Inverse wave scattering with discontinuous wave speed

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The inverse scattering problem on the line is studied for the generalized Schrödinger equation $(d^2\psi/dx^2)+k^2H(x)^2\psi=Q(x)\psi$, where $H(x)$ is a positive, piecewise continuous function with positive limits $H_\pm$ as $x \to \pm\infty$. This equation, in the frequency domain, describes the wave propagation in a nonhomogeneous medium, where $Q(x)$ is the restoring force and $1/H(x)$ is the variable wave speed changing abruptly at various interfaces. A related Riemann–Hilbert problem is formulated, and the associated singular integral equation is obtained and proved to be uniquely solvable. The solution of this integral equation leads to the recovery of $H(x)$ in terms of the scattering data consisting of $Q(x)$, a reflection coefficient, either of $H_\pm$, and the bound state energies and norming constants. Some explicitly solved examples are provided. © *1995 American Institute of Physics.*

I. INTRODUCTION

Consider the one-dimensional generalized Schrödinger equation

$$\psi''(k,x)+k^2H(x)^2\psi(k,x)=Q(x)\psi(k,x), \quad x\in\mathbf{R}, \tag{1.1}$$

where the prime denotes the derivative with respect to the space coordinate. This equation, in the frequency domain, describes the propagation of waves in a nonhomogeneous medium where k^2 is energy, $1/H(x)$ is the wave speed, and $Q(x)$ is the restoring force per unit length. Our assumptions on $H(x)$ and $Q(x)$ in this article will vary, but several key results will be proved under the same set of assumptions listed here for future reference:

(H1) $H(x)$ is strictly positive and piecewise continuous with jump discontinuities at x_n for $n=1,\dots,N$ such that $x_1<\cdots<x_N$.

(H2) $H(x) \to H_\pm$ as $x \to \pm\infty$, where $H_\pm$ are positive constants.

(H3) $H-H_\pm\in L^1(\mathbf{R}^\pm)$, where $\mathbf{R}^-=(-\infty,0)$ and $\mathbf{R}^+=(0,+\infty)$.

(H4) H' is absolutely continuous on (x_n,x_{n+1}) and $2H''H-3(H')^2\in L^1_1(x_n,x_{n+1})$ for $n=0,\dots,N$, where $x_0=-\infty$ and $x_{N+1}=+\infty$.

(H5) $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in[0,1]$, where $L^1_\beta(I)$ is the space of measurable functions $f(x)$ on I such that $\int_I dx(1+|x|)^\beta|f(x)|<\infty$.

The scattering solutions of Eq. (1.1) are those behaving like $e^{ikH_\pm x}$ or $e^{-ikH_\pm x}$ as $x \to \pm\infty$. Such solutions occur when $k^2>0$. There are two linearly independent scattering solutions of Eq. (1.1) satisfying the boundary conditions

$$f_l(k,x)=e^{ikH_+x}+o(1), \quad f_l'(k,x)=ikH_+e^{ikH_+x}+o(1), \quad x \to +\infty, \tag{1.2}$$

$$f_r(k,x)=e^{-ikH_-x}+o(1), \quad f_r'(k,x)=-ikH_-e^{-ikH_-x}+o(1), \quad x \to -\infty. \tag{1.3}$$

We will call $f_l(k,x)$ the Jost solution of Eq. (1.1) from the left and $f_r(k,x)$ the Jost solution from the right. We have

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$$f_l(k,x)=\frac{1}{T_l(k)}\,e^{ikH_-x}+\frac{L(k)}{T_l(k)}\,e^{-ikH_-x}+o(1), \quad x\to -\infty, \tag{1.4}$$

$$f_r(k,x)=\frac{1}{T_r(k)}\,e^{-ikH_+x}+\frac{R(k)}{T_r(k)}\,e^{ikH_+x}+o(1), \quad x\to +\infty, \tag{1.5}$$

where $T_l(k)$ and $T_r(k)$ are the transmission coefficients from the left and from the right, respectively, and $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix associated with Eq. (1.1) is defined as

$$\mathbf{S}(k)=\begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k)\end{bmatrix}.$$

The solutions of Eq. (1.1) belonging to $L^2(\mathbf{R})$ are called the bound state solutions; to be more precise, here we need to use $L^2(\mathbf{R},H(x)^2dx)$ instead of $L^2(\mathbf{R},dx)$, but these two spaces are equivalent due to the fact that $H(x)$ is a strictly positive, bounded function. Such solutions may occur only at finitely many negative values of k^2, known as the bound state energies. Associated with each bound state is a positive constant called the bound state norming constant. The purpose of this article is to recover $H(x)$ from the scattering data consisting of $Q(x)$, one of the reflection coefficients, either of $H_\pm$, and the bound state energies and norming constants. In the special case when $Q(x)=0$ and $H_\pm=1$, Grinberg[1] proposed a method to recover $H(x)$ with jump discontinuities; in Ref. 2 the case $Q(x)=0$ and $H_\pm\neq 1$ was considered. When $Q(x)=0$, the solution of the inverse problem for Eq. (1.1) is greatly simplified because in that case there are no bound states and the reduced reflection coefficients defined in Eq. (4.1) are strictly less than 1 in absolute value; in fact, as noted in Refs. 1 and 2 no abstract Fredholm theory is needed to solve the resulting singular integral equation when $Q(x)=0$. When $H(x)$ is twice continuously differentiable, the solution of the inverse problem can be found[3] by using a Liouville transformation defined on the whole real axis. However, when $H(x)$ has discontinuities, the Liouville transformation (3.1) can only be used locally, i.e., only on each interval (x_n,x_{n+1}). In fact, the nonexistence of a global Liouville transformation greatly complicates the analysis of the inverse scattering problem. A discussion of the inverse problem when $H(x)$ has discontinuities was also given in Ref. 4, where a different approach was used and an incomplete solution was given.

The discontinuities in $H(x)$ correspond to abrupt changes in the properties of the medium in which the wave propagates. In the inversion procedure described in Refs. 1 and 2 and in the present article, neither the locations of the discontinuities of $H(x)$ nor the jumps in $H(x)$ at such locations are given as part of the scattering data; on the contrary, these locations and jumps are recovered by the inversion method.

In order to extend our solution of the inverse problem from $Q\in L_2^1(\mathbf{R})$ to $Q\in L_{1+\alpha}^1(\mathbf{R})$ with $\alpha\in(0,1)$, we relate the key singular integral equation (5.21) to the solution of the vector Riemann–Hilbert problem (7.4)–(7.7) whose unique solvability is proved by studying factorizations of almost periodic matrix functions.[5,6] Although Riemann–Hilbert problems have been found useful in solving various other inverse problems,[7–11] to the best of our knowledge the Riemann–Hilbert problem described in Eqs. (7.4)–(7.7) does not appear elsewhere in the inverse scattering literature. In setting up Eqs. (7.4)–(7.7) we follow ideas outlined in Ref. 12, where several types of integral equations are studied by reducing them to Riemann–Hilbert problems of the type (7.4)–(7.7).

In regard to the problems with discontinuous coefficients, we remark that Sabatier and his co-workers[13–16] studied the scattering for the impedance-potential equation and that Krueger studied[17,18] the inverse scattering problem for $u_{xx}-u_{tt}+c_1(x)u_x+c_2(x)u_t+c_3(x)u=0$, where $x,t\in\mathbf{R}$ and the coefficients c_1, c_2, c_3 are sectionally continuous functions with support in a finite interval. Krueger also considered $u_{xx}-\varepsilon(x)u_{tt}=0$ when $\varepsilon(x)$ is constant for $x<0$ and sectionally continuous for $x>0$, and he developed[19] an iterative method to recover $\varepsilon(x)$ when the incoming and reflected waves are given.

This article is organized as follows. In Sec. II we establish the analyticity of the Jost solutions of Eq. (1.1) and obtain some estimates on these solutions. In Sec. III we analyze a sequence of Schrödinger equations related to Eq. (1.1) in the intervals (x_n,x_{n+1}) for $n=0,\dots,N$. The results in Secs. II and III are used in Sec. IV to obtain certain properties of the scattering matrix $\mathbf{S}(k)$ needed to solve the inverse scattering problem. In Sec. V we give the solution of our inverse problem; $H(x)$ is recovered by using in Eq. (5.24) the unique solution of our singular integral equation (5.21). In Sec. VI our inversion procedure is illustrated by two examples. The unique solvability of Eq. (5.21) is established in Sec. VII. In Sec. VIII we give the recovery of $H(x)$ when bound states are present and illustrate the inversion procedure by an example. Finally, in the Appendix we give the proof of the second part of Theorem 4.2, establishing the behavior of the scattering matrix as $k \to 0$.

II. SCATTERING SOLUTIONS

In this section we show that the Jost solutions can be extended analytically in k from the real axis to the upper-half complex plane. We let $\mathbf{C}^+$ denote the upper-half complex plane and $\overline{\mathbf{C}^+} = \mathbf{C}^+\cup\mathbf{R}$. Similarly, $\mathbf{C}^-$ denotes the lower-half complex plane and $\overline{\mathbf{C}^-} = \mathbf{C}^-\cup\mathbf{R}$.

Let $[f;g]=fg'-f'g$ denote the Wronskian. Using Eqs. (1.2)–(1.5) we obtain

$$[f_l(k,x);f_r(k,x)]=-2ik\,\frac{H_+}{T_r(k)}=-2ik\,\frac{H_-}{T_l(k)}, \tag{2.1}$$

$$[f_l(k,x);f_r(-k,x)]=2ikH_-\,\frac{L(k)}{T_l(k)}=-2ikH_+\,\frac{R(-k)}{T_r(-k)}, \tag{2.2}$$

$$[f_l(-k,x);f_l(k,x)]=2ikH_+=\frac{2ikH_-}{T_l(k)T_l(-k)}\,[1-L(k)L(-k)], \tag{2.3}$$

$$[f_r(-k,x);f_r(k,x)]=-2ikH_-=-\frac{2ikH_+}{T_r(k)T_r(-k)}\,[1-R(k)R(-k)]. \tag{2.4}$$

From Eqs. (1.1), (1.2), and (1.3) we see that the Jost solutions satisfy

$$f_l(-k,x)=\overline{f_l(k,x)},\quad f_r(-k,x)=\overline{f_r(k,x)},\quad k\in\mathbf{R}, \tag{2.5}$$

where the overbar denotes complex conjugation. Hence from Eqs. (2.1) and (2.2) we have

$$\mathbf{S}(-k)=\overline{\mathbf{S}(k)},\quad k\in\mathbf{R}. \tag{2.6}$$

The scattering matrix $\mathbf{S}(k)$ is not unitary unless $H_+=H_-$. However, from Eqs. (2.1)–(2.4) it follows that

$$T_r(k)L(-k)+R(k)T_r(-k)=0,\quad H_+T_l(k)=H_-T_r(k), \tag{2.7}$$

$$T_r(k)T_l(-k)+L(k)L(-k)=T_r(-k)T_l(k)+R(k)R(-k)=1, \tag{2.8}$$

$$\det\mathbf{S}(k)=T_l(k)T_r(k)-L(k)R(k)=\frac{T_l(k)}{T_l(-k)}=\frac{T_r(k)}{T_r(-k)}, \tag{2.9}$$

where det denotes the matrix determinant.

From Eqs. (1.1) and (1.2) we have

$$f_l(k,x)=e^{ikH_+x}+\frac{1}{kH_+}\int_x^\infty dz[\sin kH_+(z-x)][k^2\{H_+^2-H(z)^2\}+Q(z)]f_l(k,z). \quad (2.10)$$

Similarly, from Eqs. (1.1) and (1.3) we have

$$f_r(k,x)=e^{-ikH_-x}+\frac{1}{kH_-}\int_{-\infty}^x dz[\sin kH_-(x-z)][k^2\{H_-^2-H(z)^2\}+Q(z)]f_r(k,z). \quad (2.11)$$

Let us define the Faddeev functions, $m_l(k,x)$ from the left and $m_r(k,x)$ from the right, as

$$m_l(k,x)=e^{-ikH_+x}f_l(k,x), \quad m_r(k,x)=e^{ikH_-x}f_r(k,x). \quad (2.12)$$

From Eqs. (1.2) and (1.3) it is seen that

$$m_l(k,x)=1+o(1), \quad m_l'(k,x)=o(1), \quad x\to+\infty,$$

$$m_r(k,x)=1+o(1), \quad m_r'(k,x)=o(1), \quad x\to-\infty.$$

From Eqs. (2.10)–(2.12) we obtain

$$m_l(k,x)=1-\frac{1}{2ikH_+}\int_x^\infty dz[1-e^{2ikH_+(z-x)}][k^2\{H_+^2-H(z)^2\}+Q(z)]m_l(k,z), \quad (2.13)$$

$$m_r(k,x)=1-\frac{1}{2ikH_-}\int_{-\infty}^x dz[1-e^{2ikH_-(x-z)}][k^2\{H_-^2-H(z)^2\}+Q(z)]m_r(k,z). \quad (2.14)$$

Theorem 2.1:

(i) Assume $Q\in L_1^1(\mathbf{R})$ and $H-H_\pm\in L^1(\mathbf{R}^\pm)$. Then, for each fixed $x\in\mathbf{R}$, the Faddeev functions $m_l(k,x)$, $m_r(k,x)$ and their derivatives $m_l'(k,x)$, $m_r'(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Consequently, the Jost solutions $f_l(k,x)$, $f_r(k,x)$ and their derivatives $f_l'(k,x)$, $f_r'(k,x)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Moreover, the following estimates hold:

$$|m_l(k,x)|\le(1+\max\{0,-x\})e^{E_+(k,x)}, \quad k\in\overline{\mathbf{C}^+}, \quad (2.15)$$

$$|m_r(k,x)|\le(1+\max\{0,x\})e^{E_-(k,x)}, \quad k\in\overline{\mathbf{C}^+}, \quad (2.16)$$

$$|m_l'(k,x)|\le e^{E_+(k,x)}\int_x^\infty dz(1+\max\{0,-z\})[|k|^2|H_+^2-H(z)^2|+|Q(z)|], \quad k\in\overline{\mathbf{C}^+}, \quad (2.17)$$

$$|m_r'(k,x)|\le e^{E_-(k,x)}\int_{-\infty}^x dz(1+\max\{0,z\})[|k|^2|H_-^2-H(z)^2|+|Q(z)|], \quad k\in\overline{\mathbf{C}^+}, \quad (2.18)$$

where

$$E_\pm(k,x)=\pm\int_x^{\pm\infty} dz\left[\frac{|k|}{H_\pm}|H_\pm^2-H(z)^2|+(1+|z|)|Q(z)|\right]. \quad (2.19)$$

(ii) Assume $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in[0,1]$ and $H-H_\pm\in L^1(\mathbf{R}^\pm)$. If $\alpha\in[0,1)$, then, as $k\to 0$ in $\overline{\mathbf{C}^+}$ we have

$$m_l(k,x)-m_l(0,x)=o(|k|^\alpha),\quad m_l'(k,x)-m_l'(0,x)=o(|k|^\alpha)$$

uniformly on $x\geqslant a$, and

$$m_r(k,x)-m_r(0,x)=o(|k|^\alpha),\quad m_r'(k,x)-m_r'(0,x)=o(|k|^\alpha)$$

uniformly on $x\leqslant a$, for any $a\in\mathbf{R}$. If $\alpha=1$, then the same relations hold, but with $O(k)$ on the right-hand sides. Moreover, if $\alpha=1$, then $\dot m_l(0,x)$ and $\dot m_r(0,x)$ exist, where the overdot denotes the derivative with respect to k.

Proof: (i) The proof of the analyticity and continuity of $m_l(k,x)$, $m_r(k,x)$, $m_l'(k,x)$, and $m_r'(k,x)$ is similar to the proof of Theorem 2.1 in Ref. 20 and is based on iteration of Eqs. (2.13) and (2.14). We omit the details. To prove Eq. (2.15) we note that by Eq. (2.13)

$$|m_l(k,x)|\leqslant 1+\int_x^\infty dz\left[\frac{|k|}{H_+}\,|H_+^2-H(z)^2|+(z-x)|Q(z)|\right]|m_l(k,z)|, \tag{2.20}$$

where we have used the estimate $|1-e^{2ikH_+(z-x)}|\leqslant 2$ on the term involving $H_+-H(z)$ and used $|1-e^{2ikH_+(z-x)}|\leqslant 2|k|H_+|z-x|$ on the term involving $Q(z)$. Defining $\hat m_l(k,x)=m_l(k,x)/(1+\max\{0,-x\})$, and using

$$(z-x)\,\frac{(1+\max\{0,-z\})}{(1+\max\{0,-x\})}\leqslant 1+|z|,\quad z\geqslant x,$$

from Eq. (2.20) we obtain

$$|\hat m_l(k,x)|\leqslant\frac{1}{1+\max\{0,-x\}}+\int_x^\infty dz\left[\frac{|k|}{H_+}\,|H_+^2-H(z)^2|+(1+|z|)|Q(z)|\right]|\hat m_l(k,z)|.$$

Now Eq. (2.15) follows from Gronwall's inequality. Similarly one derives Eq. (2.16). Inequalities (2.17) and (2.18) are obtained by differentiating Eqs. (2.13) and (2.14), respectively, and using Eqs. (2.15) and (2.16). With the help of Eq. (2.12), for each fixed x, one obtains the analyticity in $\mathbf{C}^+$ and continuity in $\overline{\mathbf{C}^+}$ of the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ and their derivatives $f_l'(k,x)$ and $f_r'(k,x)$. Hence, the proof of (i) is complete.

To prove (ii), note that from Eq. (2.13) we have

$$m_l(k,x)-m_l(0,x)=A_1(k,x)+A_2(k,x)+A_3(k,x), \tag{2.21}$$

where

$$A_1(k,x)=-\int_x^\infty dz\left[\frac{k}{2iH_+}\,(1-e^{2ikH_+(z-x)})\right][H_+^2-H(z)^2]m_l(0,z),$$

$$A_2(k,x)=-\int_x^\infty dz\left[\frac{1+2ikH_+(z-x)-e^{2ikH_+(z-x)}}{2ikH_+(z-x)}\right](z-x)Q(z)m_l(0,z),$$

$$A_3(k,x)=-\frac{1}{2ikH_+}\int_x^\infty dz[1-e^{2ikH_+(z-x)}][k^2\{H_+^2-H(z)^2\}+Q(z)][m_l(k,z)-m_l(0,z)].$$

Using

$$\left|\frac{1-e^{2ikH_+(z-x)}}{2iH_+}\right| \leqslant C\,\frac{|k|(z-x)}{1+|k|(z-x)} \leqslant C\,\frac{|k|(z-a)}{1+|k|(z-a)} \tag{2.22}$$

for some constant C, for all $x \geqslant a$ we obtain

$$|A_1(k,x)| \leqslant C|k| \int_a^\infty dz\, \frac{|k|(z-a)}{1+|k|(z-a)}\, |H_+^2 - H(z)^2||m_l(0,z)| = o(k). \tag{2.23}$$

Using

$$\left|\frac{1+2ikH_+(z-x)-e^{2ikH_+(z-x)}}{2ikH_+(z-x)}\right| \leqslant C\,\frac{|k|(z-x)}{1+|k|(z-x)} \leqslant C\,\frac{|k|(z-a)}{1+|k|(z-a)}, \tag{2.24}$$

we obtain

$$|A_2(k,x)| \leqslant C|k|^\alpha \int_a^\infty dz\, \frac{|k|^{1-\alpha}(z-a)^{1-\alpha}}{1+|k|(z-a)}\, (z-a)^{1+\alpha}|Q(z)||m_l(0,z)| = o(|k|^\alpha). \tag{2.25}$$

By using Gronwall's inequality we see that $m_l(k,x) - m_l(0,x) = o(|k|^\alpha)$ uniformly on $x \geqslant a$. The estimates involving $m_r(k,x)$, $m_l'(k,x)$, and $m_r'(k,x)$ are obtained in a similar manner.

Now suppose $\alpha=1$. From Eqs. (2.21)–(2.25) it is seen that the error terms are $O(k)$. The proof of the existence of $\dot{m}_l(0,x)$ and $\dot{m}_r(0,x)$ is complicated by the fact that under the assumption $H-H_\pm \in L^1(\mathbf{R}^\pm)$ we cannot simply differentiate Eqs. (2.13) and (2.14) under the integral sign. Since the detailed estimates are standard but lengthy, we only outline the proof. Let $\Delta(k,x) = [m_l(k,x) - m_l(0,x)]k^{-1}$. Then using Eqs. (2.21)–(2.25) and Gronwall's inequality, we show that $|\Delta(k,x)| \leqslant C'$ uniformly for $x \geqslant a$ and k near 0. Using this bound and the integral equation satisfied by $\Delta(k,x)$ we estimate the difference $|\Delta(k_1,x) - \Delta(k_2,x)|$ and show that it is Cauchy as $k_1, k_2 \to 0$. Thus $\Delta(k,x)$ has a limit as $k \to 0$ and hence $\dot{m}_l(0,x)$ exists. A similar argument works for $\dot{m}_r(0,x)$. ■

The statements in (ii) of Theorem 2.1 also apply to the Jost solutions and their derivatives, but in that case the uniformity in x is only valid when x is restricted to a bounded interval.

