .90			ALBEDO IS .95		
μ	H	H'	H''	H	H'	H''	H	H'	H''
0.000	1.00000	∞	$-\infty$	1.00000	∞	$-\infty$	1.00000	∞	$-\infty$
.005	1.01255	2.12082	-76.53455	1.01480	2.52432	-85.13581	1.01618	2.77963	-89.08317
.010	1.02242	1.85819	-37.47978	1.02660	2.23308	-41.41093	1.02923	2.47573	-43.07129
.015	1.03131	1.70731	-24.62430	1.03732	2.06680	-27.06217	1.04115	2.30320	-28.00268
.020	1.03957	1.60164	-18.26298	1.04735	1.95092	-19.98197	1.05235	2.18356	-20.58152
.025	1.04736	1.52049	-14.48055	1.05688	1.86231	-15.78353	1.06303	2.09247	-16.18913
.050	1.08191	1.27296	-7.04906	1.09968	1.59419	-7.58719	1.11151	1.81961	-7.65311
.100	1.13881	1.03124	-3.45623	1.17214	1.33525	-3.69198	1.19523	1.56102	-3.65011
.200	1.22864	.79312	-1.70750	1.29143	1.07971	-1.85175	1.33734	1.31013	-1.81190
.300	1.30059	.65546	-1.12266	1.39135	.92842	-1.25431	1.46045	1.16158	-1.23927
.400	1.36109	.55951	-.82361	1.47850	.81951	-.95159	1.57095	1.05307	-.95779
.500	1.41326	.48699	-.63980	1.55603	.73435	-.76441	1.67179	.96645	-.78673
.600	1.45899	.42962	-.51484	1.62588	.66473	-.63514	1.76471	.89397	-.66927
.700	1.49954	.38289	-.42438	1.68935	.60621	-.53955	1.85092	.83160	-.58213
.800	1.53583	.34401	-.35608	1.74740	.55610	-.46560	1.93129	.77691	-.51407
.900	1.56854	.31117	-.30292	1.80079	.51260	-.40656	2.00650	.72835	-.45896
1.000	1.59822	.28307	-.26057	1.85010	.47443	-.35831	2.07712	.68481	-.41318
	α_0=1.38197	α_1= .73582	α_2= .50322	α_0=1.51949	α_1= .82532	α_2= .56945	α_0=1.63451	α_1= .90188	α_2= .62679

Table 1c. As Table 1a but for albedos 0.99 and 1.00

	ALBEDO IS .99			ALBEDO IS 1.00		
μ	H	H'	H''	H	H'	H''
0.000	1.00000	∞	$-\infty$	1.00000	∞	$-\infty$
.005	1.01775	3.07656	-91.66380	1.01875	3.27573	-91.56476
.010	1.03226	2.76523	-43.88668	1.03426	2.96606	-43.42059
.015	1.04561	2.59017	-28.27792	1.04863	2.79361	-27.71760
.020	1.05825	2.46984	-20.60838	1.06229	2.67618	-20.01350
.025	1.07036	2.37899	-16.07923	1.07544	2.58835	-15.47060
.050	1.12607	2.11243	-7.32587	1.13657	2.33707	-6.72070
.100	1.22488	1.87166	-3.28878	1.24735	2.12552	-2.72360
.200	1.39977	1.65424	-1.50805	1.45035	1.96165	-.99784
.300	1.55871	1.53307	-.99392	1.64252	1.88955	-.52080
.400	1.70750	1.44658	-.76119	1.82928	1.84887	-.31726
.500	1.84860	1.37754	-.63048	2.01278	1.82295	-.21134
.600	1.98336	1.31895	-.54663	2.19413	1.80516	-.14938
.700	2.11262	1.26739	-.48767	2.37397	1.79232	-.11019
.800	2.23700	1.22094	-.44333	2.55270	1.78270	-.08396
.900	2.35694	1.17842	-.40828	2.73059	1.77527	-.06563
1.000	2.47279	1.13907	-.37949	2.90781	1.76941	-.05238
	α_0=1.81818	α_1=1.02718	α_2= .72196	α_0=2.00000	α_1=1.15470	α_2= .82035

Table 2. The numbers of Gausspoints which are sufficient for 5-decimal accuracy for the values of μ and a of Tables 1a–1c

	$H(\mu)$	$H'(\mu)$	$H''(\mu)$
N_G	32	64	128
N_{G_1}	8	32	32
N_{G_2}	8	16	16

The values of N_G listed in Table 2 were found to be sufficient to yield the numbers in Tables 1a–1c by numerical quadrature of the integrals in Eqs. (7)–(10). The same accuracy was reached by the combinations of N_{G_1} and N_{G_2} listed in Table 2. We thus see that splitting the interval at $\mu = 0.1$ is computationally preferable. $N_G = 8$ sufficed for the computation of α_0, α_1 and α_2.

