Inverse scattering problem for the 3-D Schrödinger equation and Wiener–Hopf factorization of the scattering operator

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Sufficient conditions are given for the existence of a Wiener–Hopf factorization of the scattering operator for the 3-D Schrödinger equation with a potential having no spherical symmetry. A consequence of this factorization is the solution of a related Riemann–Hilbert problem, thus providing a solution of the 3-D inverse scattering problem.

I. INTRODUCTION

Consider the Schrödinger equation in three dimensions

$$\Delta\psi(k,x,\theta) + k^2\psi(k,x,\theta) = V(x)\psi(k,x,\theta), \qquad (1.1)$$

where Δ is the Laplacian, $x\in\mathbf{R}^3$ is the space coordinate, $\theta\in S^2$ is a unit vector in $\mathbf{R}^3$, and $k^2\in\mathbf{R}$ is energy. The potential $V(x)$ is assumed to decrease to zero sufficiently fast as $|x|\to\infty$. However, we do not assume any spherical symmetry on the potential. As $|x|\to\infty$, the wave function $\psi(k,x,\theta)$ behaves as

$$\psi(k,x,\theta) = e^{ik\theta\cdot x} + \frac{e^{ik|x|}}{|x|}A\left(k,\frac{x}{|x|},\theta\right) + o\left(\frac{1}{|x|}\right), \qquad (1.2)$$

where $A(k,\theta,\theta')$ is the scattering amplitude. The scattering operator $S(k,\theta,\theta')$ is then defined by

$$S(k,\theta,\theta') = \delta(\theta-\theta') - (k/2\pi i)A(k,\theta,\theta'), \qquad (1.3)$$

where δ is the Dirac delta distribution on S^2. In operator notation (1.3) is written as

$$S(k) = \mathbf{I} - (k/2\pi i)A(k),$$

where the operators are defined on $L^2(S^2)$, the Hilbert space of complex-valued, square-integrable functions on the unit sphere S^2 in $\mathbf{R}^3$ with the usual inner product $\langle\cdot,\cdot\rangle$.

The direct scattering problem is to obtain $S(k,\theta,\theta')$ when $V(x)$ is given. The inverse scattering problem, however, is to recover $V(x)$ when $S(k,\theta,\theta')$ is known. Since the main source of information about molecular, atomic, and subatomic particles consists of collision experiments, solving the inverse scattering problem is equivalent to determining the forces between particles from scattering data.

For one-dimensional and radial Schrödinger equations, the inverse scattering problem is fairly well understood (at least for certain classes of potentials).[1] In higher dimensions, however, the situation is quite different. The solution methods developed in higher dimensions include the Newton–Marchenko method,[2–4] the Gel'fand–Levitan method,[2–5] the $\bar\partial$ method,[6–9] the generalized Jost–Kohn method,[10–13] and a method that uses the Green's function of Faddeev.[14–16] There are still many open problems in multidimensional inverse scattering, and the methods developed are still far from being complete. A comprehensive review of the methods and related open problems in multidimensional inverse scattering can be found in Newton's recent book[17] or in Ref. 1.

The principal idea behind both the Newton–Marchenko and Gel'fand–Levitan methods is to formulate the inverse scattering problem as a Riemann–Hilbert boundary value problem, to transform this Riemann–Hilbert problem into a nonhomogeneous integral equation where the kernel and the nonhomogeneous term contain the Fourier transform of the scattering data, and to obtain the potential from the solution of the resulting integral equation. In this paper we present a solution of the 3-D inverse scattering problem by establishing a Wiener–Hopf factorization for the scattering operator and thus solving the corresponding Riemann–Hilbert problem. The usual theory of Wiener–Hopf factorization, however, deals with scalar functions and square matrix functions. Here, we need the Wiener–Hopf factorization of an operator function in an infinite-dimensional setting, and for this we draw on some results by Gohberg and Leiterer.[18]

The present paper is organized as follows. In Sec. II we define the class of potentials (which we will name the Newton class) for which corresponding scattering operators have a Wiener–Hopf factorization. In Sec. III we give some estimates on the scattering amplitude and its derivative and establish the Hölder continuity of the scattering operator. In Sec. IV we define the Wiener–Hopf factorization for operator-valued functions and prove its existence for scattering operators corresponding to potentials in the Newton class. In Sec. V we solve a related Riemann–Hilbert problem using the Wiener–Hopf factorization of the scattering operator. In Sec. VI the solution of the inverse scattering problem is given. Also in this section, for potentials in the Newton class having no bound states, we give the necessary and sufficient conditions for the existence and uniqueness of the Jost operator in terms of the partial indices of the scattering operator. In Sec. VII we summarize the main results of the paper and give the conclusion.

II. ESTIMATES ON THE SCATTERING OPERATOR

We first identify the class of potentials for which all of the results in this paper are valid. Except for the third condition given in the following definition, these conditions are standard assumptions on the potential.[17] The second condi-

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tion is much weaker than the usual assumptions.[17] The third condition is needed only twice: first to establish a uniform operator bound for the derivative of the scattering amplitude, and second to use an interpolation argument. Note that all four conditions used below are only sufficient conditions and might possibly be weakened.

Definition 2.1: A potential $V(x)$ is said to belong to the **Newton class** if $V(x)$ is real valued and measurable and satisfies

(i) $\exists a,b>0$ such that

$$\int_{\mathbf{R}^3} dx|V(x)|\left(\frac{|x|+|y|+a}{|x-y|}\right)^2 \leqslant b, \quad \forall y \in \mathbf{R}^3. \tag{2.1}$$

(ii) $\exists c>0,\ s>\frac{1}{2}$ such that $\forall x \in \mathbf{R}^3$

$$|V(x)| \leqslant c/(1+|x|^2)^s. \tag{2.2}$$

(iii) $\exists \gamma>0$ and $\beta \in (0,1]$ such that

$$\int_{\mathbf{R}^3} dx|x|^\beta |V(x)| \leqslant \gamma. \tag{2.3}$$

(iv) The point $k=0$ is not an exceptional point.[19] This condition is satisfied if at zero energy there are neither bound states nor half-bound states.

Remark 2.2: If $V(x)$ satisfies (2.1), we have

$$\int_{\mathbf{R}^3} dx|V(x)| < \infty,$$

$$\int_{\mathbf{R}^3} dx \frac{|V(x)|}{|x-y|} < \infty, \quad \forall y \in \mathbf{R}^3,$$

$$\int_{\mathbf{R}^3} dx \frac{|V(x)|}{|x-y|^2} < \infty, \quad \forall y \in \mathbf{R}^3,$$

$$\int\int_{\mathbf{R}^3\times\mathbf{R}^3} dx\, dy \frac{|V(x)V(y)|}{|x-y|} < \infty,$$

$$\|V\|_R = \left(\int\int_{\mathbf{R}^3\times\mathbf{R}^3} dx\, dy \frac{|V(x)V(y)|}{|x-y|^2}\right)^{1/2} < \infty.$$

The last integral defines the Rollnik norm of the potential. The real potentials with a finite Rollnik norm make up the Rollnik class. The number of bound states n_B for potentials in the Rollnik class is finite[20,21] and $n_B \leqslant \|V\|_R^2/(16\pi^2)$.

Remark 2.3: In (2.2), whenever $s>\frac{3}{4}$, the potential $V \in L^2(\mathbf{R}^3)$. If $s>\frac{1}{2}$, there are no nonzero real exceptional points and hence no positive-energy bound states.[22]

The kernel of the scattering operator $A(k)$ has the representation

$$A(k,\theta,\theta') = -\frac{1}{4\pi}\int_{\mathbf{R}^3} dx\, V(x)e^{-ik\theta\cdot x}\psi(k,x,\theta'), \tag{2.4}$$

where $\psi(k,x,\theta)$ is the solution of the Schrödinger equation. The 3-D Lippmann–Schwinger equation corresponding to the Schrödinger equation satisfying (1.2) is given by

$$\psi(k,x,\theta) = e^{ik\theta\cdot x} - \frac{1}{4\pi}\int_{\mathbf{R}^3} dy \frac{e^{ik|x-y|}}{|x-y|} V(x)\psi(k,y,\theta). \tag{2.5}$$

Iterating (2.5) three times, we obtain

$$\psi_1(k,x,\theta) = e^{ik\theta\cdot x},$$

$$\psi_j(k,x,\theta) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} dy \frac{e^{ik|x-y|}}{|x-y|} V(y)\psi_{j-1}(k,y,\theta), \quad j=2,3,$$

$$\psi_4(k,x,\theta) = \psi(k,x,\theta) - \sum_{j=1}^{3}\psi_j(k,x,\theta).$$

Then we can write (2.4) as[17]

$$A(k,\theta,\theta') = -\frac{1}{4\pi}\sum_{j=1}^{4} A_j(k,\theta,\theta'), \tag{2.6}$$

where

$$A_j(k,\theta,\theta') = \int_{\mathbf{R}^3} dx\, V(x)e^{-ik\theta\cdot x}\psi_j(k,x,\theta'), \quad j=1,2,3,4. \tag{2.7}$$

Proposition 2.4: If the potential $V(x)$ satisfies the first and fourth conditions in the Newton class, the corresponding scattering amplitude $A(k)$ is a continuous operator function in $k \in \mathbf{R}$ on $L^2(S^2)$.