If $\alpha=1$, using Eq. (2.10) as $k \to 0$ and Eq. (2.12), we find that $\dot{f}_l(0,x)$ obeys the integral equation

$$\dot{f}_l(0,x) = iH_+x + \int_x^\infty dz(z-x)Q(z)\dot{f}_l(0,z). \tag{2.26}$$

This integral equation can be solved by iteration, and it is used in Sec. V (Theorem 5.4). Using Eqs. (2.20) and (2.26) we see that $[\dot{f}_l(0,x); f_l(0,x)] = -iH_+$ and hence we have

$$\left(\frac{\dot{f}_l(0,x)}{f_l(0,x)}\right)' = \frac{iH_+}{f_l(0,x)^2}, \tag{2.27}$$

which will be used in Secs. VI and VIII.

III. A LOCAL LIOUVILLE TRANSFORMATION

In this section we relate the Jost solutions of Eq. (1.1) to solutions of a sequence of Schrödinger equations on the intervals (x_j, x_{j+1}) for $j=0,\ldots,N$. We use the fact that although $H(x)$ is discontinuous at x_j, the Jost solutions of Eq. (1.1) and their x-derivatives are continuous even at these points. The results here will be used in the next section to establish the properties of the scattering matrix.

Under the Liouville transformation

$$y=y(x)=\int_0^x ds\ H(s), \quad \psi(k,x)=\frac{1}{\sqrt{H(x)}}\ \phi(k,y), \tag{3.1}$$

the generalized Schrödinger equation (1.1) is transformed into

$$\frac{d^2\phi(k,y)}{dy^2}+k^2\phi(k,y)=V(y)\phi(k,y), \tag{3.2}$$

where

$$V(y(x))=\frac{H''(x)}{2H(x)^3}-\frac{3}{4}\frac{H'(x)^2}{H(x)^4}+\frac{Q(x)}{H(x)^2}. \tag{3.3}$$

Since $H(x)$ is assumed to have jump discontinuities at x_j for $j=1,\ldots,N$, the quantity $V(y)$ is undefined at $y_j=y(x_j)$. However, $V(y)$ is well defined in each of the intervals (y_j,y_{j+1}) for $j=0,\ldots,N$; thus, we can only use the Liouville transformation locally, i.e., only on each interval (x_j,x_{j+1}). Since $H(x)$ is strictly positive, it follows that $y_0=y(x_0)=-\infty$ and $y_{N+1}=y(x_{N+1})=+\infty$.

Let $V_{j,j+1}(y)$ be the potential defined by

$$V_{j,j+1}(y)=\begin{cases} V(y), & y\in(y_j,y_{j+1}),\\ 0, & \text{elsewhere,}\end{cases} \tag{3.4}$$

where $V(y)$ is the quantity in Eq. (3.3). We have $V_{j,j+1}\in L^1_1(\mathbf{R})$, which is satisfied because $2H''H-3(H')^2\in L^1_1(x_j,x_{j+1})$ for $j=0,\ldots,N$, i.e., because (H4) is satisfied. The Faddeev function from the left, $Y_{l;j,j+1}(k,y)$, associated with the potential $V_{j,j+1}(y)$ satisfies the differential equation

$$Y''_{l;j,j+1}(k,y)+2ikY'_{l;j,j+1}(k,y)=V_{j,j+1}(y)Y_{l;j,j+1}(k,y),$$

with the boundary conditions $Y_{l;j,j+1}(k,y_{j+1})=1$ and $Y'_{l;j,j+1}(k,y_{j+1})=0$; thus we see that $Y_{l;j,j+1}(k,y)$ satisfies the integral equation

$$Y_{l;j,j+1}(k,y)=1+\frac{1}{2ik}\int_y^{y_{j+1}}dt[e^{2ik(t-y)}-1]V_{j,j+1}(t)Y_{l;j,j+1}(k,t).$$

Recall that the prime denotes the derivative with respect to the space coordinate, and hence $Y'(k,y)$ denotes $dY(k,y)/dy$. Similarly, the Faddeev function from the right, $Y_{r;j,j+1}(k,y)$, associated with the potential $V_{j,j+1}(y)$ satisfies the differential equation

$$Y''_{r;j,j+1}(k,y)-2ikY'_{r;j,j+1}(k,y)=V_{j,j+1}(y)Y_{r;j,j+1}(k,y),$$

with the boundary conditions $Y_{r;j,j+1}(k,y_j)=1$ and $Y'_{r;j,j+1}(k,y_j)=0$; hence $Y_{r;j,j+1}(k,y)$ satisfies the integral equation

$$Y_{r;j,j+1}(k,y)=1+\frac{1}{2ik}\int_{y_j}^{y} dt[e^{2ik(y-t)}-1]V_{j,j+1}(t)Y_{r;j,j+1}(k,t).$$

Let $\mathbf{s}_{j,j+1}(k)$ be the scattering matrix for the Schrödinger equation with the potential $V_{j,j+1}(y)$ such that

$$\mathbf{s}_{j,j+1}(k)=\begin{bmatrix} t_{j,j+1}(k) & r_{j,j+1}(k) \\ l_{j,j+1}(k) & t_{j,j+1}(k) \end{bmatrix},$$

where $t_{j,j+1}(k)$ is the transmission coefficient and $r_{j,j+1}(k)$ and $l_{j,j+1}(k)$ are the reflection coefficients from the right and from the left, respectively. The following facts about the entries of $\mathbf{s}_{j,j+1}(k)$ and the Faddeev functions are well known (Refs. 21–24): $k/t_{j,j+1}(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and

$$t_{j,j+1}(k)=1+O(1/k), \quad k\to\infty \text{ in } \overline{\mathbf{C}^+}. \tag{3.5}$$

Then we have

$$Y_{l;j,j+1}(k,y)=\begin{cases} \dfrac{1}{t_{j,j+1}(k)}[1+l_{j,j+1}(k)e^{-2iky}], & y\leqslant y_j, \quad j=1,\dots,N, \quad k\in\overline{\mathbf{C}^+}, \\ \dfrac{1}{t_{0,1}(k)}[1+l_{0,1}(k)e^{-2iky}]+o(1), & y\to-\infty, \quad j=0, \quad k\in\mathbf{R}, \end{cases}$$

$$Y'_{l;j,j+1}(k,y)=\begin{cases} -2ik\,\dfrac{l_{j,j+1}(k)}{t_{j,j+1}(k)}\,e^{-2iky}, & y\leqslant y_j, \quad j=1,\dots,N, \quad k\in\overline{\mathbf{C}^+}, \\ -2ik\,\dfrac{l_{0,1}(k)}{t_{0,1}(k)}\,e^{-2iky}+o(e^{-2iky}), & y\to-\infty, \quad j=0, \quad k\in\mathbf{R} \end{cases}$$

and furthermore, for $j\geqslant 1$, $[l_{j,j+1}(k)/t_{j,j+1}(k)]e^{-2iky}$ is analytic in $\mathbf{C}^+$ and for $y\leqslant y_j$ it vanishes when $k\to\infty$ in $\overline{\mathbf{C}^+}$. In fact, we have

$$Y_{l;j,j+1}(k,y)-1=o(1), \quad Y'_{l;j,j+1}(k,y)=o(1), \quad k\to\infty \text{ in } \overline{\mathbf{C}^+} \tag{3.6}$$

uniformly in $y\in\mathbf{R}$. Similarly, we obtain

$$Y_{r;j,j+1}(k,y)=\begin{cases} \dfrac{1}{t_{j,j+1}(k)}[1+r_{j,j+1}(k)e^{2iky}], & y\geqslant y_{j+1}, \quad j=0,\dots,N-1, \quad k\in\overline{\mathbf{C}^+}, \\ \dfrac{1}{t_{N,N+1}(k)}[1+r_{N,N+1}(k)e^{2iky}]+o(1), & y\to+\infty, \quad j=N, \quad k\in\mathbf{R}, \end{cases}$$

$$Y'_{r;j,j+1}(k,y)=\begin{cases} 2ik\,\dfrac{r_{j,j+1}(k)}{t_{j,j+1}(k)}\,e^{2iky}, & y\geqslant y_{j+1}, \quad j=0,\dots,N-1, \quad k\in\overline{\mathbf{C}^+}, \\ 2ik\,\dfrac{r_{N,N+1}(k)}{t_{N,N+1}(k)}\,e^{2iky}+o(e^{2iky}), & y\to+\infty, \quad j=N, \quad k\in\mathbf{R}. \end{cases}$$

Furthermore, for $j\leqslant N-1$, $[r_{j,j+1}(k)/t_{j,j+1}(k)]e^{2iky}$ is analytic in $\mathbf{C}^+$ and for $y\geqslant y_{j+1}$ it vanishes when $k\to\infty$ in $\overline{\mathbf{C}^+}$. In fact, we have

$$Y_{r;j,j+1}(k,y)-1=o(1), \quad Y'_{r;j,j+1}(k,y)=o(1), \quad k\to\infty \text{ in } \overline{\mathbf{C}^+} \tag{3.7}$$

uniformly in $y\in\mathbf{R}$. For each fixed $y\in\mathbf{R}$, the functions $Y_{l;j,j+1}(k,y)$, $Y_{r;j,j+1}(k,y)$, $Y'_{l;j,j+1}(k,y)$, and $Y'_{r;j,j+1}(k,y)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$.

Let us define

$$\eta_{j,j+1}(k,x)=\frac{1}{\sqrt{H(x)}}\,e^{iky}Y_{l;j,j+1}(k,y),\quad \xi_{j,j+1}(k,x)=\frac{1}{\sqrt{H(x)}}\,e^{-iky}Y_{r;j,j+1}(k,y). \tag{3.8}$$

Then $\eta_{j,j+1}(k,x)$ and $\xi_{j,j+1}(k,x)$ are two linearly independent solutions of Eq. (1.1) in the interval (x_j,x_{j+1}) for $j=0,\dots,N$. After using $dy/dx=H(x)$, for $x\in(x_j,x_{j+1})$ we have

$$\eta'_{j,j+1}(k,x)=\left(ik\sqrt{H(x)}-\frac{H'(x)}{2H(x)^{3/2}}\right)e^{iky}Y_{l;j,j+1}(k,y)+\sqrt{H(x)}e^{iky}Y'_{l;j,j+1}(k,y), \tag{3.9}$$

$$\xi'_{j,j+1}(k,x)=\left(-ik\sqrt{H(x)}-\frac{H'(x)}{2H(x)^{3/2}}\right)e^{-iky}Y_{r;j,j+1}(k,y)+\sqrt{H(x)}e^{-iky}Y'_{r;j,j+1}(k,y). \tag{3.10}$$

Moreover, by letting $y\to+\infty$ we obtain

$$[Y_{l;n,n+1}(k,y);Y_{r;n,n+1}(k,y)]-2ikY_{l;n,n+1}(k,y)Y_{r;n,n+1}(k,y)=\frac{-2ik}{t_{n,n+1}(k)}. \tag{3.11}$$

Let us define the Wronskian matrix

$$F(k,x)=\begin{bmatrix} f_l(k,x) & f_r(k,x)\\ f'_l(k,x) & f'_r(k,x)\end{bmatrix}, \tag{3.12}$$

where $f_l(k,x)$ and $f_r(k,x)$ are the Jost solutions of Eq. (1.1). From Eq. (2.12) and Theorem 2.1 it follows that for each fixed $x\in\mathbf{R}$, the matrix $F(k,x)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$. Let us also define

$$\Gamma_{j,j+1}(k,x)=\begin{bmatrix} \eta_{j,j+1}(k,x) & \xi_{j,j+1}(k,x)\\ \eta'_{j,j+1}(k,x) & \xi'_{j,j+1}(k,x)\end{bmatrix},\quad j=0,\dots,N. \tag{3.13}$$

Using Eqs. (3.8)–(3.11) we obtain

$$\det\Gamma_{n,n+1}(k,x)=-\frac{2ik}{t_{n,n+1}(k)}. \tag{3.14}$$

In the interval (x_j,x_{j+1}), we have

$$F(k,x)=\Gamma_{j,j+1}(k,x)A_{j,j+1}(k), \tag{3.15}$$

where

$$A_{j,j+1}(k)=\begin{bmatrix} a_{j,j+1}(k) & c_{j,j+1}(k)\\ b_{j,j+1}(k) & d_{j,j+1}(k)\end{bmatrix}$$

is to be determined using Eqs. (3.8)–(3.10). Since $f_l(k,x)$, $f_r(k,x)$, $f'_l(k,x)$, and $f'_r(k,x)$ are continuous at each x_j, the following matching conditions must be satisfied:

$$\Gamma_{j-1,j}(k,x_j-0)A_{j-1,j}(k)=\Gamma_{j,j+1}(k,x_j+0)A_{j,j+1}(k),\quad j=1,\dots,N. \tag{3.16}$$

Let us define

$$A_\pm = \pm \int_0^{\pm\infty} ds[H_\pm - H(s)], \tag{3.17}$$

$$A = A_- + A_+ = \int_{-\infty}^0 ds[H_- - H(s)] + \int_0^\infty ds[H_+ - H(s)]. \tag{3.18}$$

Note that from Eqs. (3.1), (3.17), and (3.18), we have

$$y = H_+ x - A_+ + o(1), \quad x \to +\infty, \tag{3.19}$$

$$y = H_- x + A_- + o(1), \quad x \to -\infty. \tag{3.20}$$

Hence, using Eqs. (1.2) and (3.8) in (3.15) we obtain

$$a_{N,N+1}(k) = \sqrt{H_+} e^{ikA_+}, \quad b_{N,N+1}(k) = 0, \tag{3.21}$$

and similarly, using Eqs. (1.3) and (3.8) in (3.15), we obtain

$$d_{0,1}(k) = \sqrt{H_-} e^{ikA_-}, \quad c_{0,1}(k) = 0. \tag{3.22}$$

Using Eqs. (3.16) and (3.21) we have

$$\begin{bmatrix} a_{j,j+1}(k) \\ b_{j,j+1}(k) \end{bmatrix} = \left(\prod_{n=j}^{N-1} \Gamma_{n,n+1}(k,x_{n+1}-0)^{-1} \Gamma_{n+1,n+2}(k,x_{n+1}+0) \right) \begin{bmatrix} \sqrt{H_+} e^{ikA_+} \\ 0 \end{bmatrix}, \tag{3.23}$$

and using Eqs. (3.16) and (3.22) we obtain

$$\begin{bmatrix} c_{j,j+1}(k) \\ d_{j,j+1}(k) \end{bmatrix} = \left(\prod_{n=j}^{1} \Gamma_{n,n+1}(k,x_n+0)^{-1} \Gamma_{n-1,n}(k,x_n-0) \right) \begin{bmatrix} 0 \\ \sqrt{H_-} e^{ikA_-} \end{bmatrix}. \tag{3.24}$$

Thus, from Eqs. (3.12), (3.15), and (3.23), for $x \in (x_j, x_{j+1})$ with $0 \leq j \leq N-1$ we have

$$\begin{bmatrix} f_l(k,x) \\ f_l'(k,x) \end{bmatrix} = \Gamma_{j,j+1}(k,x) \left(\prod_{n=j}^{N-1} \Gamma_{n,n+1}(k,x_{n+1}-0)^{-1} \Gamma_{n+1,n+2}(k,x_{n+1}+0) \right) \begin{bmatrix} \sqrt{H_+} e^{ikA_+} \\ 0 \end{bmatrix}, \tag{3.25}$$

and similarly from Eqs. (3.12), (3.15), and (3.24), for $1 \leq j \leq N$ we obtain

$$\begin{bmatrix} f_r(k,x) \\ f_r'(k,x) \end{bmatrix} = \Gamma_{j,j+1}(k,x) \left(\prod_{n=j}^{1} \Gamma_{n,n+1}(k,x_n+0)^{-1} \Gamma_{n-1,n}(k,x_n-0) \right) \begin{bmatrix} 0 \\ \sqrt{H_-} e^{ikA_-} \end{bmatrix}. \tag{3.26}$$

The notation in Eqs. (3.24) and (3.26) means that n decreases from j to 1. For future use we define the matrix

$$\mathscr{G}(k) = \prod_{n=0}^{N-1} \Gamma_{n,n+1}(k,x_{n+1}-0)^{-1} \Gamma_{n+1,n+2}(k,x_{n+1}+0). \tag{3.27}$$

IV. PROPERTIES OF THE SCATTERING MATRIX

In this section we establish some properties of the scattering matrix that are needed in later sections. Let us define

$$\sigma(k)=\begin{bmatrix} \tau(k) & \rho(k) \\ \ell(k) & \tau(k) \end{bmatrix},$$

where

$$\tau(k)=\sqrt{\frac{H_+}{H_-}}\,T_l(k)e^{ikA}=\sqrt{\frac{H_-}{H_+}}\,T_r(k)e^{ikA},\quad \rho(k)=R(k)e^{2ikA_+},\quad \ell(k)=L(k)e^{2ikA_-}. \tag{4.1}$$

Then from Eqs. (2.6)–(2.9) it follows that $\sigma(k)$ is unitary. We will call $\sigma(k)$ the reduced scattering matrix for Eq. (1.1), and hence $\tau(k)$ will be called the reduced transmission coefficient and $\rho(k)$ and $\ell(k)$ the reduced reflection coefficients. Using Eqs. (1.4), (1.5), (3.19), (3.20), (3.25), (3.26), and (4.1), we have

$$\frac{1}{\tau(k)}=\frac{1}{t_{0,1}(k)}\,[1\quad 0]\mathcal{S}(k)\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{t_{N,N+1}(k)}\,[0\quad 1]\mathcal{S}(k)^{-1}\begin{bmatrix}0\\1\end{bmatrix}, \tag{4.2}$$

$$\frac{\ell(k)}{\tau(k)}=\begin{bmatrix}\dfrac{l_{0,1}(k)}{t_{0,1}(k)} & 1\end{bmatrix}\mathcal{S}(k)\begin{bmatrix}1\\0\end{bmatrix}, \tag{4.3}$$

$$\frac{\rho(k)}{\tau(k)}=\begin{bmatrix}1 & \dfrac{r_{N,N+1}(k)}{t_{N,N+1}(k)}\end{bmatrix}\mathcal{S}(k)^{-1}\begin{bmatrix}0\\1\end{bmatrix}, \tag{4.4}$$

where $\mathcal{S}(k)$ is defined by Eq. (3.27). Note that from Eqs. (3.14) and (3.27) we obtain $\det \mathcal{S}(k)=t_{0,1}(k)/t_{N,N+1}(k)$ and hence from Eqs. (4.2)–(4.4) we have

$$\mathcal{S}(k)=\begin{bmatrix} \dfrac{t_{0,1}(k)}{\tau(k)} & \dfrac{t_{0,1}(k)}{t_{N,N+1}(k)}\,\dfrac{r_{N,N+1}(k)-\rho(k)}{\tau(k)} \\ \dfrac{\ell(k)-l_{0,1}(k)}{\tau(k)} & \dfrac{\tau(k)}{t_{N,N+1}(k)}+\dfrac{\ell(k)-l_{0,1}(k)}{t_{N,N+1}(k)}\,\dfrac{r_{N,N+1}(k)-\rho(k)}{\tau(k)} \end{bmatrix}.$$

Theorem 4.1: Assume $Q\in L^1_1(\mathbf{R})$ and $H-H_\pm\in L^1(\mathbf{R}^\pm)$. Then

(i) $k/\tau(k)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$; $\tau(k)$ is continuous at $k=0$, and either $\tau(0)\neq 0$ or $\tau(k)$ vanishes linearly as $k\to 0$. The bound state energies for Eq. (1.1) correspond to the (simple) zeros of $k/\tau(k)$ in $\mathbf{C}^+$ and can only occur on the imaginary axis in $\mathbf{C}^+$. There is never a bound state at $k=0$.

(ii) $\rho(k)$ and $\ell(k)$ are continuous for $k\in\mathbf{R}$. Either $|\rho(k)|=|\ell(k)|<1$ for all $k\in\mathbf{R}$, or $|\rho(k)|=|\ell(k)|<1$ for $k\neq 0$ and $\rho(0)=\ell(0)=-1$.

Proof: The analyticity of $k/\tau(k)$ in $\mathbf{C}^+$ and the continuity in $\overline{\mathbf{C}^+}\setminus\{0\}$ follow from Eqs. (2.1), (2.12), (4.1), and Theorem 2.1. The continuity and asymptotic behavior of $\tau(k)$ near $k=0$ and the resulting dichotomy are established in the Appendix in connection with the proof of Theorem 4.2. The assertions about the bound states are proven as in the proof of Proposition 5.1 of Ref. 20. We omit the details. The proof of (ii) follows from part (i), Eqs. (2.1), (2.2), and the unitarity of $\sigma(k)$. ■

We will refer to the case when $\tau(0)=0$ [$\tau(0) \neq 0$] as the *generic [exceptional]* case. By Eq. (2.1) the exceptional case occurs if and only if the Jost solutions $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent, i.e., if, for some nonzero constant γ, we have

$$f_l(0,x)=\gamma f_r(0,x). \tag{4.5}$$

Theorem 4.2: Assume $H-H_\pm \in L^1(\mathbf{R}^\pm)$ and $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in[0,1)$. Then (i) In the generic case

$$\rho(k)=-1+o(|k|^\alpha), \quad \ell(k)=-1+o(|k|^\alpha), \quad k\to 0 \quad \text{in} \quad \mathbf{R}, \tag{4.6}$$

$$\tau(k)=ick+o(|k|^{1+\alpha}), \quad k\to 0 \quad \text{in} \quad \overline{\mathbf{C}^+}, \tag{4.7}$$

where c is a nonzero real constant.