If the albedo of single scattering is close to zero or one, $H(\mu)$, $H'(\mu)$ and $H''(\mu)$ may be obtained from series expansions which are discussed in the Appendix.

In Figs. 2 and 3 we have plotted $H'(\mu)$ and $H''(\mu)$ as functions of μ for various values of the albedo of single scattering. It is clear from these plots that both functions vary strongly for $\mu \lesssim 0.1$ and $a \geq 0.2$.

One advantage of having tables of $H'(\mu)$ and $H''(\mu)$ along with $H(\mu)$ is that it enables rapid and simple interpolation, which is a frequently occurring problem in radiative transfer. Writing for $\mu > 0$ and $0 < |h| < \mu$

$$H(\mu + h) = H(\mu) + H'(\mu)h + \tfrac{1}{2}H''(\mu)h^2 + \cdots \tag{23}$$

we find e.g. from Table 1c for $a = 1$ that $H(0.015 + 0.005) = 1.06225$. Direct linear interpolation between the values for 0.015 and 0.025 would have given 1.06204 which is further removed from $H(0.020) = 1.06229$ as tabulated in Table 1c than the value obtained by interpolation based on Eq. (23). Van de Hulst (1980, Sect. 8.3) tabulated 4-decimal numbers for $H'(1)$ and 3-decimal numbers for $\frac{1}{2}H''(1)$ which he obtained by numerical differentiation of tables for H-functions published by Stibbs and Weir (1959). Van de Hulst's table for $\frac{1}{2}H''(1)$ would need some minor corrections to make it accurate within one unit of the last decimal given.

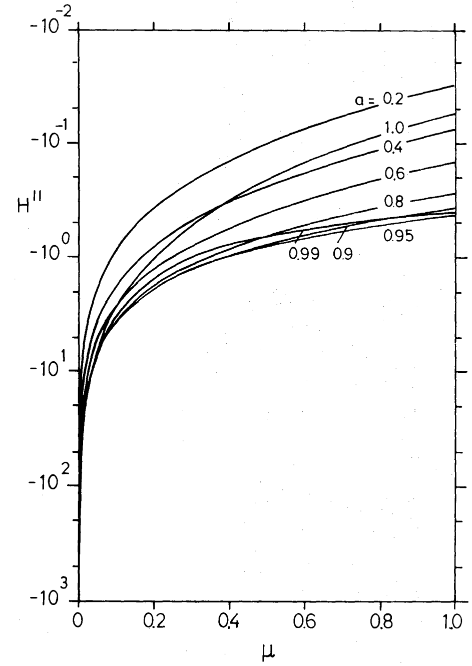

Fig. 3. As Fig. 1 but for the second derivative of the H-function with respect to μ

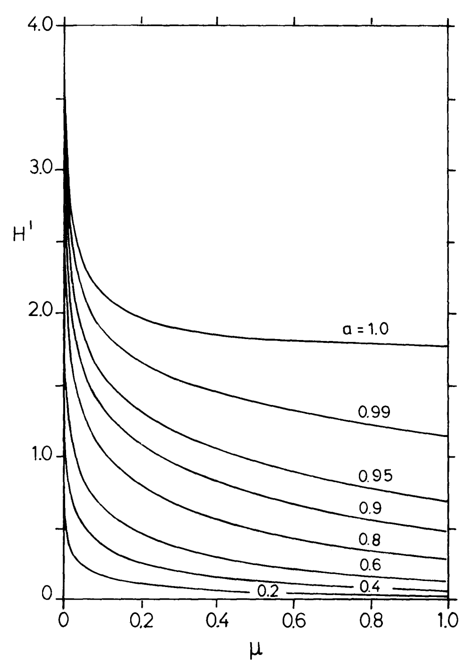

Fig. 2. As Fig. 1 but for the first derivative of the H-function with respect to μ

3. Plane-parallel atmospheres

The differential properties of H-functions may be used to understand the angular dependence of the radiation emerging from a star or planet. Some aspects of this topic will be considered in this section.