Proof: From (2.7) we obtain the estimates

$$|A_1(k,\theta,\theta')| \leqslant \int_{\mathbf{R}^3} dx|V(x)|,$$

$$|A_2(k,\theta,\theta')| \leqslant \frac{1}{4\pi}\int\int_{\mathbf{R}^3\times\mathbf{R}^3} dx\, dy \frac{|V(x)V(y)|}{|x-y|},$$

$$|A_3(k,\theta,\theta')| \leqslant \frac{1}{(4\pi)^2}\int_{\mathbf{R}^3} dx|V(x)| \times\left\{\int_{\mathbf{R}^3} dy \frac{|V(y)|}{|x-y|}\left[\int_{\mathbf{R}^3} dz \frac{|V(z)|}{|x-z|}\right]\right\},$$

and hence, using Remark 2.2 and Lebesgue's dominated convergence theorem, (2.1) is sufficient to conclude that $A_j(k)$ is continuous for $j=1,2,3$ in the operator norm on $L^2(S^2)$. The continuity of $A_4(k)$ follows[17] under the sufficient condition (2.1) and the fourth condition in the Newton class. ■

The next result is due to Weder.[23] A proof convenient to our present problem is provided by Newton.[17]

Proposition 2.5: If the potential $V(x)$ satisfies (2.1) and (2.2) with $s>1$, and the fourth condition in the definition of the Newton class, $\exists C>0$ such that $\|kA(k)\| \leqslant C/(1+|k|)$ for all $k \in \mathbf{R}$, where the norm is the operator norm on $L^2(S^2)$.

The following proposition generalizes Proposition 2.5 under a much weaker condition.

Proposition 2.6: If the potential $V(x)$ satisfies (2.1) and (2.2) with $\frac{1}{2}<s<1$, and the fourth condition in the definition of the Newton class, $\exists E>0$ such that $\|kA(k)\| \leqslant E/(1+|k|)^{2-2s}$ for all $k \in \mathbf{R}$, where the norm is the operator norm on $L^2(S^2)$.

Proof: When $|k| \leqslant 1$, using (1.3) and the unitarity of $S(k)$ we obtain

$$\|kA(k)\| \leqslant 2\pi(\|S(k)\|+1) \leqslant 4\pi\cdot 2^{2-2s}/(1+|k|)^{2-2s}.$$

When $|k| \geqslant 1$, we proceed as follows. According to the lemma due to Vega,[24] $\forall g \in L^2(S^2)$, we have

$$\left[\int_{\mathbf{R}^3} dx|(\sigma^\dagger(-1)g)(x)|^2(1+|x|^2)^{-s}\right]^{1/2} \leqslant c\|g\|, \quad \forall s>\tfrac{1}{2},$$

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where c is a constant and

$$(\sigma^\dagger(k)g)(x)=\int_{S^2}d\theta\, e^{-ik\theta\cdot x}g(\theta).$$

Replacing x by kx in Vega's lemma and using $|k|\geqslant 1$, we obtain

$$\int_{\mathbf{R}^3}dx|(\sigma^\dagger(k)g)(x)|^2(1+|x|^2)^{-s}$$
$$\leqslant c^2\|g\|^2/|k|^{3-2s}\leqslant 2^{3-2s}c^2\|g\|^2/(1+|k|)^{3-2s}. \quad (2.8)$$

Next, we apply the representation for the scattering amplitude[17,23]

$$A(k)=-(1/4\pi)\sigma(k)V^{1/2}[\mathbf{I}-L(k)]^{-1}|V|^{1/2}\sigma^\dagger(k), \quad (2.9)$$

where $\sigma(k)$ is the adjoint of $\sigma^\dagger(k)$, V is the potential $V(x)$, $V^{1/2}=\operatorname{sgn}(V)|V|^{1/2}$, and $L(k)$ is the operator whose kernel

$$L(k,x,y)=-(1/4\pi)|V(x)|^{1/2}e^{ik|x-y|}V(y)^{1/2}/|x-y|,$$

is closely related to the kernel of the Lippmann–Schwinger equation (2.5). It is known that $[\mathbf{I}-L(k)]^{-1}$ is uniformly bounded in k in operator norm.[17] Hence, if $V(x)$ satisfies (2.2) with $s>\frac{1}{2}$, using (2.8) and (2.9) we obtain $\|kA(k)\|\leqslant E/(1+|k|)^{2-2s}$, for some constant E. ∎

The choice $\beta=1$ in the next three propositions may seem to be a step backward at first; however, using the interpolation in Proposition 2.10, the results of Propositions 2.8 and 2.9 will be strengthened to include $\beta\in(0,1]$. The next proposition gives the uniform boundedness of the derivative of the scattering amplitude.

Proposition 2.7: If the potential $V(x)$ satisfies (2.1) and (2.3) with $\beta=1$, and the fourth condition in the definition of the Newton class, $\exists B>0$ such that $\|dA(k)/dk\|\leqslant B$ for all $k\in\mathbf{R}$, where the norm is the operator norm on $L^2(S^2)$.

Proof: From (2.7) we obtain by direct computation

$$\left|\frac{\partial A_1(k,\theta,\theta')}{\partial k}\right|\leqslant\int_{\mathbf{R}^3}dx|xV(x)|,$$

$$\left|\frac{\partial A_2(k,\theta,\theta')}{\partial k}\right|\leqslant\frac{1}{2\pi}\int_{\mathbf{R}^3}dy|V(y)|\times\left[\int_{\mathbf{R}^3}dx\left(\frac{|x|+|y|}{|x-y|}\right)^2|V(x)|\right],$$

$$\left|\frac{\partial A_3(k,\theta,\theta')}{\partial k}\right|\leqslant 2\int_{\mathbf{R}^3}dy|V(y)|\left\{\int_{\mathbf{R}^3}dz|V(z)|\times\int_{\mathbf{R}^3}dx\frac{|V(x)|}{|x-z|}\right\}+2\int dy|V(y)|\times\left\{\int_{\mathbf{R}^3}dx|V(x)|\left[\int_{\mathbf{R}^3}dz\frac{|zV(z)|}{|z-x|}\right]\right\},$$

and hence, using Lebesgue's dominated-convergence theorem, the first and third conditions in the Newton class are sufficient for the differentiability of $A_j(k)$ with respect to k in the operator norm on $L^2(S^2)$ and the uniform boundedness of its derivative for $k\in\mathbf{R}$, for $j=1,2,3$. The uniform boundedness $\|dA_4(k)/dk\|\leqslant B_4$ has already been established[17] using the first and fourth conditions in the definition of the Newton class. Note that in the above proof, the only place where we used (2.3) was the bound on $\|dA_1(k)/dk\|$. ∎

The Möbius transformation $k\mapsto\xi=(k-i)/(k+i)$ maps the extended real axis $\mathbf{R}_\infty$ onto the unit circle $\mathbf{T}$, the upper-half complex plane $\mathbf{C}^+$ onto the unit disk $\mathbf{T}^+$, and the lower-half plane $\mathbf{C}^-$ onto the exterior of the unit disk $\mathbf{T}^-$, where ∞ is considered to be a point of $\mathbf{T}^-$. Let $\tilde{S}(\xi)=S(k)$ under this transformation, and let us adopt this notation throughout the paper.

Let Γ be a Borel set in the complex plane $\mathbf{C}$. Consider an operator-valued function $W:\Gamma\to\mathscr{L}(L^2(S^2))$, where $\mathscr{L}(L^2(S^2))$ is the space of bounded linear operators acting on $L^2(S^2)$. Then the quantity $|||W|||_\alpha$, which is given as

$$|||W|||_\alpha=\sup_{t\in\Gamma}\|W(t)\|+\sup_{t_1\neq t_2\in\Gamma}\frac{\|W(t_1)-W(t_2)\|}{|t_1-t_2|^\alpha},$$

where $\|\cdot\|$ is the operator norm on $L^2(S^2)$ and $\alpha\in(0,1]$, defines a complete norm on the Banach space $\mathscr{H}_\alpha[\Gamma;\mathscr{L}(L^2(S^2))]$ of Hölder-continuous operator functions[18,25] with exponent α.