(ii) In the exceptional case

$$\tau(k)=\frac{2\sqrt{H_-H_+}\gamma}{H_-\gamma^2+H_+}+o(|k|^\alpha), \quad k\to 0 \quad \text{in} \quad \overline{\mathbf{C}^+}, \tag{4.8}$$

$$\rho(k)=\frac{H_+-H_-\gamma^2}{H_-\gamma^2+H_+}+o(|k|^\alpha), \quad k\to 0 \quad \text{in} \quad \mathbf{R}, \tag{4.9}$$

$$\ell(k)=\frac{H_-\gamma^2-H_+}{H_-\gamma^2+H_+}+o(|k|^\alpha), \quad k\to 0 \quad \text{in} \quad \mathbf{R}, \tag{4.10}$$

where γ is the constant defined in Eq. (4.5). Both (i) and (ii) remain valid for $\alpha=1$, provided we replace the error terms by $O(k)$.

Proof: Equations (4.6) and (4.7) follow from Eqs. (2.1), (2.2), (4.1), and Theorem 2.1. We also find that

$$c=-\frac{2\sqrt{H_+H_-}}{[f_l(0,x);f_r(0,x)]}. \tag{4.11}$$

The proof of (ii) is given in the Appendix. ■

Next we consider the large k behavior of the reduced scattering matrix. Using Eqs. (3.5)–(3.10), we obtain

$$\Gamma_{n,n+1}(k,x_{n+1}-0)^{-1}\Gamma_{n+1,n+2}(k,x_{n+1}+0) = \begin{bmatrix} \alpha_{n+1}(1+o(1)) & \beta_{n+1}e^{-2iky_{n+1}}(1+o(1)) \\ \beta_{n+1}e^{2iky_{n+1}}(1+o(1)) & \alpha_{n+1}(1+o(1)) \end{bmatrix} \tag{4.12}$$

as $k\to\infty$ in $\overline{\mathbf{C}^+}$, where

$$\alpha_n=\frac{1}{2}\frac{H(x_n-0)+H(x_n+0)}{\sqrt{H(x_n-0)H(x_n+0)}}, \quad \beta_n=\frac{1}{2}\frac{H(x_n-0)-H(x_n+0)}{\sqrt{H(x_n-0)H(x_n+0)}}. \tag{4.13}$$

Let us define

$$\mathbf{q}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{4.14}$$

$$E(k,x_n)=\begin{bmatrix} \alpha_n & \beta_n e^{-2iky_n} \\ \beta_n e^{2iky_n} & \alpha_n \end{bmatrix}. \tag{4.15}$$

Note that $E(-k,x_n)=\mathbf{q}E(k,x_n)\mathbf{q}$, and we also have $\overline{E(-k,x_n)}=E(k,x_n)$ for $k\in\mathbf{R}$. Hence $\Pi_{n=1}^{N} E(k,x_n)$ has the form

$$\prod_{n=1}^{N} E(k,x_n)=\begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix}, \tag{4.16}$$

where for $k\in\mathbf{R}$, $a(-k)=\overline{a(k)}$ and $b(-k)=\overline{b(k)}$. By Eqs. (4.13) and (4.15)

$$\det E(k,x_n)=\alpha_n^2-\beta_n^2=1, \tag{4.17}$$

and hence from Eq. (4.16) we obtain

$$\det\left(\prod_{n=1}^{N} E(k,x_n)\right)=|a(k)|^2-|b(k)|^2=1, \quad k\in\mathbf{R}. \tag{4.18}$$

By using induction on n it follows from Eqs. (4.13), (4.15), and (4.16) that

$$a(k)=\prod_{n=1}^{N} \alpha_n+\sum_{s} \gamma_s e^{2ikb_s}, \tag{4.19}$$

where the summation runs over a finite number of terms, and where γ_s and b_s are real constants. Moreover, $b_s>0$ owing to the fact that each b_s is a sum of terms of the form y_j-y_i with $j>i$. If $N=1$, i.e., if $H(x)$ has only one discontinuity, the summation in Eq. (4.19) is absent and $a(k)$ is constant. From Eqs. (3.27), (4.2), (4.12), and (4.19) we obtain

$$\frac{1}{\tau(k)}=a(k)+o(1), \quad k\to\infty \quad \text{in} \quad \overline{\mathbf{C}^+}. \tag{4.20}$$

Proposition 4.3: Suppose assumptions (H1)–(H5) are satisfied. Then $|a(k)|\geqslant 1$ in $\overline{\mathbf{C}^+}$.

Proof: From Eq. (4.13) it is seen that $\alpha_n\geqslant 1$, where the equality holds if and only if $H(x_n-0)=H(x_n+0)$. This proves the proposition when $N=1$, since then $a(k)=\alpha_1$. So we can assume $N\geqslant 2$. We first claim that $1/a(k)$ is the reduced transmission coefficient for a Schrödinger equation with $Q(x)=0$ and a piecewise constant function $H(x)=H_0(x)$ with jumps at $x_{0,1},\dots,x_{0,N}$; let $\tau_0(k)$ be the corresponding reduced transmission coefficient. Note that if $Q(x)=0$ and $H(x)$ is piecewise constant, then the potentials $V_{j,j+1}(y)$ in Eq. (3.4) are zero. Therefore, if we evaluate $1/\tau_0(k)$ by using Eq. (4.20), the $o(1)$-terms are all zero and $1/\tau_0(k)$ is of the form (4.19). The choice of $H_0(x)$ is not unique and we can find infinitely many $H_0(x)$ all of which lead to the same $\tau_0(k)$. For example, let

$$H_0(x)=\begin{cases} H_{0,-}, & x\in(-\infty,x_{0,1}) \\ \dfrac{H_{0,-}}{\Pi_{j=1}^{n}(\alpha_j+\beta_j)^2}, & x\in(x_{0,n},x_{0,n+1}), \quad n=1,\dots,N, \end{cases}$$

where $H_{0,-}>0$ is an arbitrary positive constant, $x_{0,1}$ is arbitrary, and

$$x_{0,n+1}=x_{0,n}+\frac{y_{n+1}-y_n}{H_{0,-}}\prod_{j=1}^{n}(\alpha_j+\beta_j)^2,\quad n=1,\ldots,N-1.$$

It is straightforward to check that $\alpha_{0,n}=\alpha_n$, $\beta_{0,n}=\beta_n$, and $y_{0,n+1}-y_{0,n}=y_{n+1}-y_n$, where $\alpha_{0,n}$, $\beta_{0,n}$, and $y_{0,n}$ are given by Eqs. (4.13) and (3.1) with $H(x)$ replaced by $H_0(x)$. Hence $a(k)=1/\tau_0(k)$. Using the analog of Proposition 5.4 of Ref. 20, we obtain that Eq. (1.1) with $Q(x)=0$ and $H(x)=H_0(x)$ has no bound states. Hence, by Theorem 4.1, $\tau_0(k)$ does not have any poles in $\overline{\mathbf{C}^+}$ and so $a(k)\neq 0$ in $\overline{\mathbf{C}^+}$. Furthermore, $a(k)$ is an entire function of order 1. Let $\varrho(r)=\min_{|k|=r}|a(k)|$ for $r>0$. Then, by a result in Ref. 25 (Theorem 2.7.4) we have $\varrho(r)\neq o(\exp(-r^{1+\epsilon}))$ as $r\to\infty$ for any $\epsilon>0$. In other words, there is a sequence $r_n\to\infty$ and $\delta>0$ such that $\varrho(r_n)\geqslant\delta\exp(-r_n^{1+\epsilon})$ as $n\to\infty$. Thus

$$\left.\frac{1}{|a(k)|}\right|_{|k|=r_n}\leqslant\frac{1}{\delta}\exp(r_n^{1+\epsilon}).\tag{4.21}$$

For $k\in\mathbf{R}$ we have the bound $1/|a(k)|=|\tau_0(k)|\leqslant 1$, and on the positive imaginary axis we have $1/|a(k)|\leqslant M$ for some constant M. Together with Eq. (4.21) this implies by using a Phragmén–Lindelöf theorem (Ref. 25, Theorem 1.4.2) that $1/|a(k)|\leqslant\max\{1,M\}$ in the first quadrant of the complex plane. A similar argument applies to the second quadrant; hence $1/|a(k)|\leqslant\max\{1,M\}$ in $\overline{\mathbf{C}^+}$. Another application of a Phragmén–Lindelöf theorem (Ref. 25, Theorem 1.4.3) yields $1/|a(k)|\leqslant 1$ in $\overline{\mathbf{C}^+}$. ■

Proposition 4.3 implies that we can write Eq. (4.20) in the form

$$\frac{1}{\tau(k)}=a(k)[1+o(1)],\quad k\to\infty\quad\text{in}\quad\overline{\mathbf{C}^+}.\tag{4.22}$$

In an analogous manner, from Eqs. (3.27), (4.4), (4.12), (4.18), and (4.22) we get

$$\rho(k)=\frac{-b(k)}{a(k)}+o(1),\quad k\to\pm\infty.\tag{4.23}$$

Theorem 4.4: Under assumptions (H1)–(H5), if Eq. (1.1) does not have any bound states, $|\tau(k)|\leqslant 1$ in $\overline{\mathbf{C}^+}$.

Proof: Assume $|\tau(k_0)|>1$ for some $k_0\in\overline{\mathbf{C}^+}$. By taking reciprocal expressions in Eq. (4.22) and using Proposition 4.3 we have $|\tau(k)-1/|a(k)||=o(1)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Hence there exists a number $R>|k_0|$ such that $|\tau(k)-1/|a(k)||<|\tau(k_0)|-1$ for $|k|=R$, $k\in\overline{\mathbf{C}^+}$. Since $|a(k)|\geqslant 1$ by Proposition 4.3, we conclude that $|\tau(k)|<|\tau(k_0)|$ on the semicircle $|k|=R$ with $k\in\overline{\mathbf{C}^+}$, and also on $\mathbf{R}$ since $|\tau(k)|\leqslant 1$ there. This contradicts the maximum modulus principle. Hence $|\tau(k)|\leqslant 1$ in $\overline{\mathbf{C}^+}$. ■

Let AP^W stand for the algebra of all complex-valued functions $f(k)$ on $\mathbf{R}$ which are of the form $f(k)=\Sigma_{j=-\infty}^{\infty}f_je^{ik\lambda_j}$, where $f_j\in\mathbf{C}$ and $\lambda_j\in\mathbf{R}$ for all j and $\Sigma_j|f_j|<\infty$. Then the closure of AP^W in $L^\infty(\mathbf{R})$ is the algebra AP of almost periodic functions.

Theorem 4.5: Under assumptions (H1)–(H5), we have

(i) $a(k)$, $b(k)$, $1/a(k)$, and $b(k)/a(k)$ belong to AP^W.

(ii) $\limsup_{k\to\pm\infty}|\rho(k)|<1$.

Proof: Since by Eq. (4.19) and Proposition 4.3 for $k\in\overline{\mathbf{C}^+}$ we have $1\leqslant|a(k)|\leqslant C<\infty$ for some constant C, we see that $a(k)$ is an invertible element of $L^\infty(\mathbf{R})$. Since the invertible elements of AP^W are exactly the elements of AP^W that are invertible in $L^\infty(\mathbf{R})$ (cf. Ref. 26, Corollary 1 of Sec. 29.9), $a(k)$ is invertible in AP^W. Hence $1/a(k)$ and $b(k)/a(k)$ belong to AP^W, proving part (i). Using Eq. (4.18) we get

$$\left|\frac{b(k)}{a(k)}\right| \leq \sqrt{1-\frac{1}{C^2}}<1, \quad k\in\mathbf{R}. \tag{4.24}$$

Now part (ii) follows from Eq. (4.23). ■

When $H(x)$ is discontinuous, although $\tau(k)$ does not converge to a constant as $k \to \infty$ in $\overline{\mathbf{C}^+}$, it is still possible to construct $\tau(k)$ explicitly and uniquely in terms of $|\rho(k)|$ and the bound state energies, thus generalizing a procedure found in Ref. 22 for the case $H(x)\equiv 1$. We will not give the details here, but we refer the reader to Sec. 2 of Ref. 27 where this construction is given.

We conclude this section with a result that will be needed in Sec. VIII. Let $\mathrm{sgn}(x)$ denote the sign function, i.e., $\mathrm{sgn}(x)=1$ when $x>0$ and $\mathrm{sgn}(x)=-1$ when $x<0$. Let $\mathcal{N}$ denote the number of bound states of Eq. (1.1). Then we have

Proposition 4.6: $\mathrm{sgn}(c)=(-1)^{\mathcal{N}+1}$ and $\mathrm{sgn}(\gamma)=(-1)^{\mathcal{N}}$, where c and γ are the constants in Eqs. (4.7) and (4.5), respectively.

Proof: Let $k=is$ with $s>0$. Then $1/\tau(is)$ is real and, by Theorem 4.1 (i), has $\mathcal{N}$ simple zeros corresponding to the $\mathcal{N}$ bound states. Moreover, by Eqs. (4.13), (4.19), and (4.21), $1/\tau(is) \to \Pi_{n=1}^N \alpha_n \geq 1$ as $s \to +\infty$. Thus $(-1)^{\mathcal{N}}/\tau(is)>0$ for $s>0$ small enough. Therefore, in the generic case, we find that $c=-\lim_{s\to 0} \tau(is)/s$ is positive if $\mathcal{N}$ is odd and negative if $\mathcal{N}$ is even. In the exceptional case we may set $s=0$ so that $(-1)^{\mathcal{N}}/\tau(0)>0$, and by using Eq. (4.8) we see that γ is negative if $\mathcal{N}$ is odd and positive if $\mathcal{N}$ is even. ■

V. SOLUTION OF THE INVERSE PROBLEM

In this section, when there are no bound states we obtain $H(x)$ from the scattering data consisting of $Q(x)$, one of the reflection coefficients, and either of $H_\pm$. The recovery of $H(x)$ is accomplished by using in Eq. (5.24) the unique solution of the singular integral equation (5.21). The proof of the unique solvability of Eq. (5.21) will be given in Sec. VII. In Sec. VIII the method described here will be generalized to the case when there are bound states.

Let us write the Jost solutions of Eq. (1.1) in the form

$$f_l(k,x)=\sqrt{\frac{H_+}{H(x)}}\, e^{iky+ikA_+}Z_l(k,y), \tag{5.1}$$

$$f_r(k,x)=\sqrt{\frac{H_-}{H(x)}}\, e^{-iky+ikA_-}Z_r(k,y), \tag{5.2}$$

where y is the quantity in Eq. (3.1) and $A_\pm$ are the constants in Eq. (3.17). The functions $Z_l(k,y)$ and $Z_r(k,y)$ defined in Eqs. (5.1) and (5.2), respectively, will be called the Faddeev functions associated with Eq. (3.2); in particular, we will call $Z_l(k,y)$ the Faddeev function from the left and $Z_r(k,y)$ the Faddeev function from the right. From Eq. (3.3) we see that $e^{iky}Z_l(k,y)$ and $e^{-iky}Z_r(k,y)$ satisfy Eq. (3.2) on every interval (y_j,y_{j+1}) with $j=0,\dots,N$.

Proposition 5.1: Under assumptions (H1)–(H5) the quantities $Z_l(k,y)$ and $Z_r(k,y)$ defined in Eqs. (5.1) and (5.2), respectively, as well as $Z_l'(k,y)$ and $Z_r'(k,y)$ are analytic in $k\in\mathbf{C}^+$ for each $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$. When $x \to +\infty$ in Eq. (5.1) we have $Z_l(k,y) \to 1$, and when $x \to -\infty$ in Eq. (5.2) we have $Z_r(k,y) \to 1$.

Proof: The analyticity in k for $Z_l(k,y)$, $Z_r(k,y)$, $Z_l'(k,y)$, and $Z_r'(k,y)$ follows from the analyticity of $f_l(k,x)$, $f_r(k,x)$, $f_l'(k,x)$, and $f_r'(k,x)$ via Eq. (2.12) and Theorem 2.1. Using Eqs. (1.2), (3.19), (5.1), and that $H(x) \to H_+$ as $x \to +\infty$, we see that $Z_l(k,y) \to 1$ as $x \to +\infty$. Note that the $o(1)$ term in Eq. (3.19) is equal to $-\int_x^\infty ds[H(s)-H_+]$ and hence goes to zero as

$x \to +\infty$. Similarly, using Eqs. (1.3), (3.20), (5.2), and that $H(x) \to H_-$ as $x \to -\infty$, we see that $Z_r(k,y) \to 1$ as $x \to -\infty$. ■

Note that in Proposition 5.1, if we also require $H'(x) \to 0$ as $x \to \pm\infty$, we then have $Z_l'(k,y) \to 0$ when $x \to +\infty$, and $Z_r'(k,y) \to 0$ as $x \to -\infty$.

For any $\delta \in (0,\pi/2)$ let S_δ denote the sector $S_\delta = \{k \in \overline{\mathbf{C}^+} : \delta \leqslant \arg k \leqslant \pi - \delta\}$.

Theorem 5.2: Under assumptions (H1)–(H5), for each fixed $y \in \mathbf{R} \setminus \{y_1,\ldots,y_N\}$ the functions $Z_l(k,y)$ and $Z_r(k,y)$ are bounded in $\overline{\mathbf{C}^+}$. Moreover, as $k \to \infty$ in S_δ the following asymptotic relations hold:

$$Z_l(k,y) = \left(\prod_{n=j+1}^{N} \alpha_n \right) + o(1), \quad y \in (y_j, y_{j+1}), \quad j = 0,\ldots,N-1, \tag{5.3}$$

$$Z_l(k,y) = 1 + o(1), \quad y \in (y_N, +\infty), \tag{5.4}$$

$$Z_r(k,y) = \left(\prod_{n=1}^{j} \alpha_n \right) + o(1), \quad y \in (y_j, y_{j+1}), \quad j = 1,\ldots,N, \tag{5.5}$$

$$Z_r(k,y) = 1 + o(1), \quad y \in (-\infty, y_1), \tag{5.6}$$

$$Z_l'(k,y) = o(1), \quad Z_r'(k,y) = o(1), \quad y \in \mathbf{R} \setminus \{y_1,\ldots,y_N\}. \tag{5.7}$$

Proof: We set

$$\left(\prod_{n=j}^{N-1} \Gamma_{n,n+1}(k,x_{n+1}-0)^{-1} \Gamma_{n+1,n+2}(k,x_{n+1}+0) \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} A(k,y_{j+1},\ldots,y_N) \\ B(k,y_{j+1},\ldots,y_N) \end{bmatrix}, \tag{5.8}$$

where $j = 0,\ldots,N-1$. As $k \to \infty$ in $\overline{\mathbf{C}^+}$, $A(k,y_{j+1},\ldots,y_N)$ and $B(k,y_{j+1},\ldots,y_N)$ behave as

$$A(k,y_{j+1},\ldots,y_N) = \left(\prod_{n=j+1}^{N} \alpha_n \right) (1 + o(1)) + \sum_{P \subset C_{j+1}} c_P(k,y_{j+1},\ldots,y_N) \exp\left(2ik \sum_{(r,s) \in P} (y_s - y_r) \right), \tag{5.9}$$

$$B(k,y_{j+1},\ldots,y_N) = \sum_{n=j+1}^{N} c_n(k,y_{j+1},\ldots,y_N) e^{2iky_n} + \sum_{n=j+1}^{N-2} e^{2iky_n} \left(\sum_{P \subset C_{n+1}} d_P(k,y_{n+1},\ldots,y_N) \exp\left(2ik \sum_{(r,s) \in P} (y_s - y_r) \right) \right). \tag{5.10}$$

Here C_n, $n = 1,\ldots,N-1$ with $N \geqslant 2$, is the set of ordered pairs $\{r,s\}$ such that $r,s \in \{n,\ldots,N\}$ and $r < s$. The letter P denotes any nonempty subset of C_n. The coefficients $c_P(k,y_{j+1},\ldots,y_N)$, $c_n(k,y_{j+1},\ldots,y_N)$, and $d_P(k,y_{n+1},\ldots,y_N)$ depend on k through the $o(1)$ terms in Eq. (4.12) and on y_s through Eq. (4.13) for $s = 1,\ldots,N$. Therefore, these coefficients have finite limits as $k \to \infty$

in $\overline{\mathbf{C}^+}$. It is also understood that the sum in Eq. (5.9) is absent if $j=N-1$. Similarly, the second sum in Eq. (5.10) is absent if $j=N-2$ or $j=N-1$. The proof of Eqs. (5.9) and (5.10) is straightforward by using downward induction on j, starting with $j=N-1$. The details are omitted. From Eqs. (3.8), (3.13), (3.25), (5.1), and (5.8), we get for $y\in(y_j,y_{j+1})$

$$Z_l(k,y)=Y_{l;j,j+1}(k,y)A(k,y_{j+1},\ldots,y_N)+e^{-2iky}Y_{r;j,j+1}(k,y)B(k,y_{j+1},\ldots,y_N).$$

Moreover, from Eq. (3.13), (3.15), and (3.21), we get $Z_l(k,y)=Y_{l;N,N+1}(k,y)$ whenever $y\in(y_N,+\infty)$.