Let us consider a planetary atmosphere modeled as a semi-infinite homogeneous layer illuminated at the top by a parallel beam of radiation making an angle $\arccos \mu_0$ with the normal ($0 \leq \mu_0 \leq 1$) and having a flux πF per unit area perpendicular to the beam. For isotropic scattering the emergent intensity may be written as

$$I(\gamma) = \frac{a}{4} F \frac{\mu_0}{\mu + \mu_0} H(\mu) H(\mu_0) \tag{24}$$

where $\gamma = \arccos \mu$ is the angle between the outward normal and the direction of the emergent radiation. From Eq. (24) we find

$$\frac{dI(\gamma)}{d\gamma} \equiv I'(\gamma) = -\frac{a}{4} F \frac{\mu_0}{\mu + \mu_0} H(\mu_0) \left[H'(\mu) - \frac{H(\mu)}{\mu + \mu_0} \right] \sin \gamma . \tag{25}$$

Assuming $\mu_0 \neq 0$ it is clear from Eq. (25) and the properties of $H(\mu)$ and $H'(\mu)$ that the first derivative of $I(\gamma)$ with respect to γ always vanishes if $\gamma = 0$ and tends to $-\infty$ as $\gamma \to \frac{1}{2}\pi$. Figures 4–6 show the dependence of $I(\gamma)/F$ on γ for $a = 0.6$, 0.99 and 1.0,

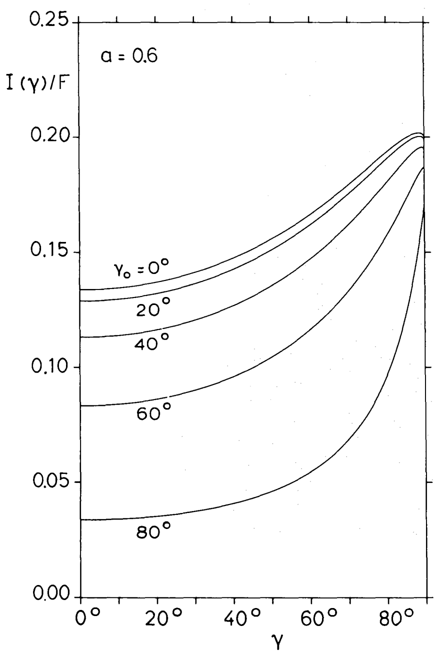

Fig. 4. The angular dependence of light reflected by a semi-infinite isotropically scattering atmosphere which is illuminated by a parallel beam of radiation. The angles of incidence and emergence with respect to the normal are γ_0 and γ, respectively. Here $a = 0.60$

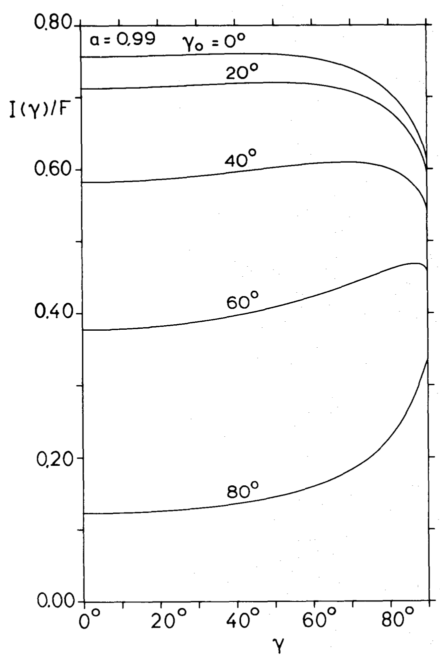

Fig. 5. Same as Fig. 4 but for $a = 0.99$

respectively, and various values of γ_0. It is clear from these figures that the slope at $\gamma = \frac{1}{2}\pi$ can easily be misjudged especially when the intensity increases for increasing γ but the corresponding curve shows no maximum for $0 < \gamma < \frac{1}{2}\pi$. To understand why some of the curves show a maximum for $0 < \gamma < \frac{1}{2}\pi$ and others do not we must investigate whether for given $\mu_0 \in (0, 1]$