Proposition 2.8: The $\tilde{S}(\xi)$ is Hölder continuous on the unit circle $\mathbf{T}$ with exponent $\frac{1}{4}$ if the potential $V(x)$ is in the Newton class with $s>1$ in (2.2) and $\beta=1$ in (2.3).

Proof: We have to show that $\exists M>0$ such that $\|\tilde{S}(\xi_1)-\tilde{S}(\xi_2)\|\leqslant M|\xi_1-\xi_2|^{1/4}$ for all $\xi_1,\xi_2\in\mathbf{T}$. Using

$$\|\tilde{S}(\xi_1)-\tilde{S}(\xi_2)\|=(1/2\pi)\|k_1A(k_1)-k_2A(k_2)\|$$

and

$$\xi_1-\xi_2=2i(k_1-k_2)/(k_1+i)(k_2+i),$$

we have

$$\frac{\|\tilde{S}(\xi_1)-\tilde{S}(\xi_2)\|}{|\xi_1-\xi_2|^\epsilon}=\frac{1}{2\pi}\frac{1}{2^\epsilon}(k_1^2+1)^{\epsilon/2}(k_2^2+1)^{\epsilon/2}\times\|k_2A(k_2)-k_1A(k_1)\|/|k_1-k_2|^\epsilon.$$

Because of the symmetry in k_1 and k_2, it is sufficient to show that $\lambda(k,\delta)$ is bounded by a constant independent of k and δ for all $\delta>0$ and $-\infty<k<\infty$, where

$$\lambda(k,\delta)=(k^2+1)^{\epsilon/2}[(k+\delta)^2+1]^{\epsilon/2}\times\|(k+\delta)A(k+\delta)-kA(k)\|(1/\delta^\epsilon).$$

In our proof we will use Propositions 2.5 and 2.7 and the constants C and B given there.

When $|k|\leqslant 1\leqslant\delta$, using $k^2+1\leqslant 2$, $(k+\delta)^2+1\leqslant 5\delta^2$, and

$$\|(k+\delta)A(k+\delta)-kA(k)\|\leqslant\|(k+\delta)A(k+\delta)\|+\|kA(k)\|\leqslant 2C,$$

we obtain $\lambda(k,\delta)\leqslant 2\cdot 10^{\epsilon/2}C$.

When $|k|\leqslant 1$, $\delta\leqslant 1$, using $k^2+1\leqslant 2$, $(k+\delta)^2+1\leqslant 5$,

$$\|(k+\delta)A(k+\delta)-kA(k)\|^{1-\epsilon}\leqslant(2C)^{1-\epsilon},$$

and

$$\|(k+\delta)A(k+\delta)-kA(k)\|^\epsilon\leqslant\left(\|A(k+\delta)\|+2(|k|+\delta)\max_{k\in\mathbf{R}}\left\|\frac{dA(k)}{dk}\right\|\right)^\epsilon\delta^\epsilon\leqslant(C+4B)^\epsilon\delta^\epsilon,$$

we obtain $\lambda(k,\delta)\leqslant 10^{\epsilon/2}(2C)^{1-\epsilon}(C+4B)^\epsilon$.

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When $1\leqslant|k|\leqslant\delta$, using $k^2+1\leqslant 2k^2$, $(k+\delta)^2+1\leqslant 5\delta^2$, and

$$\|(k+\delta)A(k+\delta)-kA(k)\|\leqslant 2C/(1+|k|),$$

we obtain $\lambda(k,\delta)\leqslant 2C\cdot 10^{\epsilon/2}[|k|^\epsilon/(1+|k|)]$.

When $1\leqslant\delta\leqslant|k|$, using

$$1/\delta^\epsilon\leqslant 1,\ k^2+1\leqslant 2k^2,(k+\delta)^2+1\leqslant 5k^2,$$

and

$$\|(k+\delta)A(k+\delta)-kA(k)\|\leqslant 2C/(1+|k|),$$

we obtain $\lambda(k,\delta)\leqslant 2C\cdot 10^{\epsilon/2}[|k|^{2\epsilon}/(1+|k|)]$.

When $\delta\leqslant 1\leqslant|k|$, using $k^2+1\leqslant 2k^2,(k+\delta)^2+1\leqslant 5k^2$,

$$\|(k+\delta)A(k+\delta)-kA(k)\|^{1-\epsilon}$$
$$\leqslant(2C)^{1-\epsilon}/(1+|k|)^{1-\epsilon},$$

and

$$\|(k+\delta)A(k+\delta)-kA(k)\|^\epsilon$$
$$\leqslant\delta^\epsilon\Big(\|A(k+\delta)\|+2(|k|+\delta)\max_{k\in\mathbf{R}}\Big|\Big|\frac{dA(k)}{dk}\Big|\Big|\Big)^\epsilon$$
$$\leqslant\delta^\epsilon(C+4B)^\epsilon|k|^\epsilon,$$

we obtain

$$\lambda(k,\delta)\leqslant(2C)^{1-\epsilon}\cdot 10^{\epsilon/2}(C+4B)^\epsilon[|k|^{3\epsilon}/(1+|k|)^{1-\epsilon}].$$

Hence, whenever $0\leqslant\epsilon\leqslant\frac{1}{4}$, we have $\lambda(k,\delta)\leqslant M$, where M is a constant independent of k and δ. ■

Under weaker assumptions on the potential, we can modify Proposition 2.8 to obtain the following result.

Proposition 2.9: The $\tilde{S}(\xi)$ is Hölder continuous with exponent $2(1-s)/(5-2s)$ if the potential $V(x)$ belongs to the Newton class with some $s\in(\frac{1}{2},1)$ in (2.2) and $\beta=1$ in (2.3).

Proof: The only place in the proof of Proposition 2.8 where we have used Proposition 2.5 are the three cases $1\leqslant|k|\leqslant\delta$, $1\leqslant\delta\leqslant|k|$, and $\delta\leqslant 1\leqslant|k|$. In these three cases, we must use the result in Proposition 2.6 instead of the result in Proposition 2.5. This is accomplished by replacing $2C/(1+|k|)$ by $2E/(1+|k|)^{2-2s}$ in the proof of Proposition 2.8. We have the following.

Using

$$\|(k+\delta)A(k+\delta)-kA(k)\|\leqslant 2E/(1+|k|)^{2-2s},$$

we obtain $\lambda(k,\delta)\leqslant 2E\cdot 10^{\epsilon/2}[|k|^\epsilon/(1+|k|)^{2-2s}]$ when $1\leqslant|k|\leqslant\delta$, and

$$\lambda(k,\delta)\leqslant 2E\cdot 10^{\epsilon/2}[|k|^{2\epsilon}/(1+|k|)^{2-2s}],$$

when $1\leqslant\delta\leqslant|k|$.

When $\delta\leqslant 1\leqslant|k|$, we use

$$\|(k+\delta)A(k+\delta)-kA(k)\|^\epsilon$$
$$\leqslant\delta^\epsilon\Big[\|A(k+\delta)\|+2(|k|+\delta)\max_{k\in\mathbf{R}}\Big|\Big|\frac{dA(k)}{dk}\Big|\Big|\Big]^\epsilon$$
$$\leqslant\delta^\epsilon(E+4B)^\epsilon|k|^\epsilon,$$

and

$$\|(k+\delta)A(k+\delta)-kA(k)\|^{1-\epsilon}\leqslant\frac{(2E)^{1-\epsilon}}{(1+|k|)^{(1-\epsilon)(2-2s)}},$$

to obtain

$$\lambda(k,\delta)\leqslant(2E)^{1-\epsilon}\cdot 10^{\epsilon/2}(E+4B)^\epsilon$$
$$\times|k|^{3\epsilon}/(1+|k|)^{(1-\epsilon)(2-2s)}.$$

Hence, whenever $0\leqslant\epsilon\leqslant 2(1-s)/(5-2s)$, we have $\lambda(k,\delta)\leqslant M$, where M is a constant independent of k and δ. ■

Proposition 2.10: The $A_1(k)$ defined in (2.7) is Hölder continuous with exponent β whenever the potential $V(x)$ belongs to the Newton class with the constant β in (2.3).