Now Eqs. (5.3) and (5.4) follow from Eqs. (3.7), (5.9), (5.10), and using $y<y_{j+1}$. The proof of Eqs. (5.5), (5.6), and (5.7) is analogous. ■

Theorem 5.3: Under assumptions (H1)–(H5) the functions $Z_l(k,y)$ and $Z_r(k,y)$ defined in Eqs. (5.1) and (5.2), respectively, satisfy for $k\in\mathbf{R}$ and $y\in\mathbf{R}\setminus\{y_1,\ldots,y_N\}$

$$\begin{bmatrix} Z_l(-k,y) \\ Z_r(-k,y) \end{bmatrix}=\begin{bmatrix} \tau(k) & -\rho(k)e^{2iky} \\ -\ell(k)e^{-2iky} & \tau(k) \end{bmatrix}\begin{bmatrix} Z_r(k,y) \\ Z_l(k,y) \end{bmatrix}. \tag{5.11}$$

Proof: The physical solutions $\psi_l(k,x)=T_l(k)f_l(k,x)$ and $\psi_r(k,x)=T_r(k)f_r(k,x)$ satisfy[4]

$$\begin{bmatrix} \psi_l(k,x) \\ \psi_r(k,x) \end{bmatrix}=\begin{bmatrix} T_l(k) & L(k) \\ R(k) & T_r(k) \end{bmatrix}\begin{bmatrix} \psi_r(-k,x) \\ \psi_l(-k,x) \end{bmatrix}, \quad k\in\mathbf{R}, \tag{5.12}$$

and hence using Eqs. (4.1), (5.1), and (5.2) in Eq. (5.12) we obtain Eq. (5.11). ■

From Eq. (5.11) we have

$$\tau(k)Z_r(k,y)=Z_l(-k,y)+\rho(k)e^{2iky}Z_l(k,y), \quad k\in\mathbf{R}, \quad y\in\mathbf{R}\setminus\{y_1,\ldots,y_N\}, \tag{5.13}$$

$$\tau(0)Z_r(0,y)=Z_l(0,y)+\rho(0)Z_l(0,y), \quad y\in\mathbf{R}\setminus\{y_1,\ldots,y_N\}. \tag{5.14}$$

Let us define

$$F_+(k,x,y)=\frac{1}{k\sqrt{H(x)}}\left[\tau(k)Z_r(k,y)-\tau(0)Z_r(0,y)\right], \tag{5.15}$$

$$F_-(k,x,y)=\frac{1}{k\sqrt{H(x)}}\left[Z_l(-k,y)-Z_l(0,y)\right]. \tag{5.16}$$

Then from Eqs. (5.13)–(5.16), for $k\in\mathbf{R}$, $x\in\mathbf{R}\setminus\{x_1,\ldots,x_N\}$, and $y\in\mathbf{R}\setminus\{y_1,\ldots,y_N\}$ we obtain

$$F_+(k,x,y)-F_-(k,x,y)=-\rho(k)e^{2iky}F_-(-k,x,y)+\frac{1}{k}\left[\rho(k)e^{2iky}-\rho(0)\right]\frac{Z_l(0,y)}{\sqrt{H(x)}}. \tag{5.17}$$

Since we assume there are no bound states, by Theorems 4.4 and 5.2, for fixed $x\in\mathbf{R}\setminus\{x_1,\ldots,x_N\}$ and $y\in\mathbf{R}\setminus\{y_1,\ldots,y_N\}$, the functions $F_\pm(k,x,y)$ have analytic extensions in k to $\mathbf{C}^\pm$, and as $k\to\infty$ in $\overline{\mathbf{C}^\pm}$, $F_\pm(k,x,y)\to 0$. The behavior of $F_\pm(k,x,y)$ at $k=0$ depends on the falloff of $Q(x)$ and $H(x)-H_\pm$ at infinity. If $Q\in L^1_{1+\alpha}(\mathbf{R})$ with $\alpha\in(0,1)$ and $H-H_\pm\in L^1(\mathbf{R}^\pm)$, then by Theorem 2.1 (ii), (5.1), and (5.2), we have

$$Z_l(k,y)-Z_l(0,y)=o(|k|^\alpha), \quad Z_r(k,y)-Z_r(0,y)=o(|k|^\alpha), \quad k\to 0 \quad \text{in} \quad \overline{\mathbf{C}^+}.$$

Also $\tau(k)-\tau(0)=o(|k|^{\alpha})$ by Eq. (4.7) and (4.8). As indicated by Theorem 5.2, $Z_l(k,y)$ and $Z_r(k,y)$ are bounded for $k \in \overline{\mathbf{C}^+}$, and hence it follows with the help of Eqs. (5.15) and (5.16) that $F_{\pm}(k,x,y)$ belong to the Hardy spaces $\mathbf{H}^p_{\pm}(\mathbf{R})$ for $p<1/(1-\alpha)$; if $\alpha=1$ a similar argument shows that $F_{\pm}(k,x,y)$ belong to $\mathbf{H}^p_{\pm}(\mathbf{R})$ for all $p\in(1,\infty)$. Recall that the Hardy spaces $\mathbf{H}^p_{\pm}(\mathbf{R})$ are the spaces of analytic functions $f(k)$ on $\mathbf{C}^{\pm}$ for which $\sup_{\epsilon>0}\int_{-\infty}^{\infty} dk|f(k\pm i\epsilon)|^p$ is finite. Associated with these spaces are the projection operators $\Pi_{\pm}$ projecting $L^p(\mathbf{R})$ onto $\mathbf{H}^p_{\pm}(\mathbf{R})$. They are given by

$$(\Pi_{\pm}f)(k)=\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k\mp i0}f(s). \tag{5.18}$$

It is known[28] that $\Pi_{\pm}$ are bounded and complementary projections on $L^p(\mathbf{R})$ when $1<p<\infty$. Applying Π_+ and Π_- to Eq. (5.17) and using $\Pi_{\pm}F_{\pm}(k,x,y)=F_{\pm}(k,x,y)$ we obtain for $k\in\mathbf{R}$

$$F_{\pm}(k,x,y)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k\mp i0}\left[\frac{\rho(s)e^{2isy}-\rho(0)}{s}\frac{Z_l(0,y)}{\sqrt{H(x)}}-\rho(s)e^{2isy}F_-(-s,x,y)\right]. \tag{5.19}$$

From Eq. (5.19) we see that for $k\in\mathbf{R}$, $x\in\mathbf{R}\backslash\{x_1,\dots,x_N\}$, and $y\in\mathbf{R}\backslash\{y_1,\dots,y_N\}$, the function $F_-(k,x,y)$ obeys the singular integral equation

$$\begin{aligned}F_-(k,x,y)=&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k+i0}\frac{\rho(s)e^{2isy}-\rho(0)}{s}\frac{Z_l(0,y)}{\sqrt{H(x)}}\\&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s+k-i0}\rho(-s)e^{-2isy}F_-(s,x,y).\end{aligned} \tag{5.20}$$

Let us write Eq. (5.20) in the form

$$X(k,x,y)=X_0(k,x,y)+(\mathscr{O}_yX)(k,x,y), \tag{5.21}$$

where $X(k,x,y)=F_-(k,x,y)$ and

$$X_0(k,x,y)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k+i0}\frac{\rho(s)e^{2isy}-\rho(0)}{s}\frac{Z_l(0,y)}{\sqrt{H(x)}}, \tag{5.22}$$

$$(\mathscr{O}_yX)(k,x,y)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s+k-i0}\rho(-s)e^{-2isy}X(s,x,y). \tag{5.23}$$

In Sec. VII we will discuss the solvability of Eq. (5.21) in $\mathbf{H}^p_{\pm}(\mathbf{R})$. Here we will only describe the recovery of $H(x)$ from the solution $X(k,x,y)$ of Eq. (5.21). The next proposition establishes a connection between $X(k,x,y)$ and $y(x)$.

Theorem 5.4: Suppose assumptions (H1)–(H5) hold with $\alpha=1$ in (H5). Then the Jost solution $f_l(0,x)$ is determined by $Q(x)$ alone and $\dot f_l(0,x)$ is determined by $Q(x)$ and H_+ alone. Furthermore, we have

$$X(0,x,y)=\frac{i}{\sqrt{H_+}}[i\dot f_l(0,x)+f_l(0,x)(y+A_+)], \tag{5.24}$$

where A_+ is the constant in Eq. (3.17).

Proof: From Eq. (2.13) we see that $f_l(0,x)$ is a solution of the integral equation

$$f_l(0,x)=1+\int_x^{\infty} dz(z-x)Q(z)f_l(0,z). \tag{5.25}$$

Since Eq. (5.25) can be solved uniquely by iteration, $f_l(0,x)$ is determined by $Q(x)$ alone. Similarly, using Eq. (2.26) $\dot{f}_l(0,x)$ is determined by $Q(x)$ and H_+ alone. Furthermore, from Eq. (2.26) it follows that $\dot{f}_l(0,x)$ is purely imaginary. From Eqs. (4.1), (5.1), and (5.16) we have

$$F_-(k,x,y)=\frac{1}{k\sqrt{H_+}}e^{ik(y+A_+)}[f_l(-k,x)-f_l(0,x)]+\frac{1}{k\sqrt{H_+}}f_l(0,x)[e^{ik(y+A_+)}-1], \tag{5.26}$$

and hence letting $k\to 0$ in Eq. (5.26), since $X(0,x,y)=F_-(0,x,y)$, we obtain Eq. (5.24). ■

Note that since $\dot{f}_l(0,x)$ is imaginary, the term in brackets in Eq. (5.24) is real. Hence $X(0,x,y)$ is purely imaginary. In order to find $X(k,x,y)$ we need to know $X_0(k,x,y)$. Now it follows from Eqs. (2.12), (5.1), and (5.2) that

$$\frac{Z_l(0,y)}{\sqrt{H(x)}}=\frac{1}{\sqrt{H_+}}f_l(0,x),\quad \frac{Z_r(0,y)}{\sqrt{H(x)}}=\frac{1}{\sqrt{H_-}}f_r(0,x), \tag{5.27}$$

and hence we see that $X_0(k,x,y)$ in Eq. (5.22) is completely determined by $\rho(k)$, $Q(x)$, and H_+. Provided Eq. (5.21) has a unique solution, $X(k,x,y)$ is also completely determined by $\rho(k)$, $Q(x)$, and H_+. Once $X(k,x,y)$ has been obtained, the value of A_+ is determined by setting $x=0$ and $y=0$ in Eq. (5.24), so that

$$A_+=-i\,\frac{\sqrt{H_+}X(0,0,0)+\dot{f}_l(0,0)}{f_l(0,0)}. \tag{5.28}$$

Note that[24] since there are no bound states, $f_l(0,x)>0$ for all $x\in\mathbf{R}$. Then $y(x)$ is found by solving Eq. (5.24) for y in terms of x. Finally, $H(x)$ is obtained by using $H(x)=dy/dx$.

We remark that if, in addition to $\rho(k)$ and $Q(x)$, H_- is known instead of H_+, then we can first compute H_+ as follows. In the exceptional case (i.e., when $\rho(0)\neq -1$), from Eq. (4.9) we have

$$H_+=\gamma^2H_-\,\frac{1+\rho(0)}{1-\rho(0)}. \tag{5.29}$$

In the generic case (i.e., when $\rho(0)=-1$) we first compute $|\tau(k)|=\sqrt{1-|\rho(k)|^2}$ for $k\in\mathbf{R}$ and then find $|c|=\lim_{k\to 0}|\tau(k)|/|k|$, where c is the constant given in Eq. (4.11). Thus

$$H_+=\frac{|c|^2|[f_l(0,x);f_r(0,x)]|^2}{4H_-}. \tag{5.30}$$

Proposition 5.4 no longer applies if $Q\in L_1^1(\mathbf{R})$, but $Q\notin L_2^1(\mathbf{R})$. Then $F_-(k,x,y)$ will in general diverge as $k\to 0$. So far we have only worked out some cases in which this divergence is of the form of certain inverse power laws. To be specific, we assume that for some $\epsilon\in(0,1)$

$$\lim_{k\to 0} k^{1-\epsilon}F_-(k,x,y)=\frac{1}{\sqrt{H_+}}F_0(x,y) \tag{5.31}$$

exists for every $x \in \mathbf{R} \backslash \{x_1,\ldots,x_N\}$ and $y \in \mathbf{R} \backslash \{y_1,\ldots,y_N\}$. Here and in the sequel it suffices to consider $k \geqslant 0$. Some conditions on $Q(x)$ that guarantee a behavior like (5.31) will be given below. To find an expression for $F_0(x,y)$ in terms of Jost solutions we insert Eq. (5.1) in Eq. (5.16) and obtain

$$k^{1-\epsilon}F_-(k,x,y) = \frac{1}{\sqrt{H(x)}} k^{-\epsilon}[Z_l(-k,y) - Z_l(0,y)]$$
$$= \frac{1}{\sqrt{H_+}} k^{-\epsilon}[f_l(-k,x) - f_l(0,x) + f_l(-k,x)(e^{iky+ikA_+} - 1)].$$

Letting $k \to 0$ in the above equations, we see that $F_0(x,y)$ in Eq. (5.31) is of the form

$$F_0(x,y) = f_0(x), \tag{5.32}$$

where

$$f_0(x) = \lim_{k \to 0} \frac{f_l(-k,x) - f_l(0,x)}{k^\epsilon}. \tag{5.33}$$

Provided the function $f_0(x)$ can be determined from $Q(x)$ alone, Eq. (5.32) takes the place of Eq. (5.24). This is the content of the next theorem.

Theorem 5.5: Suppose $Q \in L_1^1(\mathbf{R})$ and $\lim_{x \to +\infty} x^{2+\epsilon}Q(x) = q_0$ exists for some ϵ in the interval $0 < \epsilon < 1$. Then $y(x)$ and thus $H(x)$ (i.e., dy/dx) can be obtained by solving

$$F_0(x,y) = -q_0 H_+^\epsilon 2^\epsilon [\epsilon(\epsilon+1)]^{-1} \Gamma(1-\epsilon) e^{(i/2)\pi\epsilon} f_l(0,x) \tag{5.34}$$

for $y(x)$, where Γ denotes the Gamma function and $F_0(x,y)$ is defined in Eq. (5.31).

Proof: We denote by $f_{l,+}(k,x)$ the Jost solution from the left associated with the Schrödinger equation

$$\psi''(k,x) + k^2 H_+^2 \psi(k,x) = Q(x)\psi(k,x), \quad x \in \mathbf{R}. \tag{5.35}$$

The reason for considering Eq. (5.35) is that when we write

$$f_l(-k,x) - f_l(0,x) = \Delta_1 + \Delta_2, \tag{5.36}$$

with $\Delta_1 = f_l(-k,x) - f_{l,+}(-k,x)$ and $\Delta_2 = f_{l,+}(-k,x) - f_l(0,x)$, we can estimate Δ_1 and the small-k asymptotics of Δ_2 have already been worked out in Ref. 29. We begin with Δ_1. By means of the variation of constants formula we obtain

$$f_l(k,x) = f_{l,+}(k,x) + \frac{k}{2iH_+} \int_x^\infty dz [f_{l,+}(k,x)\overline{f_{l,+}(k,z)} - \overline{f_{l,+}(k,x)} f_{l,+}(k,z)]$$
$$\times [H_+^2 - H(z)^2] f_l(k,z), \tag{5.37}$$

where we used $[f_{l,+}(k,x); \overline{f_{l,+}(k,x)}] = -2ikH_+$. Since the Jost solutions appearing in Eq. (5.37) are all bounded, it follows from Eqs. (2.5) and (5.37) that for some constant C we have

$$|\Delta_1| \leqslant Ck \int_x^\infty dz |H_+^2 - H(z)^2| = O(k), \quad k \to 0. \tag{5.38}$$

To estimate Δ_2 we consider Eq. (5.35) with $Q(x)$ replaced by $Q_{\varpi}(x)=Q(x+\varpi)$, where ϖ is arbitrary. Let $f_{l,+,\varpi}(k,x)$ denote the Jost solution from the left associated with $Q_{\varpi}(x)$. Then $f_{l,+,\varpi}(k,x)$ and $f_{l,+}(k,x)$ are related by

$$f_{l,+,\varpi}(k,x)=e^{-ikH_{+}\varpi}f_{l,+}(k,x+\varpi). \tag{5.39}$$

Using Eq. (5.39) with $x=0$ and k replaced by $-k$, and $f_l(0,\varpi)=f_{l,+}(0,\varpi)=f_{l,+,\varpi}(0,0)$, we can write

$$f_{l,+}(-k,\varpi)-f_l(0,\varpi)=(e^{-ikH_{+}\varpi}-1)f_{l,+,\varpi}(-k,0)+[f_{l,+,\varpi}(-k,0)-f_{l,+,\varpi}(0,0)]. \tag{5.40}$$

The first term on the right-hand side is $O(k)$ as $k\to 0$. To estimate the second term, we note that $Q_{\varpi}(x)$ has the same asymptotic form as $Q(x)$, i.e., $\lim_{x\to+\infty} x^{2+\epsilon}Q_{\varpi}(x)=q_0$. Using the results of Ref. 29 [Theorem 3.1 (i)] together with Eq. (2.5), we have

$$f_{l,+,\varpi}(-k,0)-f_{l,+,\varpi}(0,0)=a_0e^{(i/2)\pi\epsilon}f_l(0,\varpi)H_+^{\epsilon}k^{\epsilon}+o(|k|^{\epsilon}), \tag{5.41}$$

where

$$a_0=-q_0 2^{\epsilon}[\epsilon(\epsilon+1)]^{-1}\Gamma(1-\epsilon). \tag{5.42}$$

Replacing ϖ by x and inserting Eq. (5.41) in Eq. (5.40), we get

$$\Delta_2=a_0e^{(i/2)\pi\epsilon}f_l(0,x)H_+^{\epsilon}k^{\epsilon}+o(|k|^{\epsilon}). \tag{5.43}$$

Substituting Eqs. (5.38) and (5.43) in Eq. (5.36) and evaluating the limit in Eq. (5.33), we obtain

$$f_0(x)=a_0e^{(i/2)\pi\epsilon}f_l(0,x)H_+^{\epsilon}. \tag{5.44}$$

Now Eq. (5.34) follows from Eqs. (5.32), (5.42), and (5.44). ■

VI. EXAMPLES

In this section we illustrate the inversion method described in the previous section by two examples. In the case of no bound states, we will solve Eqs. (5.21) and (5.24) starting from a given reduced reflection coefficient $\rho(k)$ both in the generic and exceptional cases. We will also discuss the connection between $\rho(k)$ and $R(k)$ or $L(k)$ in conjunction with the solution of the inverse problem. Even though the reflection coefficients in these examples are simple, the solution of the inverse problem involves extensive calculations.

We begin with the observation that the operator $\mathcal{O}_y:\mathbf{H}^2_-(\mathbf{R})\to\mathbf{H}^2_-(\mathbf{R})$ in Eq. (5.23) has the form

$$\mathcal{O}_y=\Pi_-(\rho(\cdot)e^{2i(\cdot)y}\mathcal{J}), \tag{6.1}$$

where $\mathcal{J}$ is the reversion map $(\mathcal{J}f)(k)=f(-k)$. Furthermore, $X_0(k,x,y)$ can be written as

$$X_0(k,x,y)=-\frac{f_l(0,x)}{\sqrt{H_+}}\,\Pi_-\left(\frac{\rho(s)e^{2isy}-\rho(0)}{s}\right), \tag{6.2}$$

where we have also used Eq. (5.27).

Example 6.1: Suppose there are no bound states and

$$\rho(k)=\rho_0e^{ik\beta},\quad \rho_0,\beta\in\mathbf{R},\quad |\rho_0|<1. \tag{6.3}$$

Hence we are in the exceptional case. We also assume that (H1)–(H5) hold with $\alpha=1$, and hence $Q\in L_2^1(\mathbf{R})$. Since $(1/k)[e^{ik(2y+\beta)}-1]$ belongs to $\mathbf{H}_+^2(\mathbf{R})$ when $2y+\beta\geqslant 0$ and to $\mathbf{H}_-^2(\mathbf{R})$ when $2y+\beta\leqslant 0$, from Eq. (6.2) we obtain

$$X_0(k,x,y)=\begin{cases} 0, & 2y+\beta\geqslant 0 \\ -\rho_0\dfrac{f_l(0,x)}{\sqrt{H_+}}\dfrac{e^{ik(2y+\beta)}-1}{k}, & 2y+\beta<0. \end{cases} \tag{6.4}$$

Considering the operator $\mathscr{O}_y$, we first note that $\mathscr{O}_y=0$ for $2y+\beta\geqslant 0$. When $2y+\beta<0$, using the Fourier transform $\mathscr{F}:\mathbf{H}_-^2(\mathbf{R})\to L^2(0,\infty)$ defined by

$$(\mathscr{F}g)(t)=\int_{-\infty}^{\infty}dk\, e^{ikt}g(k), \tag{6.5}$$

we obtain

$$(\mathscr{F}\mathscr{O}_y\mathscr{F}^{-1}h)(t)=\begin{cases} \rho_0 h(-t-2y-\beta), & 0<t<-(2y+\beta) \\ 0, & t>-(2y+\beta). \end{cases}$$

The spectrum of $\mathscr{F}\mathscr{O}_y\mathscr{F}^{-1}$ (and hence that of $\mathscr{O}_y$) consists of the three points $-\rho_0$, 0, and ρ_0, each of which is an eigenvalue of infinite multiplicity. Indeed, let N_-, N_0, and N_+ denote the corresponding eigenspaces of $\mathscr{F}\mathscr{O}_y\mathscr{F}^{-1}$. Then

$$N_{\mp}=\{h\in L^2(0,\infty):h(t)=\mp h(-t-2y-\beta),\quad h(t)=0\quad \text{for } t>-(2y+\beta)\},$$

$$N_0=\{h\in L^2(0,\infty):h(t)=0\quad \text{for } 0<t<-(2y+\beta)\}.$$

So, $\mathscr{O}_y$ is bounded and self-adjoint, but not compact. Taking the Fourier transform of $X_0(k,x,y)$ we obtain

$$(\mathscr{F}X_0(\cdot,x,y))(t,x,y)=\begin{cases} 2\pi i\rho_0\dfrac{f_l(0,x)}{\sqrt{H_+}}, & 0<t<-(2y+\beta) \\ 0, & t>-(2y+\beta), \end{cases}$$

and hence $\mathscr{F}X_0\in N_+$. Therefore, the solution to Eq. (5.21) is given by

$$X(k,x,y)=\frac{X_0(k,x,y)}{1-\rho_0}. \tag{6.6}$$

Now we distinguish the two cases $\beta\geqslant 0$ and $\beta<0$.