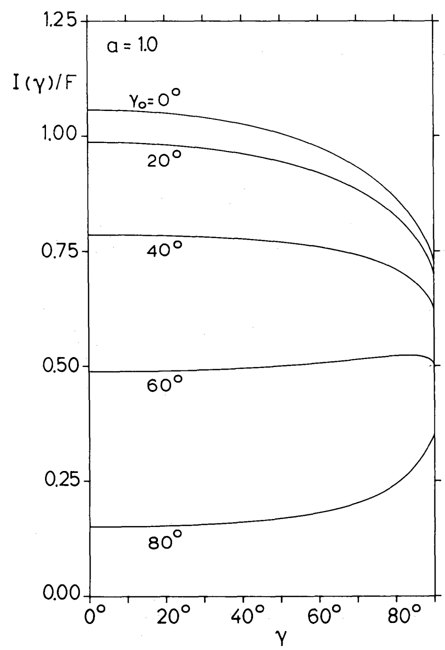

Fig. 6. Same as Fig. 4 but for $a = 1.00$

the expression between square brackets on the right-hand side of Eq. (25) has a zero $\mu \in (0, 1)$. Introducing the function

$$G(a, \mu) = \frac{H(a, \mu)}{H'(a, \mu)} - \mu \tag{26}$$

it is clear that μ is such a zero if

$$\mu_0 = G(a, \mu)\,. \tag{27}$$

Here we have written the dependence on a explicitly. The function $G(a, \mu)$ is monotonically increasing from $G(a, 0) = 0$ to $G(a, 1) > 0$, since

$$G'(a, \mu) = -\frac{H(\mu)H''(\mu)}{[H'(\mu)]^2} > 0\,. \tag{28}$$

Thus for given $0 < \mu_0 \leq 1$ Eq. (27) can be satisfied for precisely one $0 < \mu < 1$, provided $\mu_0 < G(a, 1)$. To investigate whether $G(a, 1) \leq 1$ we first note that

$$G(a, 1) = \left[\frac{a}{2} H(a, 1) \int_0^1 \frac{xH(a, x)}{(x+1)^2}\, dx\right]^{-1} - 1 \tag{29}$$

[cf. Eqs. (8) and (26)]. Since $H(a, \mu)$ increases monotonically with a [as follows directly from the first integral expression given by Van de Hulst (1980) in Sect. 8.3.1], $G(a, 1)$ decreases monotonically with a. Further, we find from Tables 1a–1c that $G(0.99, 1) \simeq 1.17089$ and $G(1.0, 1) \simeq 0.64338$. Consequently, there exists a unique a_0 with $0.99 < a_0 < 1$ satisfying $G(a_0, 1) = 1$. More detailed computations provided $a_0 \simeq 0.99508$. We therefore conclude that for $0 < a < a_0$ the graphs of $I(\gamma)/F$ have precisely one maximum at some $0 < \gamma < \frac{1}{2}\pi$ irrespective of the choice of $\mu_0 \in (0, 1]$ whereas for $a_0 \leq a \leq 1$ there is such a maximum only if $\mu_0 < G(a, 1)$. This behaviour is in agreement with Figures 4–6. For some of the curves the maximum lies so closely to $\gamma = 90°$ that it cannot be seen. For $a = 1$ we have a maximum only if $\mu_0 < 0.64338$, i.e. if $\gamma_0 \gtrsim 49°57'$. As a result the upper three curves in Fig. 6 show a monotonic decrease.

4. Spherical clouds

Recently Van de Hulst (1988) found an expression in closed form for the spherical reflection function of a homogeneous sphere with isotropic scattering if the radiation field is spherically symmetric. On developing the asymptotic theory (the dominant deviation for optically large spheres from the well-known theory for spheres with an infinitely large optical diameter) he introduced a linear differential operator of the second order, L, as follows. If $f(\mu)$ is an arbitrary function of μ, the cosine of an angle, then

$$\begin{aligned} Lf &= \mu\frac{d(\mu f)}{d\mu} - \frac{1}{2}(1-\mu^2)\frac{d^2(\mu f)}{d\mu^2} \\ &= \frac{1}{2}\frac{d}{d\mu}\left[(\mu^2-1)\frac{d(\mu f)}{d\mu}\right]. \end{aligned} \tag{30}$$