Proof: From (2.7) we have

$$A_1(k,\theta,\theta')=\int_{\mathbf{R}^3}dx\ V(x)e^{ik(\theta-\theta')\cdot x}.$$

Consider the operator $\mathscr{K}:V(x)\mapsto\langle A_1(k)f,g\rangle$, for some fixed $f,g\in L^2(S^2)$; i.e., consider

$$(\mathscr{K}V)(k)=\int\int\int_{\mathbf{R}^3\times S^2\times S^2}dx\,d\theta\,d\theta'\ V(x)$$
$$\times e^{ik(\theta-\theta')\cdot x}f(\theta)\ \overline{g(\theta')},$$

where the bar denotes complex conjugation. The operator $\mathscr{K}$ is linear from $L^1(\mathbf{R}^3;dx)$, the space of Lebesgue integrable functions with respect to measure dx, into $\mathscr{H}_0$, the Banach space of bounded continuous functions on $\mathbf{R}$. The same operator $\mathscr{K}$ maps $L^1(\mathbf{R}^3;(1+|x|)dx)$ into $\mathscr{H}_1$, the Banach space of bounded Hölder-continuous functions on $\mathbf{R}$ with exponent 1. An application of an interpolation theorem presented by Krein *et al.* (Theorems III.3.5 and III.3.6 of Ref. 25) leads to the result that $\mathscr{K}$ maps $L^1(\mathbf{R}^3;(1+|x|)^\beta dx)$ into $\mathscr{H}^0_\beta$, where

$$\mathscr{H}^0_\beta=\{h\in\mathscr{H}_\beta:|h(k_1)-h(k_2)|=o(|k_1-k_2|^\beta)$$
$$\text{as }|k_1-k_2|\to 0\}.$$

Since this result is true uniformly in f,g on bounded subsets of $L^2(S^2)$, $A_1(k)$ belongs to $\mathscr{H}^0_\beta[\mathbf{R};\mathscr{L}(L^2(S^2))]$. Hence, $A_1(k)$ is Hölder continuous with exponent β whenever the potential $V(x)$ belongs to the Newton class where β is the constant in (2.3). Note that, strictly speaking, in order to apply Krein's result, one must restrict the function $(\mathscr{K}V)(k)$ to $k\in I$, where $I\subset\mathbf{R}$ is a compact interval, and observe that all the norm bounds are independent of I to pass to the case where $(\mathscr{K}V)(k)$ is considered for all $k\in\mathbf{R}$, which is the case here. ■

Using Proposition 2.10, we improve the results of Propositions 2.8 and 2.9 to obtain the following result that will be used in Sec. V.

Theorem 2.11: If the potential $V(x)$ belongs to the Newton class with some $\beta\in(0,1]$ in (2.3), then on Möbius transformation the corresponding scattering operator $\tilde{S}(\xi)$ belongs to $\mathscr{H}_\mu[\mathbf{T};\mathscr{L}(L^2(S^2))]$, where $\mu=\beta/2(1+\beta)$ if $s>1$ in (2.2) and $\mu=\beta(1-s)/(\beta-s+\frac{3}{2})$ if $s\in(\frac{1}{2},1)$ in (2.2). Here, $\mathscr{H}_\mu[\mathbf{T};\mathscr{L}(L^2(S^2))]$ is the Banach space of Hölder-continuous operator functions on the unit circle $\mathbf{T}$ with exponent μ.

Proof: Using (1.3) and (2.6) we have

$$S(k_1)-S(k_2)=\frac{1}{8\pi^2 i}\sum_{j=1}^{4}[k_1A_j(k_1)-k_2A_j(k_2)].$$

As mentioned at the end of the proof of Proposition 2.7, the only place where we used (2.3) was in the uniform boundedness of $\|dA_1(k)/dk\|$. Therefore, from the proof of Proposition 2.8, we obtain that for $j=2,3,4$, the operator $kA_j(k)$ is

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Hölder continuous of exponent $\frac{1}{4}$ even for $\beta=0$ in (2.3). Hence, to prove the theorem, it is enough to redo the proof of Proposition 2.8 only for $A_1(k)$ and only in two cases; namely, when $|k|\leqslant 1, \delta\leqslant 1$ and when $\delta\leqslant 1\leqslant |k|$; i.e., it is enough to show that

$$\lambda_1(k,\delta)=(k^2+1)^{\epsilon/2}[(k+\delta)^2+1]^{\epsilon/2}$$
$$\times\|(k+\delta)A_1(k+\delta)-kA_1(k)\|(1/\delta^\epsilon)\leqslant M,$$

for $\epsilon\leqslant\beta/2(1+\beta)$ if $s>1$ in (2.2) and $\epsilon\leqslant\beta(1-s)/(\beta-s+\frac{3}{2})$ if $s\in(\frac{1}{2},1)$ in (2.2), where M is a constant independent of k and δ. Note that, whenever the potential $V(x)$ satisfies (2.3), from Proposition 2.10 we have $\|A_1(k+\delta)-A_1(k)\|\leqslant N\delta^\beta$, where N is a constant independent of k and δ. We will do the case when $s>1$ in (2.2) first.

When $\delta\leqslant 1, |k|\leqslant 1$, using $k^2+1\leqslant 2, (k+\delta)^2+1\leqslant 5$,

$$\|(k+\delta)A_1(k+\delta)-kA_1(k)\|$$
$$\leqslant |k|\cdot\|A_1(k+\delta)-A_1(k)\|+\delta\|A_1(k+\delta)\|$$
$$\leqslant |k|N\delta^\beta+\delta C\leqslant(N+C)\delta^\beta,$$

we have $\lambda_1(k,\delta)\leqslant 10^{\epsilon/2}(N+C)\delta^{\beta-\epsilon}$.

When $\delta\leqslant 1\leqslant |k|$, using $k^2+1\leqslant 2k^2, (k+\delta)^2+1\leqslant 5k^2$,

$$\|(k+\delta)A_1(k+\delta)-kA_1(k)\|^{1-\epsilon/\beta}$$
$$\leqslant(2C/(1+|k|))^{1-\epsilon/\beta},$$

and

$$\|(k+\delta)A_1(k+\delta)-kA_1(k)\|^{\epsilon/\beta}\leqslant(2|k|N\delta^\beta)^{\epsilon/\beta}$$
$$+(2\delta C)^{\epsilon/\beta}\leqslant(2|k|\delta)^{\epsilon/\beta}(N^{\epsilon/\beta}+C^{\epsilon/\beta}),$$

we have

$$\lambda_1(k,\delta)\leqslant 2\cdot 10^{1-\epsilon/\beta}(N^{\epsilon/\beta}+C^{\epsilon/\beta})$$
$$\times\delta^{\epsilon/\beta-\epsilon}|k|^{2\epsilon+\epsilon/\beta}/(1+|k|)^{1-\epsilon/\beta}.$$

Thus, whenever $\epsilon\leqslant\beta/2(1+\beta)$, $\lambda_1(k,\delta)$ is bounded by a constant independent of k and δ, and the proof for $s>1$ is complete.

If $\frac{1}{2}<s<1$ in (2.2), we basically have the same proof with only two minor modifications, which amount to replacing the denominator $(1+|k|)$ by $(1+|k|)^{2-2s}$ and the constant C by E above. As a result, we obtain the sufficient condition

$$2\epsilon+\epsilon/\beta\leqslant 2(1-\epsilon/\beta)(1-s),$$

for the uniform boundedness of $\lambda_1(k,\delta)$. Hence, we must have

$$0<\epsilon\leqslant\beta(1-s)/(\beta-s+\tfrac{3}{2}),$$

which completes the proof. ■

III. RIEMANN–HILBERT PROBLEM

In the Schrödinger equation, k appears as k^2 and hence $\psi(-k,x,\theta)$ is a solution whenever $\psi(k,x,\theta)$ is. These two solutions are related to each other as[2]

$$\psi(k,x,\theta)=\int_{S^2}d\theta'\,S(k,-\theta,\theta')\psi(-k,x,\theta'). \quad (3.1)$$

Define

$$f(k,x,\theta)=e^{-ik\theta\cdot x}\psi(k,x,\theta). \quad (3.2)$$

If the potential satisfies (2.1) and if there are no bound states, for fixed x and θ, the function $f(k,x,\theta)$ has an analytic extension in k to $\mathbf{C}^+$ and $f(k,x,\theta)=1+O(1/|k|)$ as $|k|\to\infty$ there.[2] Similarly, $f(-k,x,\theta)$ has an analytic extension in k to $\mathbf{C}^-$. Hence, using (3.1), we obtain the Riemann–Hilbert problem

$$f(k,x,\theta)=\int_{S^2}d\theta'\,e^{-ik\theta\cdot x}S(k,-\theta,\theta')$$
$$\times e^{-ik\theta'\cdot x}f(-k,x,\theta'). \quad (3.3)$$

Analogously, in the absence of bound states, we have the associated operator Riemann–Hilbert problem

$$F(k,x,\theta,\theta')=\int_{S^2}d\theta''\,e^{-ik\theta\cdot x}S(k,-\theta,\theta'')$$
$$\times e^{-ik\theta''\cdot x}F(-k,x,-\theta'',-\theta'), \quad (3.4)$$

where, for fixed x,θ,θ', the operator $F(k,x,\theta,\theta')$ has an analytic extension in k to $\mathbf{C}^+$ and $F(k,x,\theta,\theta')=\delta(\theta-\theta')+O(1/|k|)$ as $|k|\to\infty$ there. Similarly, $F(-k,x,\theta,\theta')$ has an analytic extension in k to $\mathbf{C}^-$.