Case (a): $\beta\geqslant 0$: On the interval $y>-\beta/2$, Eq. (5.24) assumes the form

$$0=i\dot{f}_l(0,x)+f_l(0,x)(y+A_+), \tag{6.7}$$

where, by Eq. (5.28) we have

$$A_+=-i\frac{\dot{f}_l(0,0)}{f_l(0,0)}. \tag{6.8}$$

Hence $y(x)=-i[\dot{f}_l(0,x)/f_l(0,x)]+i[\dot{f}_l(0,0)/f_l(0,0)]$, and so

$$H(x)=y'(x)=-i\left(\frac{\dot{f}_l(0,x)}{f_l(0,x)}\right)'. \tag{6.9}$$

From Eqs. (2.27), (3.17), and (6.9) it follows that the constant A_+ in Eq. (6.8) can be written as

$$A_+=-H_+\int_0^\infty dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}. \tag{6.10}$$

Using Eq. (2.27) in Eq. (6.9) we obtain

$$H(x)=\frac{H_+}{f_l(0,x)^2}, \quad x>\varpi_1, \tag{6.11}$$

where ϖ_1 is such that $y(\varpi_1)=-\beta/2$. It will turn out that, in the notation of assumption (H1), $\varpi_1=x_1$ is the point where $H(x)$ has a discontinuity. By Eq. (6.11), ϖ_1 is determined uniquely by the equation

$$\frac{\beta}{2H_+}=\int_{\varpi_1}^0 \frac{dx}{f_l(0,x)^2}. \tag{6.12}$$

It remains to solve Eq. (5.24) when $y<-\beta/2$, i.e., $x<\varpi_1$. From Eqs. (6.3), (6.4), and (6.6) we have

$$X(0,x,y)=-i\,\frac{\rho_0}{1-\rho_0}\,\frac{f_l(0,x)}{\sqrt{H_+}}\,(2y+\beta).$$

Solving Eq. (5.24) we get

$$y(x)=-\frac{\beta\rho_0}{1+\rho_0}-\frac{1-\rho_0}{1+\rho_0}\left(A_++i\,\frac{\dot{f}_l(0,x)}{f_l(0,x)}\right), \tag{6.13}$$

and after using Eq. (2.27)

$$H(x)=\frac{1-\rho_0}{1+\rho_0}\,\frac{H_+}{f_l(0,x)^2}, \quad x<\varpi_1. \tag{6.14}$$

Case (b): $\beta<0$: The analysis in this case is similar to that in (a), except that now the constant A_+ has to be evaluated by using the solution for $y<-\beta/2$, i.e., Eq. (6.13). The value ϖ_1, which is now positive, is determined uniquely by

$$\frac{|\beta|}{2H_+}\,\frac{1+\rho_0}{1-\rho_0}=\int_0^{\varpi_1}\frac{dx}{f_l(0,x)^2}, \tag{6.15}$$

and A_+ is given by

$$A_+=-H_+\int_0^\infty dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}-\beta\,\frac{\rho_0}{1-\rho_0}. \tag{6.16}$$

The result for $H(x)$ is the same as in part (a), Eq. (6.11), and Eq. (6.14).

We add a few more details concerning the function $H(x)$ constructed above. First we observe that $H(\varpi_1+0)=H(\varpi_1-0)$ if and only if $\rho_0=0$, i.e., if and only if the potential is reflectionless. Then $H(x)=H_+/f_l(0,x)^2$ for $x\in\mathbf{R}$.

Since we are dealing with the exceptional case, $f_l(0,x)=\gamma f_r(0,x)$ by Eq. (4.5) and $f_r(0,x)\to 1$ as $x\to-\infty$. Evaluating H_- by using Eq. (6.14) we obtain the expression already found in Eq. (5.29). Furthermore, since $f_l(0,x)$ has no zeros and $f_l(0,x)\to 1$ as $x\to+\infty$, the following estimate is valid:

$$|H(x)-H_+|=H_+\left|\frac{1-f_l(0,x)^2}{f_l(0,x)^2}\right|\leqslant C|1-f_l(0,x)|,\quad x>\varpi_1. \tag{6.17}$$

Now, from Eqs. (2.15) and (2.19) it follows that

$$|1-f_l(0,x)|\leqslant e^{E_+(0,x)}-1\leqslant C\int_x^\infty dz(1+|z|)|Q(z)|,\quad x>0. \tag{6.18}$$

Since $Q\in L_2^1(\mathbf{R})$, it follows by using an integration by parts that the right-hand side of Eq. (6.18) is in $L^1(\mathbf{R}^+)$. Together with Eq. (6.17) this implies $H-H_+\in L^1(\mathbf{R}^+)$, which is consistent with our assumption (H3). Similarly, one argues that $H-H_-\in L^1(\mathbf{R}^-)$.

The Jost solutions from the left associated with $H(x)$ can be obtained from Eqs. (5.1), (5.16), (6.4), and (6.6). We have

$$f_l(k,x)=\begin{cases} e^{ik(y+A_+)}f_l(0,x), & x>\varpi_1\\ \dfrac{f_l(0,x)}{1-\rho_0}\,e^{ik(y+A_+)}-\dfrac{\rho_0}{1-\rho_0}\,f_l(0,x)e^{-ik(y-A_++\beta)}, & x<\varpi_1. \end{cases} \tag{6.19}$$

It follows from Eqs. (1.4), (3.20), (4.1), (5.29), and (6.19) that $\tau(k)=\sqrt{1-\rho_0^2}$ and $\ell(k)=-\overline{\rho(k)}$, and one verifies that $\tau(k)=1/a(k)=1/\alpha_1$, where $a(k)$ and α_1 were defined in Eqs. (4.19) and (4.13), respectively.

We have so far discussed the recovery of $H(x)$ in terms of $\rho(k)$, $Q(x)$, and H_+ under the assumption that there are no bound states. From the viewpoint of physical applications it seems more appropriate to replace $\rho(k)$ by $R(k)$, in analogy to the standard case where $H(x)=1$. As an example, let us assume that $R(k)=R_0$ is constant with $-1<R_0<1$. From Eq. (4.1) we obtain $\rho(k)=R_0e^{2ikA_+}$, where for the moment A_+ is considered to be a parameter. So $\rho(k)$ is of the form (6.3) with $\beta=2A_+$ and $\rho_0=R_0$. Let

$$d=-\int_0^\infty dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}. \tag{6.20}$$

From Eqs. (6.10) and (6.16) we have $A_+=dH_+$ when $A_+\geqslant 0$, and $A_+=dH_+[(1-R_0)/(1+R_0)]$ when $A_+<0$. Hence $A_+\geqslant 0$ if and only if $d\geqslant 0$. If $d\geqslant 0$, the function $H(x)$ is given by Eqs. (6.11) and (6.14), where ϖ_1 is given by Eq. (6.12) which can be written as

$$-\int_0^\infty dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}=\int_{\varpi_1}^0\frac{dx}{f_l(0,x)^2}. \tag{6.21}$$

If $d<0$, the function $H(x)$ is given by Eqs. (6.11) and (6.14) with ϖ_1 determined by Eq. (6.15), i.e.,

$$\int_0^\infty dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}=\int_0^{\varpi_1}\frac{dx}{f_l(0,x)^2}. \tag{6.22}$$

Therefore, we see that corresponding to the reflection coefficient $R(k)=R_0$ there exists a one-parameter family of functions $H(x)$ with H_+ as the parameter. The point ϖ_1, where $H(x)$ may be discontinuous, is determined by Eqs. (6.21) or (6.22) depending on whether $d\geqslant 0$ or $d<0$. The sign of d is determined by $Q(x)$ alone.

Example 6.2: Suppose there are no bound states and

$$\rho(k)=\frac{\mu+i\xi k}{-\mu+ik}\,e^{ik\beta},\quad \mu>0,\quad -1<\xi<1,\quad \beta\in\mathbf{R}. \tag{6.23}$$

Note that $|\rho(k)|<1$ for $k\in\mathbf{R}\backslash\{0\}$ and $\rho(0)=-1$; thus this is the generic case. From Eq. (6.2) we obtain

$$X_0(k,x,y)=\begin{cases} 0, & 2y+\beta\geqslant 0\\ -\dfrac{f_l(0,x)}{\sqrt{H_+}}\left[\dfrac{1-e^{ik(2y+\beta)}}{k}+\dfrac{\xi+1}{k+i\mu}\,e^{ik(2y+\beta)}-\dfrac{\xi+1}{k+i\mu}\,e^{\mu(2y+\beta)}\right], & 2y+\beta<0.\end{cases}$$

Using the Fourier transform defined in Eq. (6.5), we have

$$(\mathscr{F}X_0(\cdot,x,y))(t,x,y)=\begin{cases} -2\pi i\,\dfrac{f_l(0,x)}{\sqrt{H_+}}\,[1-(\xi+1)e^{\mu(2y+\beta+t)}], & 0<t<-(2y+\beta)\\ 0, & t>-(2y+\beta).\end{cases} \tag{6.24}$$

Moreover, similarly as in Example 6.1, we have

$$\mathscr{O}_y=0,\quad 2y+\beta\geqslant 0, \tag{6.25}$$

and when $2y+\beta<0$ we have

$$(\mathscr{F}O_y\mathscr{F}^{-1}h)(t)=\begin{cases} \xi h(-t-2y-\beta)-\mu(\xi+1)e^{\mu(t+2y+\beta)}\displaystyle\int_0^{-2y-\beta-t}dx\;e^{\mu x}h(x), & 0<t<-(2y+\beta)\\ 0, & t>-(2y+\beta).\end{cases} \tag{6.26}$$

We first determine the spectrum of $\mathscr{O}_y$. From Example 6.1 we know that the spectrum of the operator $h\rightarrow\xi h(-t-2y-\beta)$ restricted to $L^2(0,-2y-\beta)$ with $2y+\beta<0$ consists of the two eigenvalues $\pm\xi$, each having infinite multiplicity. According to Eq. (6.26) this noncompact operator is perturbed by a compact integral operator. Hence, on $L^2(0,-2y-\beta)$, the spectrum of $\mathscr{O}_y$ consists of the two points $\pm\xi$ and possibly a countable number of eigenvalues λ that can accumulate only at $\pm\xi$. We also know that $|\lambda|\leqslant 1$ because $|\rho(k)|\leqslant 1$, and hence $\|\mathscr{O}_y\|\leqslant 1$ by Eq. (6.1). The following analysis confirms this expectation. For simplicity we set $\beta=0$ in the following. The case $\beta\neq 0$ can be handled by the substitution $y\rightarrow y+\beta/2$. It also suffices to consider $\mathscr{O}_y$ on the invariant subspace $L^2(0,-2y)$ with $y<0$ since on its orthogonal complement $\mathscr{O}_y$ is the zero operator. On $L^2(0,-2y)$ the eigenvalue problem for $\mathscr{F}\mathscr{O}_y\mathscr{F}^{-1}$ reads

$$\xi h(-t-2y)-\mu(\xi+1)e^{\mu(2y+t)}\int_0^{-2y-t}dx\;e^{\mu x}h(x)=\lambda h(t), \tag{6.27}$$

or equivalently

$$\xi h(t)-\mu(\xi+1)e^{-\mu t}\int_0^t dx\; e^{\mu x}h(x)=\lambda h(-t-2y). \tag{6.28}$$

Now let

$$U(t)=\int_0^t dx\; e^{\mu x}h(x). \tag{6.29}$$

Our goal is to convert the eigenvalue problem (6.27) to a second-order differential equation for $U(t)$. In order to justify the following steps we must first show that $U'(t)$ is absolutely continuous. Writing Eqs. (6.27) and (6.28) in terms of $U(t)$ we obtain

$$\xi U'(-t-2y)-\mu(\xi+1)U(-t-2y)=\lambda e^{-2\mu(t+y)}U'(t), \tag{6.30}$$

$$\xi U'(t)-\mu(\xi+1)U(t)=\lambda e^{2\mu(t+y)}U'(-t-2y). \tag{6.31}$$

Every solution of Eq. (6.30) [resp. (6.31)] that also satisfies $U(0)=0$ gives rise to a solution of Eq. (6.27) [resp. (6.28)] and vice versa. Now Eqs. (6.30) and (6.31) can be solved for $U'(t)$ and $U'(-t-2y)$ in terms of $U(t)$ and $U(-t-2y)$, provided $\lambda\neq\pm\xi$. So let us assume $\lambda\neq\pm\xi$ for the moment. As a result, $U'(t)$ can be expressed in terms of $U(t)$ and $U(-t-2y)$, both of which are absolutely continuous by Eq. (6.29). Hence $U'(t)$ is absolutely continuous. Multiplying Eq. (6.31) by ξ and using Eq. (6.30) to replace $\xi U'(-t-2y)$ in Eq. (6.31) we obtain

$$\xi^2U'(t)-\mu\xi(\xi+1)U(t)=\lambda\mu(\xi+1)e^{2\mu(t+y)}U(-t-2y)+\lambda^2U'(t). \tag{6.32}$$

Differentiating Eq. (6.32) we obtain

$$\begin{aligned}\xi^2U''(t)-\mu\xi(\xi+1)U'(t)=2\lambda\mu^2(\xi+1)e^{2\mu(t+y)}U(-t-2y)\\ -\lambda\mu(\xi+1)e^{2\mu(t+y)}U'(-t-2y)+\lambda^2U''(t).\end{aligned}$$

Now in the above equation, we replace $U(-t-2y)$ and $U'(-t-2y)$ by their equivalents in terms of $U(t)$ and $U'(t)$ by using Eqs. (6.32) and (6.31), respectively. After some simplifications we arrive at the differential equation

$$(\xi^2-\lambda^2)U''(t)+2\mu(\lambda^2-\xi^2)U'(t)+\mu^2(\xi^2-1)U(t)=0. \tag{6.33}$$

When $\lambda=\pm\xi$, we can no longer conclude that $U'(t)$ is absolutely continuous. However, Eq. (6.32) is still valid except that the terms involving $U'(t)$ cancel out, i.e., we have $U(t)=\mp e^{2\mu(t+y)}U(-t-2y)$ when $\lambda=\pm\xi$. Proceeding with this equation as with Eq. (6.32), we arrive at Eq. (6.33), but without the terms involving $U'(t)$ and $U''(t)$. Hence $U(t)=0$ and so $\pm\xi$ cannot be eigenvalues of $\mathcal{O}_y$. The roots of the characteristic equation of (6.33), i.e., the roots of

$$(\xi^2-\lambda^2)r^2+2\mu(\lambda^2-\xi^2)r+\mu^2(\xi^2-1)=0,$$

are given by $r=\mu\pm(\mu\sqrt{1-\lambda^2}/\sqrt{\xi^2-\lambda^2})$. It can be checked that when the roots are real, the resulting solution for $U(t)$ does not satisfy (6.30). Therefore, the roots must be nonreal, which means $|\xi|<|\lambda|<1$. The function $U(t)$ is then of the form $U(t)=e^{\mu t}\sin\delta t$, where $\delta=\mu\sqrt{1-\lambda^2}/\sqrt{\lambda^2-\xi^2}$. Thus

$$h(t)=\mu\sin\delta t+\delta\cos\delta t. \tag{6.34}$$

Substituting Eq. (6.34) in Eq. (6.28) we see that the following two equations must be satisfied:

$$\mu=\lambda(\mu \cos 2\delta y+\delta \sin 2\delta y), \tag{6.35}$$

$$\xi\delta=\lambda(\delta \cos 2\delta y-\mu \sin 2\delta y). \tag{6.36}$$

Solving Eq. (6.35) for λ and substituting the result in Eq. (6.36) we obtain

$$\tan 2\delta|y|=-\frac{\mu\delta(1-\xi)}{\xi\delta^2+\mu^2}. \tag{6.37}$$

This equation determines the values of δ that are allowed. Assume for the moment that $\xi>0$. From a graphical solution of Eq. (6.37) it follows that there are infinitely many solutions and that each solution δ is associated with a certain sign of $\cos 2\delta y$. We therefore distinguish the two increasing sequences δ_n^+ and δ_n^- of solutions of Eq. (6.37) such that $\cos 2\delta_n^+ y>0$ and $\cos 2\delta_n^- y<0$, respectively. From Eq. (6.35) we then see that there are two infinite sequences of eigenvalues given by

$$\lambda_n^{\pm}=\pm\frac{\sqrt{\mu^2+(\delta_n^{\pm})^2\xi^2}}{\sqrt{\mu^2+(\delta_n^{\pm})^2}}. \tag{6.38}$$

It follows that $\lambda_n^{\pm} \to \pm\xi$ as $n \to \infty$; in particular, when $\xi=0$ the eigenvalues accumulate at zero. This is in agreement with the fact that when $\xi=0$ the operator $\mathscr{O}_y$ is compact. Equations (6.37) and (6.38) remain valid when $\xi<0$. However, a special circumstance arises when $\xi\delta^2+\mu^2=0$. This occurs when $2\delta|y|=(\pi/2)+n\pi$ for $n=0,1,2\dots$, i.e., if $2\mu|y|/\sqrt{|\xi|}=(\pi/2)+n\pi$. In this case, as can be seen from Eqs. (6.35)–(6.37), $\lambda=+\sqrt{|\xi|}$ is an eigenvalue when n is even and $\lambda=-\sqrt{|\xi|}$ is an eigenvalue when n is odd.

We now turn to the solution of Eq. (5.21). Since $X_0(k,x,y)=0$ when $2y+\beta\geqslant 0$, it follows from Eqs. (6.24), (6.25), and (5.21) that $X(k,x,y)=0$ when $2y+\beta\geqslant 0$. Hence we can assume $2y+\beta<0$. From Eqs. (6.24) and (6.25) we see that $X(k,x,y)=0$ when $t>-(2y+\beta)$. For $0<t<-(2y+\beta)$, Eq. (5.21) can be written in the form

$$h(t)-\xi h(-t-2y-\beta)+\mu(\xi+1)e^{\mu(t+2y+\beta)}\int_0^{-2y-\beta-t} dx\, e^{\mu x}h(x)=g(t), \tag{6.39}$$

where $h(t)=(\mathscr{F}X(\cdot,x,y))(t)$ and

$$g(t)=-2\pi i\,\frac{f_l(0,x)}{\sqrt{H_+}}\,[1-(\xi+1)e^{\mu(2y+\beta+t)}]. \tag{6.40}$$

Proceeding with Eq. (6.39) similarly as with Eq. (6.27), using $U(t)$ from Eq. (6.29), we derive the differential equation

$$\begin{aligned}(1-\xi^2)U''(t)&+2\mu(\xi^2-1)U'(t)+\mu^2(1-\xi^2)U(t)\\ &=[\mu g(-t-2y-\beta)-\xi g'(-t-2y-\beta)]e^{\mu t}-[\mu g(t)-g'(t)]e^{\mu t}.\end{aligned} \tag{6.41}$$

Using Eq. (6.40) the right-hand side of Eq. (6.41) simplifies and we get

$$(1-\xi^2)U''(t)+2\mu(\xi^2-1)U'(t)+\mu^2(1-\xi^2)U(t)=-2\pi i\,\frac{f_l(0,x)}{\sqrt{H_+}}\,\mu(\xi^2-1). \tag{6.42}$$

The solution of the homogeneous equation corresponding to Eq. (6.42) is of the form $U(t)=c_1e^{\mu t}+c_2te^{\mu t}$. A particular solution of Eq. (6.42) is given by $U(t)=2\pi i[f_l(0,x)/(\sqrt{H_+}\mu)]$. Taking into account the condition $U(0)=0$, we get $c_1=-2\pi i[f_l(0,x)/(\sqrt{H_+}\mu)]$, and hence

$$U(t)=2\pi i\,\frac{f_l(0,x)}{\sqrt{H_+}\mu}\,(1-e^{\mu t})+c_2te^{\mu t},$$

which gives

$$h(t)=-2\pi i\,\frac{f_l(0,x)}{\sqrt{H_+}}+c_2+c_2\mu t. \tag{6.43}$$

In order to determine c_2 we substitute Eq. (6.43) in Eq. (6.39) and use Eq. (6.40). This yields

$$c_2=2\pi i\,\frac{f_l(0,x)}{\sqrt{H_+}}\,\frac{1}{1-\xi-2\mu y-\mu\beta},$$

$$h(t)=2\pi i\,\frac{f_l(0,x)}{\sqrt{H_+}}\,\frac{\xi+2\mu y+\mu\beta+\mu t}{1-\xi-2\mu y-\mu\beta}.$$

Using the inverse Fourier transform with the convention (6.5), we get

$$\begin{aligned}X(k,x,y)&=(\mathscr{F}^{-1}h)(k,x,y)\\&=\frac{1}{2\pi}\int_0^{-2y-\beta}dt\;e^{-itk}h(t)\\&=i\,\frac{f_l(0,x)}{\sqrt{H_+}}\,\frac{1}{1-\xi-2\mu y-\mu\beta}\left[i\,\frac{\xi(e^{ik(2y+\beta)}-1)}{k}+\frac{\mu(e^{ik(2y+\beta)}-1-2iyk-i\beta k)}{k^2}\right],\end{aligned} \tag{6.44}$$

and hence

$$X(0,x,y)=-i\,\frac{f_l(0,x)}{2\sqrt{H_+}}\,\frac{(2y+\beta)(2\xi+2\mu y+\mu\beta)}{1-\xi-2\mu y-\mu\beta},\quad 2y+\beta<0.$$

As in Example 6.1 we distinguish the two cases:

Case (a) $\beta\geqslant 0$: The solution of Eq. (5.24) when $y>-\beta/2$ is the same as in Example 6.1, i.e., given by Eqs. (6.11) and (6.12). When $y<-\beta/2$ the solution of Eq. (5.24) is [cf. Eqs. (2.27) and (6.10)]

$$y(x)=\frac{2H_+(\xi-1+\mu\beta)\varphi(x)-2\beta\xi-\mu\beta^2}{2[1+\xi+\mu\beta-2\mu H_+\varphi(x)]}, \tag{6.45}$$

where $\varphi(x)=\int_x^0[dz/f_l(0,z)^2]$. The denominator in Eq. (6.45) must be nonzero and $y(x)$ must grow linearly as $x\to-\infty$. Since $f_l(0,x)=-c_lx+o(x)$ as $x\to-\infty$, where $c_l=[f_l(0,x);f_r(0,x)]>0$ [cf. Eq. (2.15) in Ref. 24], $\varphi(-\infty)=\lim_{x\to-\infty}\varphi(x)$ is finite. Hence, in order for $y(x)$ to be unbounded as $x\to-\infty$, it is necessary and sufficient that

$$1+\xi+\mu\beta-2\mu H_{+}\varphi(-\infty)=0. \tag{6.46}$$

This says that H_+ is determined by $\rho(k)$ and $Q(x)$, namely,

$$H_{+}=\frac{1+\xi+\mu\beta}{2\mu\varphi(-\infty)}. \tag{6.47}$$

With the help of Eq. (6.46) we can write Eq. (6.45) as

$$y(x)=\frac{2H_{+}(\xi-1+\mu\beta)\varphi(x)-2\beta\xi-\mu\beta^2}{4\mu H_{+}[\varphi(-\infty)-\varphi(x)]}. \tag{6.48}$$

Since

$$\frac{c_l}{f_l(0,x)^2}=\left(\frac{f_r(0,x)}{f_l(0,x)}\right)', \quad \frac{c_l}{f_r(0,x)^2}=-\left(\frac{f_l(0,x)}{f_r(0,x)}\right)', \tag{6.49}$$

we have

$$c_l\varphi(x)=\frac{f_r(0,0)}{f_l(0,0)}-\frac{f_r(0,x)}{f_l(0,x)}, \quad c_l\varphi(-\infty)=\frac{f_r(0,0)}{f_l(0,0)}. \tag{6.50}$$

Hence

$$y(x)=\frac{\xi-1+\mu\beta}{2\mu}\left[\frac{f_l(0,x)}{f_r(0,x)}\frac{f_r(0,0)}{f_l(0,0)}-1\right]-\frac{\beta c_l(2\xi+\mu\beta)}{4\mu H_{+}}\left(\frac{f_l(0,x)}{f_r(0,x)}\right).$$

Differentiating this expression, using Eqs. (6.46), (6.49), and (6.50), we obtain

$$H(x)=y'(x)=\frac{1-\xi^2}{4\mu^2H_{+}}\frac{c_l^2}{f_r(0,x)^2}, \quad x<\varpi_1, \tag{6.51}$$

with ϖ_1 given in Eq. (6.12).