The function LH is of particular importance. It occurs, for instance, in the asymptotic expression for the radiance leaving the cloud under the angle $\arccos\mu$ with the normal if this cloud is exposed to the uniform incident radiance 1. Rewriting Eq. (30) we find

$$\begin{aligned} LH &= [LH](\mu) \\ &= \mu H(\mu) + (2\mu^2-1)H'(\mu) + \tfrac{1}{2}\mu(\mu^2-1)H''(\mu). \end{aligned} \tag{31}$$

Thus $LH = \mu$ if $a = 0$. For $a > 0$ we computed LH from Eq. (31) using the methods of Sect. 2 for the calculation of the derivatives. Results are shown in Table 3 for various values of the albedo of single scattering. We expect the numbers to be accurate within one unit of the last decimal given, for which either $N_G = 64$ or $N_{G_1} = 32$ and $N_{G_2} = 16$ were found to be sufficient. It may be shown that LH tends to $-\infty$ if $a > 0$ and $\mu \to 0$ [cf. Eqs. (31) and (13)–(15)]. To check the numerical results we verified the simple relations

$$\int_0^1 LH\,d\mu = \tfrac{1}{2} \tag{32}$$

and

$$\int_0^1 \mu LH\,d\mu = \alpha_2 \tag{33}$$

by numerical quadrature. These equations are readily checked by integration by parts and using Eq. (13). Complete agreement was found with some 4-decimal numbers for $\mu = 1$ obtained by numerical differentiation and reported by Van de Hulst (1988). In Fig. 7 the dependence of LH on μ is plotted for various values of the albedo of single scattering. The Appendix contains series expansions for LH if a is close to 0 or 1.

The asymptotic expressions for the spherical reflection function itself contain the function

$$Q(\mu,\mu_0) = LR^\infty + R^\infty\bar{L} \tag{34}$$

Table 3. LH as a function of μ for various values of a, the albedo of single scattering

μ	a = .20	a = .40	a = .60	a = .80	a = .90	a = .95	a = .99	a =1.00
0.000	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
.005	-.39265	-.83034	-1.32620	-1.92432	-2.30628	-2.55171	-2.84216	-3.04157
.010	-.32010	-.69131	-1.12264	-1.66022	-2.01534	-2.24961	-2.53494	-2.73804
.015	-.27604	-.60878	-1.00294	-1.50643	-1.84739	-2.07658	-2.36129	-2.56879
.020	-.24361	-.54922	-.91720	-1.39701	-1.72867	-1.95503	-2.24070	-2.45274
.025	-.21752	-.50217	-.84986	-1.31151	-1.63638	-1.86104	-2.14840	-2.36496
.050	-.12833	-.34802	-.63196	-1.03672	-1.34203	-1.56409	-1.86288	-2.10095
.100	-.01657	-.17039	-.38579	-.72566	-1.00857	-1.22959	-1.54894	-1.82346
.200	.14285	.05879	-.07445	-.32002	-.55728	-.76391	-1.09717	-1.41885
.300	.27747	.23826	.16542	.00594	-.17268	-.34520	-.65383	-.98559
.400	.40287	.39867	.37722	.30233	.19400	.07320	-.17279	-.47222
.500	.52368	.54926	.57407	.58310	.55417	.50018	.35375	.13454
.600	.64187	.69403	.76170	.85395	.91135	.93701	.92566	.83971
.700	.75839	.83503	.94311	1.11777	1.26673	1.38292	1.54054	1.64560
.800	.87380	.97345	1.12010	1.37626	1.62068	1.83659	2.19530	2.55341
.900	.98844	1.11000	1.29380	1.63051	1.97328	2.29667	2.88678	3.56381
1.000	1.10252	1.24518	1.46498	1.88129	2.32453	2.76194	3.61187	4.67722