For fixed x,θ, and θ', let $\mathbf{X}(k,x,\theta,\theta')$ denote both the analytic extension in k of $F(k,x,\theta,\theta')-\delta(\theta-\theta')$ to $\mathbf{C}^+$ and the analytic extension of $F(-k,x,-\theta,-\theta')-\delta(\theta-\theta')$ to $\mathbf{C}^-$. Then $\mathbf{X}(k,x,\theta,\theta')$ is a sectionally analytic operator-valued function of k in the complex plane with a jump on the real axis. For $k\in\mathbf{R}$, define

$$\mathbf{X}_+(k,x,\theta,\theta')=\lim_{\epsilon\to 0^+}\mathbf{X}(k+i\epsilon,x,\theta,\theta')$$
$$=F(k,x,\theta,\theta')-\delta(\theta-\theta'), \quad (3.5)$$

$$\mathbf{X}_-(k,x,\theta,\theta')=\lim_{\epsilon\to 0^+}\mathbf{X}(k-i\epsilon,x,\theta,\theta')$$
$$=F(-k,x,-\theta,-\theta')-\delta(\theta-\theta'), \quad (3.6)$$

and

$$G(k,x,\theta,\theta')=e^{-ik\theta\cdot x}S(k,-\theta,-\theta')e^{ik\theta'\cdot x}. \quad (3.7)$$

Then, in operator notation, we can write (3.4) as

$$\mathbf{X}_+(k)=G(k)\mathbf{X}_-(k)+[G(k)-\mathbf{I}], \quad (3.8)$$

where we suppress the x dependence; note that x enters (3.8) only as a parameter. The operators $\mathbf{X}_+(k)$, $\mathbf{X}_-(k)$, $G(k)$, and $\mathbf{I}$ all act on $L^2(S^2)$. Let $\hat{1}$ be the constant function on this space defined as $\hat{1}(\theta)=1$, $\forall\theta\in S^2$. Let us define

$$X_+(k)=\mathbf{X}_+(k)\hat{1}=f(k,x,\theta)-1, \quad (3.9)$$
$$X_-(k)=\mathbf{X}_-(k)\hat{1}, \quad (3.10)$$

where $f(k,x,\theta)$ is as in (3.2). Then we can write (3.3) in vector form as

$$X_+(k)=G(k)X_-(k)+[G(k)-\mathbf{I}]\hat{1}. \quad (3.11)$$

If there are bound states, the extension of $f(k,x,\theta)$ in k to $\mathbf{C}^+$ becomes meromorphic with simple poles on the imaginary axis. A pole at $k=i\gamma$ corresponds to a bound state of the Hamiltonian with energy $-\gamma^2$. It is possible to remove these simple poles from the Riemann–Hilbert problem by a reduction method.[4] Assume there is a bound state corresponding to a pole at $k=i\gamma$. Using a suitable orthogonal projection $\mathbf{B}$, we form the rational function

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$$\Pi(k) = \mathbf{I} - \mathbf{B} + [(k+i\gamma)/(k-i\gamma)]\mathbf{B}.$$

For the operators $\mathbf{X}_+(k)$, $\mathbf{X}_-(k)$, and $G(k)$, we then define the corresponding reduced operators

$$\mathbf{X}_+^{\rm red}(k) = \Pi(k)^{-1}\mathbf{X}_+(k) + [\Pi(k)^{-1} - \mathbf{I}], \quad (3.12)$$

$$\mathbf{X}_-^{\rm red}(k) = \Pi(k)\mathbf{X}_-(k) + [\Pi(k) - \mathbf{I}], \quad (3.13)$$

$$G^{\rm red}(k) = \Pi(k)^{-1}G(k)\Pi(k).$$

Thus we have

$$X_+^{\rm red}(k) = \Pi(k)^{-1}X_+(k) + [\Pi(k)^{-1} - \mathbf{I}]\hat{1}, \quad (3.14)$$

$$X_-^{\rm red}(k) = \Pi(k)X_-(k) + [\Pi(k) - \mathbf{I}]\hat{1}. \quad (3.15)$$

As a result, $\mathbf{X}_+^{\rm red}(k)$ and $X_+^{\rm red}(k)$ do not have a pole at $k = i\gamma$, and $\mathbf{X}_-^{\rm red}(k)$ and $X_-^{\rm red}(k)$ do not have a pole at $k = -i\gamma$. If there is more than one bound state, this procedure must be repeated to remove the finitely many poles corresponding to the bound states; the details can be found in Ref. 4. This eventually leads to the operator Riemann–Hilbert problem

$$\mathbf{X}_+^{\rm red}(k) = G^{\rm red}(k)\mathbf{X}_-^{\rm red}(k) + [G^{\rm red}(k) - \mathbf{I}], \quad (3.16)$$

and the vector Riemann–Hilbert problem

$$X_+^{\rm red}(k) = G^{\rm red}(k)X_-^{\rm red}(k) + [G^{\rm red}(k) - \mathbf{I}]\hat{1}. \quad (3.17)$$

Once the reduced Riemann–Hilbert problems (3.16) and (3.17) are solved, the solutions of the original Riemann–Hilbert problems (3.8) and (3.11) can be obtained using (3.12), (3.13), (3.14), and (3.15). Hence, in the following sections we will give the solutions of both the operator and vector Riemann–Hilbert problems assuming that $X_+(k)$ and $X_-(k)$, and similarly $\mathbf{X}_+(k)$ and $\mathbf{X}_-(k)$, have analytic extensions to $\mathbf{C}^+$ and $\mathbf{C}^-$, respectively.

IV. WIENER–HOPF FACTORIZATION OF THE SCATTERING MATRIX

The usual theory for the existence of Wiener–Hopf factorizations deals either with scalar functions[26] or with matrix functions.[27-30] In our case we study the Wiener–Hopf factorization of operator-valued functions. Hence, we must study Wiener–Hopf factorization in an infinite-dimensional setting[31] and use results on the existence of Wiener–Hopf factorizations of operator functions.[18,32,33]

By a **(left) Wiener–Hopf factorization** of an operator-valued function $G{:}\mathbf{R}_\infty \to \mathscr{L}(L^2(S^2))$, we mean a representation of $G(k)$ in the form

$$G(k) = G_+(k)D(k)G_-(k), \quad k \in \mathbf{R}_\infty, \quad (4.1)$$

with

$$D(k) = P_0 + \sum_{j=1}^{m} \left(\frac{k-i}{k+i}\right)^{\rho_j} P_j,$$

where

(i) $G_+(k)$ is continuous in $\mathbf{C}^+$ in the operator norm on $\mathscr{L}(L^2(S^2))$ and boundedly invertible there. Similarly, $G_-(k)$ is continuous in $\mathbf{C}^-$ in the operator norm and boundedly invertible there;

(ii) $G_+(k)$ is analytic in $\mathbf{C}^+$ and $G_-(k)$ is analytic in $\mathbf{C}^-$; and

(iii) $G_+(\infty) = G_-(\infty) = \mathbf{I}$.

The projections $P_1,...,P_m$ are finite in number, are mutually disjoint, and have rank one, while $P_0 = \mathbf{I} - \Sigma_{j=1}^m P_j$. The **(left) partial indices** $\rho_1,...,\rho_m$ are nonzero integers. In the absence of partial indices, we have $D(k) = \mathbf{I}$, in which case the Wiener–Hopf factorization is called **(left) canonical**. The partial indices depend neither on the choice of the factors $G_+(k)$ and $G_-(k)$ nor on the choice of the projections $P_1,...,P_m$. If the factorization is (left) canonical, the factors $G_+(k)$ and $G_-(k)$ are unique, as one sees by applying Liouville's theorem.