Case (b) $\beta<0$: One proceeds as in case (a) with similar modifications as in Example 6.1. The constant A_+ is now given by

$$A_{+}=-\frac{\beta(2\xi+\mu\beta)}{2(1-\xi-\mu\beta)}-H_{+}\int_0^{\infty}dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}. \tag{6.52}$$

When $y>-\beta/2$, $y(x)$ is obtained by solving Eq. (6.7), that is $y(x)=-i[\dot{f}_l(0,x)/f_l(0,x)]-A_+$; when $y<-\beta/2$ we have

$$y(x)=\frac{H_{+}(\xi-1+\mu\beta)^2\varphi(x)}{\xi^2-1-2\mu H_{+}(\xi-1+\mu\beta)\varphi(x)}. \tag{6.53}$$

The point ϖ_1 where $y(\varpi_1)=-\beta/2$ is determined by

$$\int_0^{\varpi_1}\frac{dx}{f_l(0,x)^2}=-\frac{\beta}{2H_{+}}\frac{1+\xi}{1-\xi-\mu\beta}. \tag{6.54}$$

In place of Eq. (6.46) we have from Eq. (6.53) the condition

$$\xi^2-1-2\mu H_{+}(\xi-1+\mu\beta)\varphi(-\infty)=0, \tag{6.55}$$

and so

$$H_+=\frac{\xi^2-1}{2\mu(\xi-1+\mu\beta)\varphi(-\infty)}. \tag{6.56}$$

Using Eq. (6.55) we can write Eq. (6.53) as

$$y(x)=\frac{(\xi-1+\mu\beta)\varphi(x)}{2\mu[\varphi(-\infty)-\varphi(x)]}. \tag{6.57}$$

Proceeding with Eq. (6.57) as with Eq. (6.48), using Eqs. (6.49) and (6.50), we see that $H(x)$ is given by Eq. (6.11) when $x>\varpi_1$ and by Eq. (6.51) when $x<\varpi_1$. Furthermore, from Eq. (6.51) we see that

$$H_-=\frac{(1-\xi^2)c_l^2}{4\mu^2H_+}. \tag{6.58}$$

This agrees with Eq. (5.30) if we observe that by Eq. (6.23) we have

$$|c|^2=\lim_{k\to 0}\frac{1-|\rho(k)|^2}{k^2}=\frac{1-\xi^2}{\mu^2}.$$

Moreover, we can evaluate the difference $H(\varpi_1+0)-H(\varpi_1-0)$ as follows. Write Eq. (5.24) as

$$i\frac{\dot f_l(0,x)}{f_l(0,x)}+y+A_+=-\frac{(2y+\beta)(2\xi+2\mu y+\mu\beta)}{2(1-\xi-2\mu y-\mu\beta)}.$$

Differentiating both sides of the above equation, using Eq. (2.27) and setting $y=-\beta/2$ (i.e., $x=\varpi_1$), we obtain

$$H(\varpi_1-0)=y'(\varpi_1-0)=\frac{1-\xi}{1+\xi}\frac{H_+}{f_l(0,\varpi_1)^2}.$$

Comparing this with Eq. (6.11) we obtain

$$H(\varpi_1+0)-H(\varpi_1-0)=\frac{2\xi}{1+\xi}\frac{H_+}{f_l(0,\varpi_1)^2}.$$

Hence $H(\varpi_1-0)=H(\varpi_1+0)$ if and only if $\xi=0$, i.e., if and only if $\mathscr{O}_y$ is compact. As in Example 6.1, we also see that $H-H_\pm\in L^1(\mathbf{R}^\pm)$.

For $x<\varpi_1$ the Jost solution $f_l(k,x)$ can be evaluated by using Eqs. (5.1), (5.16), and (6.44) [recall that $F_-(k,x,y)=X(k,x,y)$]. One finds

$$f_l(k,x)=\frac{if_l(0,x)}{1-\xi-2\mu y-\mu\beta}\left[\frac{ik\xi-\mu}{k}e^{-ik(y-A_++\beta)}+\frac{\mu-ik}{k}e^{ik(y+A_+)}\right],\quad x<\varpi_1. \tag{6.59}$$

Letting $x\to-\infty$ in Eq. (6.59), using $f_l(0,x)=-c_lx+o(x)$, Eqs. (1.4), (3.20), and (4.1), we obtain

$$\tau(k)=\frac{2\mu\sqrt{H_-H_+}}{c_l}\frac{ik}{ik-\mu}. \tag{6.60}$$

With the help of Eq. (6.58) we can write Eq. (6.60) as $\tau(k)=ik\sqrt{1-\xi^2}/(-\mu+ik)$. Furthermore, from Eqs. (1.4), (3.20), (4.1), and (6.59) we obtain $\ell(k)=[(\mu-ik\xi)/(-\mu+ik)]e^{-ik\beta}$.

As in Example 6.1 we discuss in a particular case the possibility of recovering $H(x)$ from $R(k)$ instead of $\rho(k)$. In contrast to Example 6.1, we will see that it is not always possible to find an $H(x)$ for a given reflection coefficient $R(k)$ and an arbitrary $Q(x)$. Assume $R(k)$ is of the form (6.23) with $\beta=0$, i.e., $R(k)=(\mu+i\xi k)/(-\mu+ik)$. In view of Eq. (4.1) we put $\rho(k)=R(k)e^{2ikA_+}$, where A_+ is considered to be a parameter. Hence, we are trying to find a function $H(x)$ whose corresponding reduced reflection coefficient is of the form (6.23) with $\beta=2A_+$. First assume that $\beta\geqslant 0$. Then $2dH_+=\beta$ by Eq. (6.10), where d is the constant defined in Eq. (6.20), and hence $d\geqslant 0$. Combining Eqs. (6.10) and (6.47) we see that $\beta=d(1+\xi+\mu\beta)/[\mu\varphi(-\infty)]$, and hence

$$\beta=\frac{(1+\xi)d}{\mu[\varphi(-\infty)-d]}. \tag{6.61}$$

Since $\beta\geqslant 0$ and $d\geqslant 0$, Eq. (6.61) implies that $d<\varphi(-\infty)$, i.e.,

$$-\int_0^\infty dx\,\frac{1-f_l(0,x)^2}{f_l(0,x)^2}<\int_{-\infty}^0\frac{dx}{f_l(0,x)^2}. \tag{6.62}$$

In view of Eq. (6.21) the inequality in Eq. (6.62) is also necessary and sufficient for ϖ_1 to exist. If Eq. (6.62) is not satisfied, there is no function $H(x)$ corresponding to the given $R(k)$ with $A_+\geqslant 0$. However, there may still be a function $H(x)$ with $A_+<0$. If Eq. (6.62) is satisfied, then a function $H(x)$ exists and it is given by Eqs. (6.11) and (6.51). By using Eqs. (6.21) and (6.61) we can write Eq. (6.47) as

$$H_+=\frac{1+\xi}{2\mu\int_{-\infty}^{\varpi_1}dx\,f_l(0,x)^{-2}}. \tag{6.63}$$

We now look for possible $H(x)$ corresponding to our reflection coefficient when $A_+<0$, i.e., $\beta<0$. From Eqs. (6.52) and (6.56) we obtain

$$\beta=\frac{(1-\xi)d}{\mu\varphi(-\infty)}. \tag{6.64}$$

Since $\beta<0$, we have $d<0$. Therefore, if $d\geqslant 0$ and Eq. (6.62) does not hold, a function $H(x)$ with $A_+<0$ does not exist either, and hence no function $H(x)$ exists at all. If $d<0$, using Eq. (6.64) in Eq. (6.54) we obtain $\int_0^{\varpi_1}[dx/f_l(0,x)^2]=-d$, and this equation always has a solution. Thus we see that when $d<0$ there always exists a function $H(x)$. The expression (6.56) for H_+ can be rewritten by using Eq. (6.64). The result is again Eq. (6.63), where now $\varpi_1>0$. We will demonstrate by a simple example that Eq. (6.62) can indeed be violated. Let $Q(x)=c\,\delta(x-1)$, where c is a positive parameter and $\delta(x)$ is the Dirac delta function. The fact that $Q\notin L_1^1(\mathbf{R})$ is not important, since we could replace the delta function by a step function concentrated near $x=1$. Then $f_l(0,x)=1$ for $x>1$ and $f_l(0,x)=1-c(x-1)$ for $x<1$. An easy computation shows that the left-hand side of (6.62) is equal to $c/(c+1)$ and the right-hand side is equal to $1/c(c+1)$. So Eq. (6.62) does not hold when $c\geqslant 1$.

VII. THE SINGULAR INTEGRAL EQUATION

In this section we study the singular integral equation (5.21). In Eq. (5.21) x and y appear only as parameters, and hence we can suppress these parameters in $X(k,x,y)$ and $X_0(k,x,y)$. Thus, we will write $X(k)=X(k,x,y)$ and $X_0(k)=X_0(k,x,y)$ and analyze

$$X(k)=(\mathscr{O}_y X)(k)=X_0(k), \quad k\in\mathbf{R}, \tag{7.1}$$

where

$$(\mathscr{O}_y X)(k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s+k-i0}\,\rho(-s)e^{-2isy}X(s), \quad k\in\mathbf{R}$$

such that $y\in\mathbf{R}$ is a parameter and $X_0(k)$ is a given function. Using the operators $\Pi_\pm$ in Eq. (5.18) and $\mathscr{T}$ in Eq. (6.1), we can write Eq. (7.1) as

$$X-\Pi_-[\rho(\cdot)e^{2i(\cdot)y}\mathscr{T}X]=X_0 \tag{7.2}$$

on a suitable Banach space of scalar functions on $\mathbf{R}$. Recall that $\Pi_\pm$ are bounded and complementary projections of $L^p(\mathbf{R})$ onto the Hardy spaces $\mathbf{H}^p_\pm(\mathbf{R})$ for $1<p<\infty$. A function $f\in\mathbf{H}^p_\pm(\mathbf{R})$ can be identified with its nontangential limits which exist and agree with f at almost every point of $\mathbf{R}$ (Ref. 30, Chap. 8). For $p=2$ the projections $\Pi_\pm$ are orthogonal. By $\mathbf{H}^\infty_\pm(\mathbf{R})$ we denote the Banach algebra of bounded analytic functions on $\mathbf{C}^\pm$, identified with their almost everywhere existing nontangential limits as $k\rightarrow\mathbf{R}$, which makes it into a closed subalgebra of $L^\infty(\mathbf{R})$.

From Eq. (7.2) it is clear that $\mathscr{O}_y$ is a bounded operator on $\mathbf{H}^p_-(\mathbf{R})$ for $1<p<\infty$. Using Eqs. (4.6) and (4.9) we can easily prove that $X_0(k,x,y)$ defined by Eq. (5.22) belongs to $\mathbf{H}^p_-(\mathbf{R})$, provided $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in(0,1]$ satisfying $\alpha>1-(1/p)$. Indeed, this is immediate from Eq. (6.2) and the identity

$$\frac{\rho(k)e^{2iky}-\rho(0)}{k}=\begin{cases} o(|k|^{\alpha-1}), & k\rightarrow 0\\ O(1/k), & k\rightarrow\pm\infty.\end{cases}$$

We will establish the unique solvability of Eq. (7.1) in $\mathbf{H}^p_-(\mathbf{R})$ for $1<p<\infty$. The proof will consist of several parts. First we prove the unique solvability of Eq. (7.1) in $\mathbf{H}^2_-(\mathbf{R})$ by a contraction argument. Then we derive a two-vector Riemann–Hilbert problem satisfied by any solution of Eq. (7.1) in $\mathbf{H}^p_-(\mathbf{R})$. The accompanying Riemann–Hilbert problem, where only the asymptotic part $-b(k)/a(k)$ of $\rho(k)$ has been retained, will be shown to be uniquely solvable by factorization of an almost periodic 2×2 matrix function. As a result, Eq. (7.1) will be a Fredholm integral equation of index zero in $\mathbf{H}^p_-(\mathbf{R})$. A Fredholm argument then leads to the unique solvability of Eq. (7.1) in $\mathbf{H}^p_-(\mathbf{R})$ with $1<p<\infty$. Its solution $X(k)$ will then belong to $\mathbf{H}^p_-(\mathbf{R})$ for every finite p satisfying $\alpha>1-(1/p)$. Consequently, the solution $X(k,x,y)$ of Eq. (5.21) will belong to $\mathbf{H}^p_-(\mathbf{R})$ for every finite p satisfying $\alpha>1-(1/p)$ whenever $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha>0$. In particular, if $Q\in L^1_2(\mathbf{R})$, the solution of Eq. (5.21) belongs to $\mathbf{H}^p_-(\mathbf{R})$ for every $p\in(1,\infty)$.

Theorem 7.1: For $1<p<\infty$, Eq. (7.1) has a unique solution $X\in\mathbf{H}^p_-(\mathbf{R})$ for every $X_0\in\mathbf{H}^p_-(\mathbf{R})$. This solution is given by $X(k)=\Sigma_{n=0}^\infty[\mathscr{O}_y^n X_0](k)$, where the series converges absolutely in the norm of $\mathbf{H}^p_-(\mathbf{R})$.

Proof (for p=2): Here we will give the proof for only $p=2$; the proof for $p\neq 2$ will be given at the end of this section. Since $\Pi_-\vartheta\mathscr{T}f=0$ for every $f\in\mathbf{H}^2_-(\mathbf{R})$ if $\vartheta\in\mathbf{H}^\infty_+(\mathbf{R})$ (in particular, for constant ϑ), we may replace $\rho(k)e^{2iky}$ by $\rho(k)e^{2iky}+\epsilon$ for any constant $\epsilon=\epsilon(y)\geqslant 0$ without changing the operator $\mathscr{O}_y$ and hence Eq. (7.1). Using Theorem 4.5 we see that there exists $\epsilon=\epsilon(y)\geqslant 0$ such that

$$\|\rho(\cdot)e^{2i(\cdot)y}+\epsilon\|_\infty=\sup_{k\in\mathbf{R}}|\rho(k)e^{2iky}+\epsilon|<1.$$

In the exceptional case, where $\|\rho\|_\infty=\sup_{k\in\mathbf{R}}|\rho(k)|<1$, we may take $\epsilon=0$. Because Π_- and $\mathscr{T}$ are both operators of unit norm, we have in the norm of $\mathbf{H}^2_-(\mathbf{R})$

$$\|\mathcal{O}_y X\| \leqslant \|X\| \sup_{k\in\mathbf{R}} |\rho(k)e^{2iky}+\epsilon|$$

Consequently, Eq. (7.1) has a unique solution $X\in\mathbf{H}^2_-(\mathbf{R})$ for every $X_0\in\mathbf{H}^2_-(\mathbf{R})$, and this solution can be obtained by iterating Eq. (7.1). ■

The exact expression for the norm of $\mathcal{O}_y$ in $\mathbf{H}^2_-(\mathbf{R})$ follows from Nehari's theorem [Ref. 31, Theorem 1.3 and the discussion following Eq. (2.1); Ref. 32, Corollary 4.7], namely,

$$\|\mathcal{O}_y\| = \inf_{\vartheta\in\mathbf{H}^\infty_+(\mathbf{R})} \sup_{k\in\mathbf{R}} |\rho(k)e^{2iky}-\vartheta(k)|. \tag{7.3}$$

Since $\overline{\rho(-k)}=\rho(k)$ for $k\in\mathbf{R}$ implies that $\mathcal{O}_y$ is self-adjoint in $\mathbf{H}^2_-(\mathbf{R})$, Eq. (7.3) also yields the spectral radius of $\mathcal{O}_y$ in $\mathbf{H}^2_-(\mathbf{R})$. Note that the expression in Eq. (7.3) is nonincreasing in $y\in\mathbf{R}$.

In the exceptional case $\|\mathcal{O}_y\|\leqslant\|\rho\|_\infty<1$ and hence the spectrum of $\mathcal{O}_y$ is bounded away from 1, uniformly in y. Therefore, $\|X\|\leqslant(1-\|\rho\|_\infty)^{-1}\|X_0\|$ and so $X(k)$ is bounded in the L^2-norm uniformly in y. In the generic case, $X(k)$ will generally not be bounded in the L^2-norm as $y\to-\infty$. This can be seen from Example 6.2. Indeed, consider Eq. (6.44) with $\beta=0$ and hence $y<0$. Since

$$\int_{-\infty}^{\infty} dk \left|\frac{e^{2iky}-1-2iyk}{k^2}\right|^2 = |y|^3 \int_{-\infty}^{\infty} du \left|\frac{e^{2iu}-1-2iu}{u^2}\right|^2,$$

for each fixed x the L^2-norm of $X(\cdot,x,y)$ diverges like $|y|^{1/2}$ as $y\to-\infty$. This divergence is to be expected in view of the fact that as a consequence of Eqs. (6.37) and (6.38), the eigenvalues of $\mathcal{O}_y$ approach ±1 as $y\to-\infty$. Hence $\|(\mathbf{I}-\mathcal{O}_y)^{-1}\|$ is unbounded as $y\to-\infty$, where we use $\mathbf{I}$ to denote the identity operator.