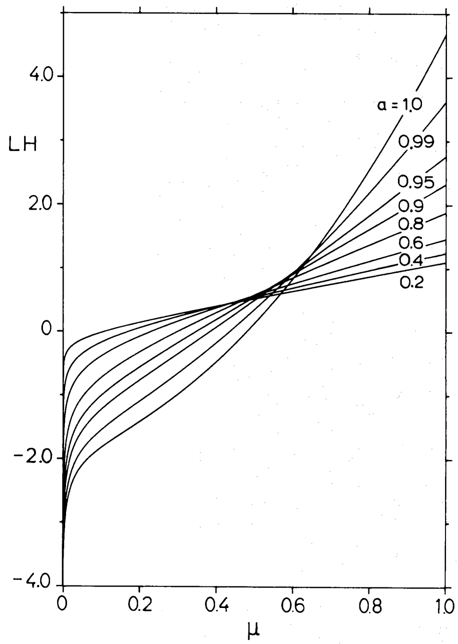

Fig. 7. Functions arising from letting a differential operator L act on H-functions for various values of the albedo of single scattering $a>0$. For $a=0$ the curves tend to the straight line $LH=\mu$ (not drawn)

where

$$R^\infty = \frac{aH(\mu)H(\mu_0)}{4(\mu+\mu_0)} \tag{35}$$

$$LR^\infty = [LR^\infty](\mu,\mu_0) = \frac{aH(\mu_0)}{4}\left[\frac{\mu_0(1+\mu\mu_0)}{(\mu+\mu_0)^3}H(\mu) + \frac{\mu^3+2\mu^2\mu_0-\mu_0}{(\mu+\mu_0)^2}H'(\mu) + \frac{\mu(\mu^2-1)}{2(\mu+\mu_0)}H''(\mu)\right] \tag{36}$$

and $R^\infty \bar{L}$ is obtained by interchanging μ and μ_0 in Eq. (36). Consequently, $Q(\mu,\mu_0)\equiv 0$ if $a=0$. For $a>0$ we calculated $Q(\mu,\mu_0)$ from the above equations employing the numerical techniques of Sect. 2 for the derivatives. The numbers shown in Tables 4a–4c are estimated to be correct within one unit of the last decimal given. For this purpose we found either $N_G=32$ or $N_{G_1}=16$ and $N_{G_2}=16$ to be sufficient. As a check we have also computed the moments

$$[QN](\mu) = \int_0^1 Q(\mu,\mu_0)\,d\mu_0 \tag{37}$$

$$[QU](\mu) = \int_0^1 2\mu_0 Q(\mu,\mu_0)\,d\mu_0 \tag{38}$$

by direct numerical integration of $Q(\mu,\mu_0)$ and compared the results with the right-hand sides of the explicit expressions

$$[QN](\mu) = \frac{a}{8\mu}H(\mu) + \frac{1}{2}\mu H'(\mu) - \frac{1}{4}(1-\mu^2)H''(\mu) \tag{39}$$

$$[QU](\mu) = \left[\mu(1-a)^{1/2} + \frac{a}{2}\alpha_1\right]H(\mu) - (1-a)^{1/2}[LH](\mu)\,. \tag{40}$$

The last two equations are readily obtained from Eqs. (34)–(38) using integration by parts. For $a=0$ we have

$$[QN](\mu) = [QU](\mu) = 0\,. \tag{41}$$

If $a=1$ $[QU](\mu)$ simplifies to

$$[QU](\mu) = \frac{H(\mu)}{\sqrt{3}}\,. \tag{42}$$

The entries in the last two columns of Tables 4a–4c have been computed from Eqs. (39)–(40). We found complete agreement with 4-decimal numbers for QU with $\mu=1$ as tabulated by Van de Hulst (1988) and based on numerical differentiation. The entries in Tables 4a–4c for $\mu=0$ require special care. They have been obtained as follows. Using Eqs. (13)–(15), (34) and (36) we readily obtain

$$Q(0,\mu_0) = Q(\mu_0,0) = -\infty, \qquad \mu_0 > 0\,. \tag{43}$$

From Eqs. (13), (14) and (39) we find immediately

$$\lim_{\mu\to 0}[QN](\mu) = +\infty\,. \tag{44}$$

Equation (40) yields for $0<a<1$

$$\lim_{\mu\to 0}[QU](\mu) = +\infty \tag{45}$$

Table 4a. The functions $Q(\mu,\mu_0)$, $[QN](\mu)$ and $[QU](\mu)$ for an albedo of single scattering 0.20