In the same way, we define a right Wiener–Hopf factorization, right partial indices, and a right canonical factorization by interchanging $G_+(k)$ and $G_-(k)$ in (4.1). The right indices may be different, both in number and in value, from the left indices, but the sum of the right indices coincides with the sum of the left indices. This sum is called the **sum index** of $G(k)$.

By using the Möbius transformation defined above Proposition 2.8, we can define the left and right Wiener–Hopf factorizations of operator functions on the unit circle $\mathbf{T}$ in the complex plane. The left and right partial indices are invariant under this Möbius transformation.

Remark 4.1: If $G(k)$ has a left Wiener–Hopf factorization of the form (4.1) with left partial indices $\rho_1,...,\rho_m$, then taking the inverses of both sides of (4.1) converts it into a right Wiener–Hopf factorization of $G(k)^{-1}$ with right partial indices $-\rho_1,..., -\rho_m$. On the other hand, if we consider the right Wiener–Hopf factorization

$$G(k) = \hat{G}_-(k)\hat{D}(k)\hat{G}_+(k), \quad k \in \mathbf{R}_\infty, \quad (4.2)$$

with

$$\hat{D}(k) = \hat{P}_0 + \sum_{j=1}^{\hat{m}} \left(\frac{k-i}{k+i}\right)^{\hat{\rho}_j} \hat{P}_j,$$

and take the adjoints on both sides of (4.1) with k replaced by its complex conjugate $\bar{k}$, we convert it into a right Wiener–Hopf factorization of $G(\bar{k})^\dagger$ with right partial indices $-\hat{\rho}_1,..., -\hat{\rho}_{\hat{m}}$. Hence, if $G(k)$ is unitary for every real k, which is the case in inverse scattering theory, the sets of left and right partial indices of $G(k)$ necessarily coincide. Moreover, the projections and factors appearing in (4.1) and (4.2) are related by

$\hat{P}_j = (P_j)^\dagger$ for $j = 1,...,m$; and $G_\pm(k)^{-1} = G_\mp(\bar{k})^\dagger$.

In the remainder of this section we will only consider left Wiener–Hopf factorizations, though our results can also be derived for right Wiener–Hopf factorizations.

Theorem 4.2: If the potential $V(x)$ is in the Newton class, the operator function $G(k)$ defined in (3.7) has a left Wiener–Hopf factorization.

Proof: According to Theorem 6.1 (or 6.2) of Ref. 18, it is sufficient to show the following:

(i) $G(k)$ is boundedly invertible for every $k \in \mathbf{R}_\infty$;

(ii) $G(k)$ is a compact perturbation of the identity for every $k \in \mathbf{R}_\infty$; and

(iii) $\tilde{G}(\xi) \in \mathscr{H}_\alpha[\mathbf{T};\mathscr{L}(L^2(S^2))]$ for some $\alpha \in (0,1)$, where $\tilde{G}(\xi)$ is the Möbius transform of $G(k)$, as explained above in Proposition 2.8.

Under these conditions there exists a left Wiener–Hopf

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factorization of $\widetilde{G}(\xi)$ with respect to the unit circle $\mathbf{T}$ that is given by

$$\widetilde{G}(\xi)=\widetilde{G}_+(\xi)\widetilde{D}(\xi)\widetilde{G}_-(\xi),$$

where

$$\widetilde{D}(\xi)=P_0+\sum_{j=1}^{m}\xi^{\rho_j}P_j,$$

$\widetilde{G}_+(\xi)\in\mathscr{H}_\alpha[\mathbf{T}^+;\mathscr{L}(L^2(S^2))]$ and is invertible there, $\widetilde{G}_-(\xi)\in\mathscr{H}_\alpha[\mathbf{T}^-;\mathscr{L}(L^2(S^2))]$ and is invertible there, and $\widetilde{G}_+(\xi)$ and $\widetilde{G}_-(\xi)$ are analytic in $\mathbf{T}^+$ and in $\mathbf{T}^-$, respectively. The inverse of the Möbius transformation given above Proposition 2.8 then yields a left Wiener–Hopf factorization for $G(k)$ of the type (4.1) where the Möbius transformed factors $\widetilde{G}_+(\xi)$ and $\widetilde{G}_-(\xi)$ as well as their inverses are Hölder continuous of exponent α in the operator norm in $\mathbf{T}^+$ and $\mathbf{T}^-$, respectively.

First, note that we can use (3.7) to write

$$G(k)=U(k)QS(k)QU(k)^\dagger, \tag{4.3}$$

where

$$(U(k)f)(\theta)=e^{-ik\theta\cdot x}f(\theta),\quad (Qf)(\theta)=f(-\theta),$$

so that $G(k)$ is unitarily equivalent to $S(k)$. Hence, $G(k)$ is boundedly invertible for every $k\in\mathbf{R}_\infty$.

Next, since $A(k,\theta,\theta')$ is bounded and continuous in all three variables, it is Hilbert–Schmidt and hence compact as an operator on $L^2(S^2)$ for every real k. As a result,

$$\mathbf{I}-G(k)=-(k/2\pi i)U(k)QA(k)QU(k)^\dagger$$

is compact for every real k, and thus $G(k)$ is a compact perturbation of the identity.

Moreover, using (4.3) as well as the unitarity of $U(k)$ and Q, we have the estimate

$$\begin{aligned}\|G(k_1)-G(k_2)\|\leqslant&\|U(k_1)-U(k_2)\|\cdot\|S(k_1)\|\\&+\|S(k_1)-S(k_2)\|+\|S(k_2)\|\\&\cdot\|U(k_1)^\dagger-U(k_2)^\dagger\|.\end{aligned}$$

Because $U(k)$ has a k derivative whose operator norm is uniformly bounded in k for every x, it is Lipschitz continuous in the operator norm with a Lipschitz constant independent of $k\in\mathbf{R}$. Further, according to Theorem 2.11 we have $\widetilde{S}(\xi)\in\mathscr{H}_\mu[\mathbf{T};\mathscr{L}(L^2(S^2))]$ for some $\mu\in(0,1)$. Hence, $\widetilde{G}(\xi)\in\mathscr{H}_\mu[\mathbf{T};\mathscr{L}(L^2(S^2))]$ for some positive μ.

Thus all three conditions needed to apply the above mentioned Gohberg–Leiterer result are satisfied, and the proof is complete. ■

Remark 4.3: Using the symmetry relation $G(-k)=QG(k)^{-1}Q$, we can prove that it is possible to choose $G_+(k)$ and $G_-(k)$ in (4.1) such that

$$G_\pm(-k)=QG_\mp(k)^{-1}Q. \tag{4.4}$$

Indeed, from (4.1) and using $D(-k)=D(k)^{-1}$ we have

$$\begin{aligned}G(k)^{-1}&=G_-(k)^{-1}D(k)^{-1}G_+(k)^{-1}\\&=QG_+(-k)D(k)^{-1}G_-(-k)Q,\end{aligned}$$

so that

$$\begin{aligned}&G_+(k)^{-1}QG_-(-k)D(k)\\&\quad=D(k)G_-(k)^{-1}QG_+(-k).\end{aligned} \tag{4.5}$$

If the factorization (4.1) is canonical, i.e., if $D(k)\equiv\mathbf{I}$, Liouville's theorem gives (4.4) directly from (4.5). If (4.1) is not a canonical factorization, we obtain

$$\begin{aligned}&((k+i)/(k-i))^{\rho_r-\rho_s}P_rG_+(k)^{-1}QG_-(-k)^{-1}P_s\\&\quad=P_rG_-(k)QG_+(-k)P_s,\end{aligned}$$

where P_r and P_s are two of the projections appearing in $D(k)$ with $r,s\in\{0,1,2,...,m\}$. If $\rho_r\leqslant\rho_s$, both sides of the last equation are equal to P_rQP_s, due to Liouville's theorem. If $\rho_r>\rho_s$, however, we have

$$\begin{aligned}&P_rG_+(k)^{-1}QG_-(-k)^{-1}P_s\\&\quad=[\phi_{rs}(k)/(k+i)^{\rho_r-\rho_s}]P_rQP_s,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&P_rG_-(k)QG_+(-k)P_s\\&\quad=[\phi_{rs}(k)/(k-i)^{\rho_r-\rho_s}]\,P_rQP_s,\end{aligned} \tag{4.7}$$

where $\phi_{rs}(k)$ is a polynomial of degree $(\rho_r-\rho_s)$ with leading coefficient 1. Using the procedure in Ref. 31, we can multiply $G_\pm(k)$ by suitable rational functions and therefore change our original factorization (4.1) in such a way that both sides of (4.6) and (4.7) reduce to P_rQP_s. Then, using $\Sigma_{r=0}^mP_r=\Sigma_{s=0}^mP_s=\mathbf{I}$, we find (4.4) for this modified factorization.