Before studying Eq. (7.1) in $\mathbf{H}^p_-(\mathbf{R})$, we derive a Riemann–Hilbert problem satisfied by any solution of Eq. (7.1) in $\mathbf{H}^p_-(\mathbf{R})$, using a procedure essentially originating from Sec. 2 of Ref. 12. Writing $X_-=X$ and $X_+(k)=\Pi_+[\rho(\cdot)e^{2i(\cdot)y}\mathcal{J}X_-](k)$, we extend Eq. (7.2) to an equation in $L^p(\mathbf{R})$, namely,

$$X_+(k)+X_-(k)-\rho(k)e^{2iky}X_-(-k)=X_0(k), \quad k\in\mathbf{R}. \tag{7.4}$$

Changing the variable in Eq. (7.4) from k to $-k$ one gets

$$X_-(-k)+X_+(-k)-\rho(-k)e^{-2iky}X_-(k)=X_0(-k), \quad k\in\mathbf{R}. \tag{7.5}$$

From Eqs. (7.4) and (7.5) we obtain the vector Riemann–Hilbert problem

$$\begin{bmatrix} 1 & -\rho(k)e^{2iky} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_+(k) \\ X_-(-k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\rho(-k)e^{-2iky} & 1 \end{bmatrix}\begin{bmatrix} X_-(k) \\ X_+(-k) \end{bmatrix} = \begin{bmatrix} X_0(k) \\ X_0(-k) \end{bmatrix}, \quad k\in\mathbf{R},$$

which is equivalent to

$$\begin{bmatrix} X_+(k) \\ X_-(-k) \end{bmatrix} + \mathbf{G}(k)\begin{bmatrix} X_-(k) \\ X_+(-k) \end{bmatrix} = \mathbf{F}(k), \quad k\in\mathbf{R},$$

where

$$\mathbf{G}(k)=\begin{bmatrix} 1-\rho(k)\rho(-k) & \rho(k)e^{2iky} \\ -\rho(-k)e^{-2iky} & 1 \end{bmatrix}, \quad \mathbf{F}(k)=\begin{bmatrix} X_0(k)+\rho(k)e^{2iky}X_0(-k) \\ X_0(-k) \end{bmatrix}.$$

By the same token, replacing $\rho(k)$ by its asymptotic part $\rho_{as}(k)=-b(k)/a(k)$ given in Eq. (4.23), we prove that any solution $X\in\mathbf{H}_-^p(\mathbf{R})$ of the singular integral equation

$$X-\Pi_-[\rho_{as}(\cdot)e^{2i(\cdot)y}\mathscr{J}X]=X_0 \tag{7.6}$$

satisfies the Riemann–Hilbert problem

$$\begin{bmatrix} X_+(k) \\ X_-(-k)\end{bmatrix}+\mathbf{G}_{as}(k)\begin{bmatrix} X_-(k) \\ X_+(-k)\end{bmatrix}=\mathbf{F}_{as}(k), \quad k\in\mathbf{R},$$

where

$$\mathbf{G}_{as}(k)=\begin{bmatrix} 1-\rho_{as}(k)\rho_{as}(-k) & \rho_{as}(k)e^{2iky} \\ -\rho_{as}(-k)e^{-2iky} & 1\end{bmatrix}, \quad \mathbf{F}_{as}(k)=\begin{bmatrix} X_0(k)+\rho_{as}(k)e^{2iky}X_0(-k) \\ X_0(-k)\end{bmatrix}. \tag{7.7}$$

Let $\mathbf{H}_\pm^{(p,2)}(\mathbf{R})=\mathbf{H}_\pm^p(\mathbf{R})\oplus\mathbf{H}_\pm^p(\mathbf{R})$, $L^{(p,2)}(\mathbf{R})=L^p(\mathbf{R})\oplus L^p(\mathbf{R})$, and $\Pi_\pm^{(2)}=\Pi_\pm\oplus\Pi_\pm$. We have

Proposition 7.2: For $1<p<\infty$, the vector Riemann–Hilbert problem

$$Y_+(k)+\mathbf{G}_{as}(k)Y_-(k)=\mathbf{F}(k), \quad k\in\mathbf{R}, \tag{7.8}$$

has a unique solution $Y_\pm\in\mathbf{H}_\pm^{(p,2)}(\mathbf{R})$ for every $\mathbf{F}\in L^{(p,2)}(\mathbf{R})$.

Proof: From Eqs. (4.24), (7.7), and $\overline{\rho_{as}(-k)}=\rho_{as}(k)$ for $k\in\mathbf{R}$, we see that the diagonal entries of $\mathbf{G}_{as}(k)$ are strictly positive. Hence (Ref. 33, Lemma III 1.1), there exists a constant $\varepsilon>0$ so that $\sup_{k\in\mathbf{R}}\|\varepsilon\mathbf{G}_{as}(k)-\mathbf{I}\|_2<1$. Here $\|\cdot\|_2$ denotes the operator norm associated with the Euclidean vector norm on $\mathbf{C}^2$, and $\mathbf{I}$ is the unit matrix. Since $\rho_{as}(k)$ and hence all elements of $\mathbf{G}_{as}(k)$ belong to the algebra AP^W [cf. Theorem 4.5(i)], there exist, according to Theorem 1 of Ref. 6 (See Ref. 5, Theorem 1, for the same result with a different proof), matrix functions $\mathbf{G}_\pm(k)$ with invertible values for $k\in\mathbf{R}$ such that all elements of $\mathbf{G}_\pm(k)$ and $\mathbf{G}_\pm(k)^{-1}$ are almost periodic functions (in AP^W) that extend to bounded analytic functions in $\mathbf{C}^\pm$ and satisfy $\mathbf{G}_{as}(k)=\mathbf{G}_+(k)\mathbf{G}_-(k)$ for $k\in\mathbf{R}$. Premultiplying Eq. (7.8) by $\mathbf{G}_+(k)^{-1}$, we obtain uniquely

$$Y_+(k)=\mathbf{G}_+(k)[\Pi_+^{(2)}\mathbf{G}_+(\cdot)^{-1}\mathbf{F}](k), \quad Y_-(k)=\mathbf{G}_-(k)^{-1}[\Pi_-^{(2)}\mathbf{G}_+(\cdot)^{-1}\mathbf{F}](k),$$

which completes the proof. ■

Proposition 7.3: For $1<p<\infty$, Eq. (7.6) has a unique solution $X\in\mathbf{H}_-^p(\mathbf{R})$ for every $X_0\in\mathbf{H}_-^p(\mathbf{R})$. This solution is given by $X(k)=\Sigma_{n=0}^\infty[(\rho_{as}(\cdot)e^{2i(\cdot)y}\mathscr{J})^nX_0](k)$, where the series converges absolutely in the norm of $\mathbf{H}_-^p(\mathbf{R})$.

Proof: Let $\mathbf{F}_{as}(k)$ be defined by Eq. (7.7) where $X_0\in\mathbf{H}_-^p(\mathbf{R})$ is a given function. Then we can write Eq. (7.8) as

$$\begin{bmatrix} 1 & -\rho_{as}(k)e^{2iky} \\ 0 & 1\end{bmatrix}Y_+(k)+\begin{bmatrix} 1 & 0 \\ -\rho_{as}(-k)e^{-2iky} & 1\end{bmatrix}Y_-(k)=\begin{bmatrix} X_0(k) \\ X_0(-k)\end{bmatrix}, \quad k\in\mathbf{R},$$

which has a unique solution $Y_\pm\in\mathbf{H}_\pm^{(p,2)}(\mathbf{R})$. Changing k to $-k$ and switching the two components of the two-vectors we find for $k\in\mathbf{R}$

$$\begin{bmatrix} 1 & -\rho_{as}(k)e^{2iky} \\ 0 & 1\end{bmatrix}\begin{bmatrix} Y_{-,2}(-k) \\ Y_{-,1}(-k)\end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -\rho_{as}(-k)e^{-2iky} & 1\end{bmatrix}\begin{bmatrix} Y_{+,2}(-k) \\ Y_{+,1}(-k)\end{bmatrix}=\begin{bmatrix} X_0(k) \\ X_0(-k)\end{bmatrix},$$

where $Y_\pm(k)=(Y_{\pm,1}(k),Y_{\pm,2}(k))^T$ with the superscript T denoting the matrix transpose. Since the solution of Eq. (7.8) is unique, we must have $Y_{+,1}(k)=Y_{-,2}(-k)$ and $Y_{+,2}(k)=Y_{-,1}(-k)$, so that one has in fact found a solution of the Riemann–Hilbert problem for $k\in\mathbf{R}$

$$\begin{bmatrix} 1 & -\rho_{\mathrm{as}}(k)e^{2iky} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_+(k) \\ X_-(-k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\rho_{\mathrm{as}}(-k)e^{-2iky} & 1 \end{bmatrix} \begin{bmatrix} X_-(k) \\ X_+(-k) \end{bmatrix} = \begin{bmatrix} X_0(k) \\ X_0(-k) \end{bmatrix}. \tag{7.9}$$

The top equation of the pair (7.9) then implies Eq. (7.6) on applying Π_-; conversely, every solution of Eq. (7.6) leads to a solution of Eq. (7.9). Consequently, Eq. (7.6) has a unique solution $X \in \mathbf{H}^p_-(\mathbf{R})$ for every $X_0 \in \mathbf{H}^p_-(\mathbf{R})$.

To see that Eq. (7.6) can in fact be solved by iteration, we replace $\rho_{\mathrm{as}}(k)$ by $c\rho_{\mathrm{as}}(k)$ for some complex number c with $|c| \le 1$. Then $\mathbf{G}_{\mathrm{as}}(k)$ is replaced by $\mathbf{G}_{\mathrm{as},c}(k) = \frac{1}{2}[\mathbf{G}_{\mathrm{as}}(k) + \mathbf{G}_{\mathrm{as}}(k)^*]$, given by

$$\mathbf{G}_{\mathrm{as},c}(k) = \begin{bmatrix} 1 - \frac{1}{2}(c^2 + \bar{c}^2)|\rho_{\mathrm{as}}(k)|^2 & \frac{1}{2}(c - \bar{c})\rho_{\mathrm{as}}(k)e^{2iky} \\ \frac{1}{2}(\bar{c} - c)\overline{\rho_{\mathrm{as}}(k)}e^{-2iky} & 1 \end{bmatrix},$$

which has a strictly positive real part if $|\mathrm{Re}\, c| < (1/\|\rho_{\mathrm{as}}\|_\infty)$. Here the superscript $*$ denotes the adjoint. Since $\|\rho_{\mathrm{as}}\|_\infty < 1$, the real part of the matrix $\mathbf{G}_{\mathrm{as},c}(k)$ exceeds $\epsilon\mathbf{I}$ for some $\epsilon > 0$ independent of $k \in \mathbf{R}$. As a result, Eq. (7.6) with $\rho_{\mathrm{as}}(k)$ replaced by $c\rho_{\mathrm{as}}(k)$ has a unique solution $X \in \mathbf{H}^p_-(\mathbf{R})$ for every $X_0 \in \mathbf{H}^p_-(\mathbf{R})$. Hence the integral operator in Eq. (7.6) has spectral radius strictly less than 1 in $\mathbf{H}^p_-(\mathbf{R})$. ■

The following proposition follows directly from the Hartman–Wintner theorem [Ref. 31, Theorem 1.4 and the discussion following (2.1); Ref. 32, Corollary 4.7]. Here we present a direct proof.

Proposition 7.4: Let $1 < p < \infty$, and let $\vartheta(k)$ be continuous in $k \in \mathbf{R}$ and vanishing as $k \to \pm\infty$. Then the operator K on $\mathbf{H}^p_-(\mathbf{R})$ defined by $KX = \Pi_- \vartheta \mathscr{F} X$ is compact.

Proof: Since $\vartheta(k)$ can be approximated in the L^∞-norm by continuous functions of compact support, it suffices to consider $\vartheta(k)$ vanishing outside $(-\delta, \delta)$ for some $\delta > 0$. Then (Ref. 28, Theorem I 4.3) the operator $\mathscr{B}$ defined as $\mathscr{B} = \vartheta S - S\vartheta$, where

$$(Sf)(k) = \frac{1}{\pi i}\mathscr{P}\int_{-\delta}^{\delta} ds\, \frac{f(s)}{s-k}$$

is compact on $L^p(-\delta, \delta)$; here $\mathscr{P}$ represents the Cauchy principal value. If we extend $\mathscr{B}$ to $L^p(\mathbf{R})$ by defining $(\mathscr{B}f)(k) = 0$ for $k \notin (-\delta, \delta)$, then $\mathscr{B} = \vartheta S - S\vartheta$ is also compact on $L^p(\mathbf{R})$. According to Plemelj's formulas, we have $S = \Pi_+ - \Pi_-$. Hence

$$\mathscr{B} = \vartheta(1 - 2\Pi_-) - (1 - 2\Pi_-)\vartheta = 2[\Pi_- \vartheta - \vartheta \Pi_-].$$

Then for $X \in \mathbf{H}^p_-(\mathbf{R})$ we have

$$\tfrac{1}{2}\Pi_- \mathscr{B}\mathscr{F}X = \Pi_-(\Pi_- \vartheta - \vartheta\Pi_-)\mathscr{F}X = \Pi_- \vartheta \Pi_+ \mathscr{F}X = \Pi_- \vartheta \mathscr{F}X = KX$$

so that K is the restriction of the compact operator $\frac{1}{2}\Pi_- \mathscr{B}\mathscr{F}$ to $\mathbf{H}^p_-(\mathbf{R})$. Hence K is compact. ■

Next we give the proof of Theorem 7.1 for $p \neq 2$.

Proof (of Theorem 7.1 for $p \neq 2$): We have $\mathscr{O}_y X = \Pi_- \rho_{\mathrm{as}}(\cdot)e^{2i(\cdot)y}\mathscr{F}X + \Pi_- \vartheta \mathscr{F}X$ with $X \in \mathbf{H}^p_-(\mathbf{R})$ for some continuous $\vartheta(k)$ vanishing as $k \to \pm\infty$. Hence, by Propositions 7.3 and 7.4 the operator $\mathbf{I} - \mathscr{O}_y$ is Fredholm of index zero, and by Theorem 7.1 with $p = 2$ the operator $\mathbf{I} - \mathscr{O}_y$ is invertible on $\mathbf{H}^2_-(\mathbf{R})$. Making $\mathbf{H}^p_-(\mathbf{R}) \cap \mathbf{H}^2_-(\mathbf{R})$ into a Banach space by equipping it with the sum of the L^p- and the L^2-norm, we see that $\mathbf{I} - \mathscr{O}_y$ is the sum of an invertible and a compact operator and hence a Fredholm operator of index zero. Since it must be injective in view of the invertibility of $\mathbf{I} - \mathscr{O}_y$ on $\mathbf{H}^2_-(\mathbf{R})$, it is invertible on $\mathbf{H}^p_-(\mathbf{R}) \cap \mathbf{H}^2_-(\mathbf{R})$. As a result, it has a dense range on $\mathbf{H}^p_-(\mathbf{R})$. Being Fredholm of index zero, $\mathbf{I} - \mathscr{O}_y$ must be invertible on $\mathbf{H}^p_-(\mathbf{R})$.

To prove that Eq. (7.6) can be solved by iteration, we repeat the above Fredholm argument with $\rho_{\mathrm{as}}(k)$ replaced by $c\rho_{\mathrm{as}}(k)$ for some complex number c with $|c|\leq 1$. Then $\mathbf{I}-c\mathcal{O}_y$ is Fredholm of index zero on $\mathbf{H}^p_-(\mathbf{R})$ and is invertible on $\mathbf{H}^2_-(\mathbf{R})$. Thus $\mathbf{I}-c\mathcal{O}_y$ is invertible on $\mathbf{H}^p_-(\mathbf{R})$ if $|c|\leq 1$. ■

VIII. BOUND STATES

When there are bound states for Eq. (1.1) with energies $-\kappa_j^2$ with $j=1,\dots,\mathcal{N}$, the reduced transmission coefficient $\tau(k)$ has simple poles on the positive imaginary axis at $k=i\kappa_j$. In this section we modify the inversion procedure described in Sec. V to include the bound states. In this case the scattering data will consist of $Q(x)$, a reflection coefficient, either of $H_\pm$, the bound state energies, and the norming constants. We assume that (H1)–(H5) hold with $\alpha\in(0,1]$ in (H5).

From Eqs. (2.1) and (4.1) we see that at the bound states, the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ are linearly dependent. Associated with the bound states are the norming constants

$$\nu_j=\left(\int_{-\infty}^{\infty} dx\, f_l(i\kappa_j,x)^2 H(x)^2\right)^{-1}, \quad j=1,\dots,\mathcal{N}.$$

The ratios $c_j=f_r(i\kappa_j,x)/f_l(i\kappa_j,x)$ are related to the norming constants by

$$\nu_j^{-1}=i\,\frac{H_-}{c_j}\,\frac{d}{dk}\left(\frac{1}{T_l(k)}\right)\bigg|_{k=i\kappa_j}, \quad j=1,\dots,\mathcal{N}. \tag{8.1}$$

From Eqs. (5.1) and (5.2) we have

$$Z_r(i\kappa_j,y)=c_j\sqrt{\frac{H_+}{H_-}}\,e^{-2\kappa_j y-\kappa_j(A_+-A_-)}Z_l(i\kappa_j,y). \tag{8.2}$$

Let us define $\tilde{\tau}(k)=(-1)^{\mathcal{N}}\tau(k)w(k)^{-1}$ and $\tilde{\rho}(k)=\rho(k)w(k)^{-1}$, where

$$w(k)=(-1)^{\mathcal{N}}\prod_{j=1}^{\mathcal{N}}\frac{k+i\kappa_j}{k-i\kappa_j}. \tag{8.3}$$

Multiplying both sides of Eq. (5.13) by $(-1)^{\mathcal{N}}w(k)^{-1}$, for $k\in\mathbf{R}$ and $y\in\mathbf{R}\backslash\{y_1,\dots,y_N\}$ we obtain

$$\tilde{\tau}(k)Z_r(k,y)=(-1)^{\mathcal{N}}w(k)^{-1}Z_l(-k,y)+(-1)^{\mathcal{N}}\tilde{\rho}(k)e^{2iky}Z_l(k,y). \tag{8.4}$$

At $k=0$ from Eq. (8.4) we have

$$\tilde{\tau}(0)Z_r(0,y)=(-1)^{\mathcal{N}}Z_l(0,y)+(-1)^{\mathcal{N}}\tilde{\rho}(0)Z_l(0,y), \quad y\in(y_j,y_{j+1}). \tag{8.5}$$

In analogy with Eqs. (5.15) and (5.16), let us define

$$G_+(k,x,y)=\frac{1}{k\sqrt{H(x)}}\,[\tilde{\tau}(k)Z_r(k,y)-\tilde{\tau}(0)Z_r(0,y)], \tag{8.6}$$

$$G_-(k,x,y)=\frac{1}{k\sqrt{H(x)}}\,[Z_l(-k,y)-Z_l(0,y)]. \tag{8.7}$$

Then from Eqs. (8.4) and (8.5), for $k\in\mathbf{R}$, $x\in\mathbf{R}\backslash\{x_1,\dots,x_N\}$, and $y\in\mathbf{R}\backslash\{y_1,\dots,y_N\}$ we obtain

$$G_+(k,x,y)-(-1)^{\mathcal{N}}w(k)^{-1}G_-(k,x,y)=\Omega(k,x,y), \tag{8.8}$$

where we have defined

$$\Omega(k,x,y)=-(-1)^{\mathcal{N}}\tilde{\rho}(k)e^{2iky}G_-(-k,x,y)+\frac{(-1)^{\mathcal{N}}}{k}\,[\tilde{\rho}(k)e^{2iky}-\tilde{\rho}(0)]\,\frac{Z_l(0,y)}{\sqrt{H(x)}}$$
$$+\frac{(-1)^{\mathcal{N}}}{k}\,[w(k)^{-1}-1]\,\frac{Z_l(0,y)}{\sqrt{H(x)}}. \tag{8.9}$$

Note that for fixed $x\in\mathbf{R}\backslash\{x_1,\ldots,x_N\}$ and $y\in\mathbf{R}\backslash\{y_1,\ldots,y_N\}$, $G_\pm(k,x,y)$ have analytic extensions in k to $\mathbf{C}^\pm$, and as $k\to\infty$ in $\overline{\mathbf{C}^\pm}$, $G_\pm(k,x,y)\to 0$. Thus we have a scalar Riemann–Hilbert problem in Eq. (8.8). Since $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in(0,1]$, we have $\Omega(\cdot,x,y)\in L^p(\mathbf{R})$ when $p<1/(1-\alpha)$; hence we have the decomposition $\Omega(k,x,y)=\Omega_+(k,x,y)-\Omega_-(k,x,y)$, where we have defined $\Omega_\pm(k,x,y)=\pm[\Pi_\pm\Omega](k,x,y)$. Thus, for fixed $x\in\mathbf{R}\backslash\{x_1,\ldots,x_N\}$ and $y\in\mathbf{R}\backslash\{y_1,\ldots,y_N\}$, $\Omega_\pm(k,x,y)$ have analytic extensions in k to $\mathbf{C}^\pm$, and as $k\to\infty$ in $\overline{\mathbf{C}^\pm}$, $\Omega_\pm(k,x,y)\to 0$. Let us write Eq. (8.8) in the form

$$\left(\prod_{j=1}^{\mathcal{N}}(k+i\kappa_j)\right)G_+(k,x,y)-\left(\prod_{j=1}^{\mathcal{N}}(k-i\kappa_j)\right)G_-(k,x,y)$$
$$=\left(\prod_{j=1}^{\mathcal{N}}(k+i\kappa_j)\right)[\Omega_+(k,x,y)-\Omega_-(k,x,y)],$$

or equivalently in the form

$$\left(\prod_{j=1}^{\mathcal{N}}(k+i\kappa_j)\right)G_+(k,x,y)-\left(\prod_{j=1}^{\mathcal{N}}(k+i\kappa_j)\right)\Omega_+(k,x,y)$$
$$=\left(\prod_{j=1}^{\mathcal{N}}(k-i\kappa_j)\right)G_-(k,x,y)-\left(\prod_{j=1}^{\mathcal{N}}(k+i\kappa_j)\right)\Omega_-(k,x,y). \tag{8.10}$$

In Eq. (8.10) the left-hand side has an analytic extension in k to $\mathbf{C}^+$ and that extension does not grow faster than a polynomial in k of degree $\mathcal{N}-1$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$; analogously, the right-hand side has a similar analytic extension in k to $\mathbf{C}^-$. Thus both sides must be equal to

$$P_{\mathcal{N}-1}(k,x,y)=\sum_{n=0}^{\mathcal{N}-1}p_n(x,y)k^n, \tag{8.11}$$

which is a polynomial in k of degree $\mathcal{N}-1$ such that the coefficients are functions of x and y. By Eqs. (8.7) and (8.9), $G_-(-k,x,y)=-\overline{G_-(k,x,y)}$ and $\Omega(-k,x,y)=-\overline{\Omega(k,x,y)}$ when $k\in\mathbf{R}$; thus $p_n(x,y)=(-1)^{\mathcal{N}+n+1}\overline{p_n(x,y)}$. From Eqs. (8.10) and (8.11) we have

$$G_-(k,x,y)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\kappa_j}{k-i\kappa_j}\right)\Omega_-(k,x,y)+\frac{P_{\mathcal{N}-1}(k,x,y)}{\prod_{j=1}^{\mathcal{N}}(k-i\kappa_j)}, \tag{8.12}$$

$$G_+(k,x,y)=\Omega_+(k,x,y)+\frac{P_{l-1}(k,x,y)}{\prod_{j=1}^{l}(k+i\kappa_j)}.$$

Using Eqs. (8.3) and (8.9) we can write Eq. (8.12) in the form

$$G_-(k,x,y)=B(k,x,y)+\frac{w(k)}{2\pi i}\int_{-\infty}^{\infty} ds\,\frac{\tilde{\rho}(-s)e^{-2isy}}{s+k-i0}\,G_-(s,x,y), \tag{8.13}$$

where we have defined

$$B(k,x,y)=\frac{P_{l-1}(k,x,y)}{\prod_{j=1}^{l}(k-i\kappa_j)}+\frac{w(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k+i0}\,[\tilde{\rho}(s)e^{2isy}-\tilde{\rho}(0)-1+w(s)^{-1}]\,\frac{Z_l(0,y)}{s\sqrt{H(x)}}. \tag{8.14}$$

In analogy with Eqs. (5.21) and (7.1), we can write the singular integral equation in (8.13) in the form

$$X(k)=B(k)+(\mathscr{L}_yX)(k), \quad k\in\mathbf{R}, \tag{8.15}$$

where we have defined $B(k)=B(k,x,y)$, $X(k)=X(k,x,y)=G_-(k,x,y)$, and

$$(\mathscr{L}_yX)(k)=\frac{w(k)}{2\pi i}\int_{-\infty}^{\infty} ds\,\frac{\tilde{\rho}(-s)e^{-2isy}}{s+k-i0}\,X(s).$$

By using the method of Sec. VII, we find that $\mathscr{L}_y$ has a spectral radius less than 1 on $\mathbf{H}^p_-(\mathbf{R})$ for every $1<p<\infty$. Since $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in(0,1]$, the function $B(k,x,y)$ given in Eq. (8.14) belongs to $\mathbf{H}^p_-(\mathbf{R})$ for $p>1/(1-\alpha)$. Thus, Eq. (8.15) is uniquely solvable in $\mathbf{H}^p_-(\mathbf{R})$ for $p>1/(1-\alpha)$, and the solution can be obtained by iteration.