ALBEDO IS .20

μ	μ_0 = .1	.2	.3	.5	.7	.9	1.0	QN	QU
0.0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	∞	∞
.1	1.30868	.59341	.33813	.15275	.08677	.05588	.04628	.47502	.16124
.2	.59341	.34753	.22922	.12263	.07712	.05341	.04559	.22941	.11347
.3	.33813	.22922	.16633	.10045	.06823	.04995	.04362	.14847	.08900
.5	.15275	.12263	.10045	.07170	.05452	.04339	.03921	.08481	.06275
.7	.08677	.07712	.06823	.05452	.04494	.03801	.03523	.05823	.04853
.9	.05588	.05341	.04995	.04339	.03801	.03369	.03186	.04384	.03953
1.0	.04628	.04559	.04362	.03921	.03523	.03186	.03037	.03890	.03616

Table 4b. As Table 4a but for albedo 0.80

ALBEDO IS .80									
μ	μ_0 = .1	.2	.3	.5	.7	.9	1.0	QN	QU
0.0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	∞	∞
.1	5.79976	2.71842	1.56291	.69215	.37215	.22002	.17255	2.04579	.71063
.2	2.71842	1.73506	1.20623	.68722	.44652	.31370	.26834	1.10343	.61463
.3	1.56291	1.20623	.94975	.63789	.46367	.35542	.31580	.78725	.55463
.5	.69215	.68722	.63789	.53273	.44739	.38139	.35412	.52436	.47121
.7	.37215	.44652	.46367	.44739	.41289	.37688	.35981	.40234	.41090
.9	.22002	.31370	.35542	.38139	.37688	.36147	.35214	.32870	.36380
1.0	.17255	.26834	.31580	.35412	.35981	.35214	.34591	.30136	.34380

Table 4c. As Table 4a but for albedo 1.00

ALBEDO IS 1.00									
μ	μ_0 = .1	.2	.3	.5	.7	.9	1.0	QN	QU
0.0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	∞	.57735
.1	7.02726	3.14753	1.69024	.62996	.29298	.18150	.16486	2.33955	.72016
.2	3.14753	2.02709	1.41643	.85092	.64041	.57311	.56813	1.34212	.83736
.3	1.69024	1.41643	1.19543	.95194	.86441	.85757	.87260	1.08630	.94831
.5	.62996	.85092	.95194	1.07658	1.18727	1.30503	1.36737	.99856	1.16208
.7	.29298	.64041	.86441	1.18727	1.44487	1.67631	1.78691	1.06529	1.37062
.9	.18150	.57311	.85757	1.30503	1.67631	2.00835	2.16487	1.18124	1.57651
1.0	.16486	.56813	.87260	1.36737	1.78691	2.16487	2.34319	1.24818	1.67883

and Eq. (42) for $a = 1$

$$\lim_{\mu\to 0} [QU](\mu) = \frac{1}{\sqrt{3}} \simeq 0.57735 . \tag{46}$$

The Appendix contains series expansions for $Q(\mu, \mu_0)$ and its moments which may be useful if a is near zero or one.

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Appendix

In this appendix we discuss the behaviour of H-functions, related functions and their moments as the albedo of single scattering approaches 0 or 1. Isotropic scattering and $\mu > 0$ are assumed. The resulting formulae give an impression of the nature of the relevant functions near $a = 0$ and $a = 1$ and may also be useful for numerical computations.

A.1. Expansions in a, useful near a = 0

We observe that $H(\mu)$ is an analytic function of a near $a = 0$, so that we may expand it in a power series in a. From Eq. (7) or Eq. (18) we easily get

$$H(\mu) = 1 + \frac{a\mu}{2} \ln\left(1 + \frac{1}{\mu}\right) + O(a^2) . \tag{A-1}$$