V. SOLUTION OF THE RIEMANN–HILBERT PROBLEM

In Sec. IV we have derived the existence of a Wiener–Hopf factorization of the operator function relevant to the Riemann–Hilbert problems (3.8) and (3.11). This result was obtained under the assumption that the potential $V(x)$ belongs to the Newton class. In this section we will use the factorization (4.1) to obtain the solutions of the Riemann–Hilbert problems (3.8) and (3.11). During the process the variable x enters as a dummy variable, which may affect the partial indices and the factors in (4.1) and hence the unique solvability properties of (3.8) and (3.11) and the explicit form of their solutions, but does not affect the way in which the solution itself is obtained. Therefore, to simplify our notation we suppress the x dependence of all vectors, operators, and partial indices.

Starting from the Wiener–Hopf factorization (4.1) of $G(k)$, we define

$$D_+(k)=P_0+\sum_{\rho_j>0}\left(\frac{k-i}{k+i}\right)^{\rho_j}P_j+\sum_{\rho_j<0}P_j$$

and

$$D_-(k)=P_0+\sum_{\rho_j>0}P_j+\sum_{\rho_j<0}\left(\frac{k-i}{k+i}\right)^{\rho_j}P_j,$$

where $P_1,...,P_m$ are the mutually disjoint, rank one projections appearing in the diagonal factor $D(k)$ and $P_0=\mathbf{I}-\Sigma_{j=1}^mP_j$.

Using (4.1), let us write (3.11) in the form

$$\begin{aligned}X_+(k)=&G_+(k)D_+(k)D_-(k)G_-(k)X_-(k)\\&+[G_+(k)D_+(k)D_-(k)G_-(k)-\mathbf{I}]\hat{1},\end{aligned} \tag{5.1}$$

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where $\hat{1}$ is the function in $L^2(S^2)$ as defined above (3.9). Then

$$D_+(k)^{-1}G_+(k)^{-1}X_+(k) = D_-(k)G_-(k)X_-(k) + [D_-(k)G_-(k) - D_+(k)^{-1}G_+(k)^{-1}]\hat{1}. \tag{5.2}$$

Premultiplying both sides by P_0 yields

$$P_0G_+(k)^{-1}X_+(k) + P_0G_+(k)^{-1}\hat{1} = P_0G_-(k)X_-(k) + P_0G_-(k)\hat{1}. \tag{5.3}$$

The left-hand side of (5.3) is analytic in $\mathbf{C}^+$, the right-hand side is analytic in $\mathbf{C}^-$, and both sides tend to $P_0\hat{1}$ as $k\to\infty$ from the appropriate half-plane. Hence, by Liouville's theorem,

$$P_0G_+(k)^{-1}X_+(k) = P_0[\mathbf{I} - G_+(k)^{-1}]\hat{1} \tag{5.4}$$

and

$$P_0G_-(k)X_-(k) = P_0[\mathbf{I} - G_-(k)]\hat{1}. \tag{5.5}$$

Similarly, premultiplying both sides of (5.2) by $(k-i)^{\rho_j}P_j$ with $\rho_j>0$ and using Liouville's theorem, we obtain

$$P_jG_+(k)^{-1}X_+(k) = P_j[\mathbf{I} - G_+(k)^{-1}]\hat{1} + [\varphi_j(k)/(k+i)^{\rho_j}]\pi_j \tag{5.6}$$

and

$$P_jG_-(k)X_-(k) = -P_jG_-(k)\hat{1} + [\varphi_j(k)/(k-i)^{\rho_j}] \times\pi_j + [(k+i)/(k-i)]^{\rho_j}P_j\hat{1}. \tag{5.7}$$

Here, π_j is a fixed nonzero vector in the range of P_j, and $\varphi_j(k)$ is an arbitrary polynomial of degree less than ρ_j. Next, premultiplication of both sides of (5.2) by P_j with $\rho_j<0$ and yet another application of Liouville's theorem yield

$$P_jG_+(k)^{-1}X_+(k) = P_j[\mathbf{I} - G_+(k)^{-1}]\hat{1} \tag{5.8}$$

and

$$P_jG_-(k)X_-(k) = -P_jG_-(k)\hat{1} + [(k+i)/(k-i)]^{\rho_j}P_j\hat{1}, \tag{5.9}$$

provided the second term on the right-hand side of (5.9) is analytic at $k=-i$. Because $\rho_j<0$, the latter happens if and only if $P_j\hat{1}=0$.

Finally, adding (5.4), (5.6), and (5.8) together as well as (5.5), (5.7), and (5.9), and using $P_0 + \Sigma_{\rho_j>0}P_j + \Sigma_{\rho_j<0}P_j = \mathbf{I}$, we obtain

$$X_+(k) = [G_+(k) - \mathbf{I}]\hat{1} + G_+(k)\sum_{\rho_j>0}\frac{\varphi_j(k)}{(k+i)^{\rho_j}}\pi_j \tag{5.10}$$

and

$$X_-(k) = [G_-(k)^{-1} - \mathbf{I}]\hat{1} + G_-(k)^{-1} \times\sum_{\rho_j>0}\frac{\varphi_j(k)\pi_j + [(k+i)^{\rho_j} - (k-i)^{\rho_j}]P_j\hat{1}}{(k-i)^{\rho_j}}, \tag{5.11}$$

provided $P_j\hat{1}=0$ whenever $\rho_j<0$. Hence, if these $(-\Sigma_{\rho_j<0}\rho_j)$ linear constraints on P_j for $\rho_j<0$ are satisfied, there is a $(\Sigma_{\rho_j>0}\rho_j)$ parameter family of solutions to (3.11), and these solutions are given by (5.10) and (5.11).

We can summarize the above results as follows.

Theorem 5.1: Let $V(x)$ be a potential in the Newton class. Then the Riemann–Hilbert problem (3.11) has a solution, if and only if $P_j\hat{1}=0$ whenever $\rho_j<0$. In that case the solutions are given by (5.10) and (5.11), where $\varphi_j(k)$ is an arbitrary polynomial of degree less than ρ_j associated with each $\rho_j>0$.

The solution of the operator Riemann–Hilbert problem (3.8) is obtained in the same way as the vector Riemann–Hilbert problem (3.11) is solved using (5.1) through (5.11). The solution of (3.8) is given by

$$\mathbf{X}_+(k) = G_+(k) - \mathbf{I} + G_+(k)\sum_{\rho_j>0}\frac{\varphi_j(k)}{(k+i)^{\rho_j}}P_j \tag{5.12}$$

and

$$\mathbf{X}_-(k) = G_-(k)^{-1} - \mathbf{I} + G_-(k)^{-1} \times\sum_{\rho_j>0}\frac{\varphi_j(k) + [(k+i)^{\rho_j} - (k-i)^{\rho_j}]}{(k-i)^{\rho_j}}P_j, \tag{5.13}$$

provided there are no negative partial indices. If there are any negative partial indices, the solution does not exist. Due to the presence of $\varphi_j(k)$ in (5.12) and (5.13), the solution is not unique unless there are no positive partial indices.

Note that, when there are no bound states, for $x=0$, the operator $[\mathbf{I} + \mathbf{X}_+(k)]$ becomes related to the 3-D Jost operator used in the 3-D Gel'fand–Levitan inversion method.[2–4] Hence, we obtain the following result.

Corollary 5.2: If the potential $V(x)$ belongs to the Newton class with no bound states, the Jost operator exists if and only if there are no partial indices of the scattering operator. In that case the Jost operator is given by

$$J(k) = QS_+(k)Q, \tag{5.14}$$

where $S_+(k)$ is the operator that is given by $G_+(k)$ evaluated at $x=0$.

VI. SOLUTION OF THE INVERSE PROBLEM

Once the Riemann–Hilbert problem posed in (3.11) is solved by the Wiener–Hopf factorization method given in Sec. V, we obtain $f(k,x,\theta)$ given in (3.2) using (3.9). If there are no bound states, from the Schrödinger equation (1.1) we then obtain the potential as

$$V(x) = \frac{(\Delta + 2ik\theta\cdot\nabla)X_+(k,x,\theta)}{1 + X_+(k,x,\theta)}. \tag{6.1}$$

Note that the right-hand side of this equation contains θ and k whereas these two variables are absent from the left-hand side. Hence, the solution of the Riemann–Hilbert problem will lead to a potential only if the right-hand side of (6.1) is independent of θ and k. Below, we show that if the so-called miracle condition[2] occurs, the right-hand side of (6.1) is independent of θ and k and becomes equal to a potential function of x.

Let the Fourier transform of $X_+(k,x,\theta)$ be given by

$$\eta(\alpha,x,\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, X_+(k,x,\theta)e^{-ik\alpha}. \tag{6.2}$$

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Since $X_+(k,x,\theta) = O(1/|k|)$ as $k \to \pm\infty$ and is analytic in k in $\mathbf{C}^+$, we have $\eta(\alpha,x,\theta) = 0$ for $\alpha < 0$. The function $\eta(\alpha,x,\theta)$ plays a major role in the 3-D Newton–Marchenko inversion theory.[2] In case the Riemann–Hilbert problem (3.11) has a unique solution, $\eta(\alpha,x,\theta)$ satisfies the equation[2]

$$\left[\Delta - 2\frac{\partial}{\partial\alpha}\theta\cdot\nabla - V(x)\right]\eta(\alpha,x,\theta) = 0, \tag{6.3}$$

where the potential is obtained as

$$V(x) = -2\theta\cdot\nabla \lim_{\alpha\to 0^+} \eta(\alpha,x,\theta), \tag{6.4}$$

provided the right-hand side of (6.4) is independent of θ. The θ independence of the right-hand side of (6.4) is known as the "miracle" condition of Newton.[2]

From (6.2) we have

$$ikX_+(k,x,\theta) = -\lim_{\alpha\to 0^+}\eta(\alpha,x,\theta) - \int_0^\infty d\alpha\, e^{ik\alpha}\frac{\partial}{\partial\alpha}\eta(\alpha,x,\theta). \tag{6.5}$$

Hence, using (6.2), (6.4), and (6.5), we obtain

$$[\Delta + 2ik\theta\cdot\nabla - V(x)]X_+(k,x,\theta) = V(x) + \int_0^\infty d\alpha\, e^{ik\alpha}\left[\Delta - 2\frac{\partial}{\partial\alpha}\theta\cdot\nabla - V(x)\right]\eta(\alpha,x,\theta).$$

Thus (6.1) is equivalent to (6.3) and (6.4) in the absence of bound states.

If there are any bound states, the above procedure can be modified to prove that the potential $V(x)$ is obtained from (6.1) if and only if (6.3) and (6.4) hold true.[34]

VII. CONCLUSION

In this paper we have established the following results. If the potential $V(x)$ belongs to the Newton class defined in Sec. II, the corresponding scattering operator has a Wiener–Hopf factorization. The related Riemann–Hilbert problem (3.11) can be solved by using these factors. The related operator Riemann–Hilbert problem (3.8) is also solvable by using the Wiener–Hopf factors. A consequence of this is the following. For potentials in the Newton class with no bound states, the Jost operator (as defined in Ref. 2) exists if and only if the corresponding scattering operator does not have any partial indices. If and only if Newton's miracle condition is satisfied, the solution of the Riemann–Hilbert problem leads to a potential.

The physical interpretation of the partial indices of the scattering operator is an open problem. It is known that the total index is related to the total number of bound states of the potential, but the relationship of each partial index to the bound states or to any physical parameters is presently not known.

A simple condition[34] that guarantees the unique solvability of the Riemann–Hilbert problems (3.8) and (3.11) is given by $\max_{k\in\mathbf{R}}\|S(k) - \mathbf{I}\| < 1$, where the norm is the operator norm on $L^2(S^2)$. When this happens, the scattering operator $S(k)$ has neither positive nor negative partial indices.

The results presented in this paper remain valid for any real, measurable potential $V(x)$ on $\mathbf{R}^n$ with $n \geqslant 2$ without real exceptional points that lead to a scattering operator $S(k)$ such that $S(k) - \mathbf{I}$ is compact for all $k\in\mathbf{R}$ and that $\tilde{S}(\xi) = S(i(1+\xi)/(1-\xi))$ is Hölder continuous in $\xi\in\mathbf{T}$.

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[1] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer, New York, 1989), 2nd ed.
[2] R. G. Newton, J. Math. Phys. **21**, 1698 (1980); **22**, 631 (1981); **23**, 693 (1982).
[3] R. G. Newton, J. Math. Phys. **22**, 2191 (1981); **23**, 693 (1982).
[4] R. G. Newton, J. Math. Phys. **23**, 2257 (1982).
[5] R. G. Newton, *Scattering Theory in Mathematical Physics*, edited by J. A. Lavita and J.-P. Marchand (Reidel, Dordrecht, The Netherlands, 1974), p. 193.
[6] A. I. Nachman and M. J. Ablowitz, Stud. Appl. Math. **71**, 243 (1984).
[7] R. Beals and R. R. Coifman, Proc. Symp. Pure Math. **43**, 45 (1985).
[8] R. Beals and R. R. Coifman, Physica D **18**, 242 (1986).
[9] R. G. Novikov and G. M. Henkin, Sov. Math. Dokl. **35**, 153 (1987) [Dokl. Akad. Nauk SSSR **292**, 814 (1987) (Russian)].
[10] R. T. Prosser, J. Math. Phys. **10**, 1819 (1969).
[11] R. T. Prosser, J. Math. Phys. **17**, 1775 (1976).
[12] R. T. Prosser, J. Math. Phys. **21**, 2648 (1980).
[13] R. T. Prosser, J. Math. Phys. **23**, 2127 (1982).
[14] L. D. Faddeev, Sov. Phys. Dokl. **10**, 1033 (1965) [Dokl. Akad. Nauk SSSR **165**, 514 (1965) (Russian)].
[15] L. D. Faddeev, J. Sov. Math. **5**, 334 (1976) [Itogi Nauki Tekh. **3**, 93 (1974) (Russian)].
[16] R. G. Newton, Inverse Problems **1**, 371 (1985).
[17] R. G. Newton, *Inverse Schrödinger Scattering in Three Dimensions* (Springer, New York, 1989).
[18] I. C. Gohberg and J. Leiterer, Math. Nachrichten **55**, 33 (1973) (Russian).
[19] R. G. Newton, J. Math. Phys. **18**, 1348 (1977).
[20] M. S. Birman, Am. Math. Soc. Transl. **53**, 23 (1966) [Mat. Sb. **55**, 125 (1961) (Russian)].
[21] J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. **47**, 122 (1961).
[22] T. Kato, Commun. Pure Appl. Math. **12**, 403 (1959).
[23] R. Weder, Inverse Problems **6**, 267 (1990).
[24] L. Vega, Proc. Am. Math. Soc. **102**, 874 (1988).
[25] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* [Transl. Math. Monographs, Vol. 54 (Am. Math. Soc., Providence, 1981)] [Nauka, Moscow, 1978 (Russian)].
[26] N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, The Netherlands, 1953) [Nauka, Moscow, 1946 (Russian)].
[27] I. C. Gohberg and M. G. Krein, Am. Math. Soc. Transl. **14**, 217 (1960) [Uspekhi Mat. Nauk **13** (2), 3 (1959) (Russian)].
[28] N. P. Vekua, *Systems of Singular Integral Equations* (Noordhoff, Groningen, The Netherlands, 1967) [Nauka, Moscow, 1950 (Russian)].
[29] I. C. Gohberg and I. A. Feldman, *Convolution Equations and Projection Methods for their Solution* [Transl. Math. Monographs, Vol. 41 (Am. Math. Soc., Providence, 1974)] [Nauka, Moscow, 1971 (Russian)].
[30] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators* (Birkhäuser OT **3**, Boston, 1981).
[31] I. C. Gohberg, Am. Math. Soc. Transl. **49**, 130 (1966) [Izvestiya Akad. Nauk SSSR, Ser. Matem. **28**, 1055 (1964) (Russian)].
[32] I. C. Gohberg and J. Leiterer, Math. Nachrichten **54**, 41 (1972) (Russian).
[33] I. C. Gohberg and J. Leiterer, Sov. Math. Dokl. **14**, 425 (1973) [Dokl. Akad. Nauk SSSR **209**, 529 (1973) (Russian)].
[34] T. Aktosun and C. van der Mee, SIAM J. Math. Anal. (to be published).

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