Differentiating both sides of Eq. (A-1) yields

$$H'(\mu) = \frac{a}{2}\left(\ln\left(1 + \frac{1}{\mu}\right) - \frac{1}{1+\mu}\right) + O(a^2) \tag{A-2}$$

$$H''(\mu) = -\frac{a}{2\mu(1+\mu)^2} + O(a^2) . \tag{A-3}$$

With the help of Eq. (31) we obtain

$$LH = \mu + a\left[\frac{3}{4} - \mu + \left(\frac{3}{2}\mu^2 - \frac{1}{2}\right) \ln\left(1 + \frac{1}{\mu}\right)\right] + O(a^2) \tag{A-4}$$

while Eqs. (34)–(36) provide

$$Q(\mu,\mu_0) = \frac{a}{4}\frac{1+\mu\mu_0}{(\mu+\mu_0)^2} + \frac{a^2}{8}\left[\frac{1+\mu\mu_0}{(\mu+\mu_0)^2}\left(\mu\ln\left(1+\frac{1}{\mu}\right)+\mu_0\ln\left(1+\frac{1}{\mu_0}\right)\right) + \frac{\mu^3+2\mu^2\mu_0-\mu_0}{(\mu+\mu_0)^2}\left(\ln\left(1+\frac{1}{\mu}\right)-\frac{1}{1+\mu}\right) + \frac{\mu_0^3+2\mu_0^2\mu-\mu}{(\mu+\mu_0)^2}\left(\ln\left(1+\frac{1}{\mu_0}\right)-\frac{1}{1+\mu_0}\right) + \frac{1}{2(\mu+\mu_0)}\left(\frac{1-\mu}{1+\mu}+\frac{1-\mu_0}{1+\mu_0}\right)\right] + O(a^3). \quad \text{(A-5)}$$

From Eqs. (39)–(40) we have immediately

$$[QN](\mu) = \frac{a}{8\mu}\left[1+2\mu^2\left(\ln\left(1+\frac{1}{\mu}\right)-\frac{1}{1+\mu}\right) + \frac{1-\mu}{1+\mu}\right] + O(a^2) \quad \text{(A-6)}$$

$$[QU](\mu) = a\left[\mu-\frac{1}{2}-\left(\mu^2-\frac{1}{2}\right)\ln\left(1+\frac{1}{\mu}\right)\right] + O(a^2). \quad \text{(A-7)}$$

A.2. Expansions in $(1-a)^{1/2}$, useful near $a = 1$

According to Yanovitskii (1968) and Van de Hulst (1980) we have the expansion

$$H(\mu) = H_c(\mu) - t\mu\sqrt{3}H_c(\mu) + O(t^2) \quad \text{(A-8)}$$

where $H_c(\mu)$ is the H-function for conservative scattering ($a = 1$) and $t = (1-a)^{1/2}$. By differentiation we find

$$H'(\mu) = H_c'(\mu) - t\sqrt{3}(H_c(\mu) + \mu H_c'(\mu)) + O(t^2) \quad \text{(A-9)}$$

$$H''(\mu) = H_c''(\mu) - t\sqrt{3}(2H_c'(\mu) + \mu H_c''(\mu)) + O(t^2) \quad \text{(A-10)}$$

which implies

$$LH = LH_c - t\sqrt{3}\left((3\mu^2-1)H_c(\mu) + \mu(3\mu^2-2)H_c'(\mu) + \tfrac{1}{2}\mu^2(\mu^2-1)H_c''(\mu)\right) + O(t^2). \quad \text{(A-11)}$$

From this identity and Eqs. (34)–(36) we obtain after some algebra

$$Q(\mu,\mu_0) = Q_c(\mu,\mu_0) - \frac{t\sqrt{3}}{4}\left(H_c(\mu)[LH_c](\mu_0) + H_c(\mu_0)[LH_c](\mu)\right) + O(t^2) \quad \text{(A-12)}$$

where $Q_c(\mu,\mu_0)$ is the value of $Q(\mu,\mu_0)$ at $a = 1$. Using Eqs. (39)–(40) we readily find

$$[QN](\mu) = [Q_cN](\mu) - \frac{t\sqrt{3}}{2}\left(\frac{1}{4}H_c(\mu) + [LH_c](\mu)\right) + O(t^2) \quad \text{(A-13)}$$

$$[QU](\mu) = \frac{1}{\sqrt{3}}H_c(\mu) - t\left(q_\infty H_c(\mu) + [LH_c](\mu)\right) + O(t^2) \quad \text{(A-14)}$$

In deriving Eq. (A-14) we have used the identity

$$\alpha_1 = \frac{2}{\sqrt{3}} - 2q_\infty t + O(t^2) \quad \text{(A-15)}$$

where $q_\infty \simeq 0.710446$ is Hopf's constant (cf. Van de Hulst, 1980, Sect. 8.3.3).

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