Note that the norming constants associated with the bound states are used to determine the coefficients in the polynomial P_{l-1}. In fact, solving Eq. (8.15) we obtain

$$G_-(-i\kappa_n,x,y)=i^{l}\,\frac{P_{l-1}(-i\kappa_n,x,y)}{\prod_{j=1}^{l}(\kappa_n+\kappa_j)}, \tag{8.16}$$

$$G_+(i\kappa_n,x,y)=\Omega_+(i\kappa_n,x,y)+(-i)^{l}\,\frac{P_{l-1}(i\kappa_n,x,y)}{\prod_{j=1}^{l}(\kappa_n+\kappa_j)}. \tag{8.17}$$

Using Eq. (8.2), from Eqs. (8.6) and (8.7) we obtain

$$G_-(-i\kappa_n,x,y)=-\frac{1}{i\kappa_n\sqrt{H(x)}}\,[Z_l(i\kappa_n,y)-Z_l(0,y)], \tag{8.18}$$

$$G_+(i\kappa_n,x,y)=\frac{1}{i\kappa_n\sqrt{H(x)}}\left[\tilde{\tau}(i\kappa_n)c_n\sqrt{\frac{H_+}{H_-}}\,e^{-2\kappa_ny-\kappa_n(A_+-A_-)}Z_l(i\kappa_n,y)-\tilde{\tau}(0)Z_r(0,y)\right]. \tag{8.19}$$

Eliminating $Z_l(i\kappa_n,y)$ in Eqs. (8.18) and (8.19) we obtain

$$G_+(i\kappa_n,x,y)=-c_n\sqrt{\frac{H_+}{H_-}}\,\tilde{\tau}(i\kappa_n)e^{-2\kappa_ny-\kappa_n(A_+-A_-)}G_-(-i\kappa_n,x,y)$$

$$+\frac{1}{i\kappa_n\sqrt{H(x)}}\left[c_n\sqrt{\frac{H_+}{H_-}}\,\tilde{\tau}(i\kappa_n)e^{-2\kappa_n y-\kappa_n(A_+-A_-)}Z_l(0,y)-\tilde{\tau}(0)Z_r(0,y)\right]. \tag{8.20}$$

Using Eqs. (5.27), (8.16), and (8.17) in Eq. (8.20) we obtain the $\mathcal{N}$ equations for $n=1,\dots,\mathcal{N}$

$$\Omega_+(i\kappa_n,x,y)+(-i)^{\mathcal{N}}\frac{P_{\mathcal{N}-1}(i\kappa_n,x,y)}{\prod_{j=1}^{\mathcal{N}}(\kappa_n+\kappa_j)}$$

$$=-i^{\mathcal{N}}c_n\sqrt{\frac{H_+}{H_-}}\,\tilde{\tau}(i\kappa_n)e^{-2\kappa_n y-\kappa_n(A_+-A_-)}\frac{P_{\mathcal{N}-1}(-i\kappa_n,x,y)}{\prod_{j=1}^{\mathcal{N}}(\kappa_n+\kappa_j)}$$

$$+\frac{1}{i\kappa_n\sqrt{H_-}}[c_n\tilde{\tau}(i\kappa_n)e^{-2\kappa_n y-\kappa_n(A_+-A_-)}f_l(0,x)-\tilde{\tau}(0)f_r(0,x)]. \tag{8.21}$$

In Eq. (8.21) the products $c_n\tilde{\tau}(i\kappa_n)$ can be expressed in terms of the norming constants ν_n using

$$\tilde{\tau}(i\kappa_n)=\frac{1}{2i\kappa_n}\prod_{j\neq n}\frac{k-i\kappa_j}{k+i\kappa_j}\left[\frac{d}{dk}\left(\frac{1}{\tau(k)}\right)\bigg|_{k=i\kappa_n}\right]^{-1}=\frac{\sqrt{H_+H_-}\,\nu_n e^{-\kappa_n A}}{2\kappa_n c_n}\prod_{j\neq n}\frac{k-i\kappa_j}{k+i\kappa_j}, \tag{8.22}$$

which follows from Eqs. (8.1) and (4.1). Using Eq. (8.22) and replacing $\tilde{\tau}(0)$ by $(-1)^{\mathcal{N}}\tau(0)$, Eq. (8.21) can be written as

$$\Omega_+(i\kappa_n,x,y)+(-i)^{\mathcal{N}}\frac{P_{\mathcal{N}-1}(i\kappa_n,x,y)}{\prod_{j=1}^{\mathcal{N}}(\kappa_n+\kappa_j)}$$

$$=-i^{\mathcal{N}}H_+\frac{\nu_n}{2\kappa_n}e^{-2\kappa_n(y+A_+)}\frac{P_{\mathcal{N}-1}(-i\kappa_n,x,y)}{\prod_{j=1}^{\mathcal{N}}(\kappa_n+\kappa_j)}$$

$$+\frac{1}{i\kappa_n\sqrt{H_-}}\left[\sqrt{H_+H_-}\frac{\nu_n}{2\kappa_n}e^{-2\kappa_n(y+A_+)}f_l(0,x)-(-1)^{\mathcal{N}}\tau(0)f_r(0,x)\right]. \tag{8.23}$$

Note that both $f_l(0,x)$ and $f_r(0,x)$ are determined by $Q(x)$ alone. In the generic case $\tau(0)=0$ and in the exceptional case $\tau(0)=(-1)^{\mathcal{N}}\sqrt{1-\rho(0)^2}$ by Proposition 4.6 and Eq. (4.8). Furthermore, H_- can be obtained from H_+, $Q(x)$, and $\rho(k)$ using Eq. (5.29) in the exceptional case and using Eq. (5.30) in the generic case. The equations (8.23) constitute a set of $\mathcal{N}$ equations for the $\mathcal{N}$ unknowns $p_0(x,y),\dots,p_{\mathcal{N}-1}(x,y)$. Note, however, that $\Omega_+(i\kappa_n,x,y)$ also depends on $p_0(x,y),\dots,p_{\mathcal{N}-1}(x,y)$ via $B(k,x,y)$ in Eq. (8.13) and $G_-(-k,x,y)$ in Eq. (8.9). Once the polynomial $P_{\mathcal{N}-1}(k,x,y)$ has been found, $H(x)$ can be obtained as described following the proof of Theorem 5.4 by using $X(0)=G_-(0,x,y)$ on the left-hand side of Eq. (5.24).

Example 8.1: Let us illustrate the inversion method outlined in this section by a simple example. Although $H(x)$ will turn out to be continuous in this example, the example shows how the inversion method works when bound states are present. A more elaborate example with a discontinuous $H(x)$ is given in Ref. 34. Suppose $\rho(k)=0$ and there is one bound state at $-\kappa^2$ with associated norming constant ν. Thus, we have the exceptional case. The solution of Eq. (8.13) is given by

$$G_-(k,x,y)=B(k,x,y)=\frac{p_0(x,y)}{k-i\kappa}, \tag{8.24}$$

and hence $G_-(-i\kappa,x,y)=i[p_0(x,y)/(2\kappa)]$. From Eq. (8.9) we have

$$\Omega(k,x,y)=\frac{2}{k+i\kappa}\,\frac{Z_l(0,y)}{\sqrt{H(x(y))}},$$

and hence we obtain

$$\Omega_+(i\kappa,x,y)=-\frac{i}{\kappa}\,\frac{Z_l(0,y)}{\sqrt{H(x(y))}}.$$

Solving Eq. (8.23) for $p_0(x,y)$, we obtain

$$p_0(x,y)=\frac{2\nu\sqrt{H_+}}{2\kappa e^{2\kappa(y+A_+)}-H_+\nu}\,f_l(0,x). \tag{8.25}$$

In deriving Eq. (8.25) we have also used the fact that

$$i\,\frac{f_l(0,x)}{\sqrt{H_+}}-i\tau(0)\,\frac{f_r(0,x)}{\sqrt{H_-}}=\frac{i\gamma f_r(0,x)}{\sqrt{H_+}}\left[\frac{H_-\gamma^2-H_+}{H_-\gamma^2+H_+}\right]=\frac{i\gamma f_r(0,x)}{\sqrt{H_+}}\,\ell(0)=0,$$

which follows from Eqs. (4.5) and (4.10). Now Eq. (5.24) takes the form

$$X(0)=G_-(0,x,y)=\frac{i}{\kappa}\,p_0(x,y)=\frac{i}{\sqrt{H_+}}\,[i\dot{f}_l(0,x)+f_l(0,x)(y+A_+)],$$

or equivalently

$$\frac{\sqrt{H_+}p_0(x,y)}{\kappa f_l(0,x)}-(y+A_+)=\frac{i\dot{f}_l(0,x)}{f_l(0,x)}. \tag{8.26}$$

Define

$$\omega=\omega(y)=e^{2\kappa(y+A_+)},\quad \zeta=\frac{H_+\nu}{2\kappa}. \tag{8.27}$$

Using Eqs. (8.25) and (8.27) we can write Eq. (8.26) as

$$f(\omega)=g(x), \tag{8.28}$$

where

$$f(\omega)=\frac{2\zeta}{\kappa}\,\frac{1}{\omega-\zeta}-\frac{1}{2\kappa}\ln\omega,\quad g(x)=i\,\frac{\dot{f}_l(0,x)}{f_l(0,x)}. \tag{8.29}$$

Here $f(\omega)$ is defined for $\omega\in(0,\zeta)\cup(\zeta,\infty)$, and on this domain $f'(\omega)<0$. Also, $f(\omega)\to+\infty$ as $\omega\to 0$, $f(\omega)\to-\infty$ as $\omega\to\zeta-0$, $f(\omega)\to+\infty$ as $\omega\to\zeta+0$, and $f(\omega)\to-\infty$ as $\omega\to+\infty$. The function $g(x)$ obeys

$$g'(x) = -\frac{H_+}{f_l(0,x)^2} < 0, \tag{8.30}$$

which is obtained from Eq. (2.27). Note that $g(x)$ has a singularity whenever $f_l(0,x)$ is zero; this follows from the expression for $g(x)$ in Eq. (8.29) provided we can show that $\dot{f}_l(0,\varpi_0) \neq 0$, where ϖ_0 is a zero of $f_l(0,x)$. Indeed, $\dot{f}_l(0,\varpi_0) \neq 0$ due to the behavior of the Jost function near $k=0$ corresponding to Eq. (1.1) restricted to the semi-infinite interval (ϖ_0,∞) [cf. Ref. 29, where the case $H(x)=1$ was considered, and the Appendix]. Together with Eq. (8.30) this implies that $g(x) \to -\infty$ as $x \to \varpi_0 - 0$ and $g(x) \to +\infty$ as $x \to \varpi_0 + 0$. Furthermore, $g(x) \to -\infty$ as $x \to +\infty$ and $g(x) \to +\infty$ as $x \to -\infty$. It follows from these properties of $f(\omega)$ and $g(x)$ that Eq. (8.28) has a solution $\omega(x)$ such that $\omega(x) \to 0$ as $x \to -\infty$ and $\omega(x) \to +\infty$ as $x \to +\infty$ [i.e., $y(x) \to -\infty$ and $y(x) \to +\infty$, respectively] if and only if $f_l(0,x)$ has exactly one zero. Note that by Eqs. (3.1) and (8.27) the function $\omega(x)$ must be continuous and monotonically increasing. The fact that $f_l(0,x)$ has exactly one zero is in agreement with the oscillation properties of $f_l(0,x)$ and with Proposition 5.4 in Ref. 20, which says that the number of bound states of Eq. (1.1) is independent of $H(x)$ and hence is the same as that for Eq. (1.1) with $H(x)=1$. So we assume $f_l(0,x)$ has one zero located at $x=\varpi_0$. The monotonicity properties of $f(\omega)$ and $g(x)$ also imply that Eq. (8.28) can be uniquely solved to obtain ω in terms of x; since $y = -A_+ + [1/(2\kappa)]\ln \omega$ by Eq. (8.27), it then follows that the solution $y(x)$ is also unique. From Eq. (8.27) we see that $d\omega/dx = 2\kappa H(x)\omega$; hence by differentiating both sides of Eq. (8.28) with respect to x and using Eq. (2.27), we obtain

$$H(x) = \left(\frac{\omega(y)-\zeta}{\omega(y)+\zeta}\right)^2 \frac{H_+}{f_l(0,x)^2}. \tag{8.31}$$

Using the initial condition $y=0$ when $x=0$ in Eq. (8.27) we obtain

$$A_+ = \frac{1}{2\kappa} \ln \omega_0, \tag{8.32}$$

where ω_0 obeys $f(\omega_0)=g(0)$. Note that Eq. (8.28) does not contain A_+ explicitly and so once ω_0 is known, A_+ can be obtained from Eq. (8.27) by setting $\omega_0 = \omega(0) = e^{2\kappa A_+}$. One can verify that this value for A_+ is consistent with Eqs. (3.17) and (8.31). For the convenience of the reader we give the detailed calculation. We have

$$\begin{aligned} A_+ &= \int_0^\infty ds[H_+ - H(s)] = \lim_{N\to\infty} \int_0^N ds[H_+ - H(s)] = H_+N - \int_0^N ds\, H(s) \\ &= H_+N - \int_{\omega_0}^{\omega(N)} d\omega\, \frac{H(s(\omega))}{\omega'(s(\omega))} = H_+N - \int_{\omega_0}^{\omega(N)} d\omega\, \frac{H(s(\omega))}{2\kappa H((\omega))\omega} \\ &= H_+N - \frac{1}{2\kappa}[\ln \omega(N) - \ln \omega_0] = \left[H_+N - \frac{1}{2\kappa}\ln \omega(N)\right] + \frac{1}{2\kappa}\ln \omega_0. \end{aligned} \tag{8.33}$$

Now by using Eqs. (8.28) and (8.29) we can write

$$H_+N - \frac{1}{2\kappa}\ln \omega(N) = i\,\frac{\dot{f}_l(0,N)}{f_l(0,N)} + H_+N - \frac{2\zeta}{\kappa}\,\frac{1}{\omega(N)-\zeta},$$

and we see that the right-hand side goes to zero as $N \to \infty$. Hence, after taking $N \to \infty$ in Eq. (8.33), we obtain Eq. (8.32). Furthermore, since the denominator in Eq. (8.25) must go to zero as $x \to \varpi_0$, using Eqs. (8.27) and (8.32) we obtain

$$\lim_{x\to\varpi_0} y(x)=y(\varpi_0)=\frac{1}{2\kappa}\ln\left(\frac{\zeta}{\omega_0}\right),$$

and from Eq. (8.31), using L'Hôpital's rule, we get

$$\lim_{x\to\varpi_0} H(x)=H(\varpi_0)=\frac{f_l'(0,\varpi_0)^2}{H_+\kappa^2},$$

and hence $H(x)$ is continuous at ϖ_0. For the Jost solutions we obtain from Eqs. (5.1), (8.7), and (8.24)

$$f_l(k,x)=e^{iky+ikA_+}\left[\sqrt{H_+}\,\frac{k}{k+i\kappa}\,p_0(x,y)+f_l(0,x)\right]. \tag{8.34}$$

Hence, letting $x\to-\infty$ in Eq. (8.34), we conclude that $L(k)=0$; using Eqs. (1.4), (3.20), and (4.5) we also obtain

$$\frac{1}{T_l(k)}=\gamma\left(\frac{-k+i\kappa}{k+i\kappa}\right)e^{ikA}, \tag{8.35}$$

where A is the constant defined in Eq. (3.18). Thus, from Eq. (4.1) we get

$$\tau(k)=-\frac{1}{\gamma}\sqrt{\frac{H_+}{H_-}}\frac{k+i\kappa}{k-i\kappa}.$$

Since $\rho(k)=0$ implies $\ell(k)=0$, we conclude from Eqs. (4.9) and (4.10) that $\gamma^2=H_+/H_-$. Since $\mathcal{N}=1$, by Proposition 4.6, $\gamma<0$ and hence $\gamma=-\sqrt{H_+/H_-}$. Hence using Eqs. (4.1) and (8.35) we get $\tau(k)=(k+i\kappa)/(k-i\kappa)$. We have also computed the potential $V(y(x))$ in Eq. (3.3). The result is (after lengthy calculations)

$$V(y)=\frac{-8\zeta\kappa^2e^{2\kappa(y-A_+)}}{(e^{2\kappa y}+\zeta e^{-2\kappa A_+})^2}.$$

This is a standard reflectionless potential (cf. Ref. 24, Example 2), as it must be the case since $\rho(k)=0$.

APPENDIX: PROOF OF THEOREM 4.2 (ii)

In this appendix we give the proof of Theorem 4.2 (ii). We first prove a proposition.

Proposition A.1: Let $\psi(k,x)$ denote the solution of Eq. (1.1) satisfying $\psi(k,0)=a$, $\psi'(k,0)=b$, where a and b are arbitrary real numbers. Suppose $\psi(0,x)$ is bounded on $x\geqslant 0$. Fix $k_0>0$. Then, for $-k_0<k<k_0$ and $x\geqslant 0$, we have

$$|\psi(k,x)-\psi(0,x)|\leqslant C_{k_0}\left[\left(\frac{kx}{1+|k|x}\right)^2+\frac{k^2x}{1+|k|x}\right]. \tag{A1}$$

Proof: Since $\psi(-k,x)=\psi(k,x)$ it suffices to consider $0\leqslant k<k_0$. By variation of parameters, $\psi(k,x)$ is a solution of the integral equation

$$\psi(k,x)=b\,\frac{\sin(kH_+x)}{kH_+}+a\,\cos(kH_+x)$$

$$+\frac{1}{kH_+}\int_0^x dz\,\sin(kH_+[x-z])[k^2\{H_+^2-H(z)^2\}+Q(z)]\psi(k,z), \tag{A2}$$

which for $k=0$ reduces to

$$\psi(0,x)=bx+a+\int_0^x dz(x-z)Q(z)\psi(0,z). \tag{A3}$$

Since $\psi(0,x)$ is bounded, Eq. (A3) implies

$$b+\int_0^\infty dz\,Q(z)\psi(0,z)=0, \tag{A4}$$

$$\psi(0,x)=a-\int_0^\infty dz\,zQ(z)\psi(0,z)+o(1),\quad x\to+\infty. \tag{A5}$$

Then, by using Eqs. (A2)–(A4), we can write $\psi(k,x)-\psi(0,x)=I_1+I_2+I_3+I_4+I_5+I_6+I_7$, where

$$I_1=\left[1-\frac{\sin(kH_+x)}{kH_+x}\right]x\int_x^\infty dz\,Q(z)\psi(0,z),$$

$$I_2=a[\cos(kH_+x)-1],$$

$$I_3=\frac{\sin(kH_+x)}{kH_+}\int_0^x dz[\cos(kH_+z)-1]Q(z)\psi(0,z),$$

$$I_4=\frac{1}{kH_+}[1-\cos(kH_+x)]\int_0^x dz\,\sin(kH_+z)Q(z)\psi(0,z),$$

$$I_5=\int_0^x dz\left[1-\frac{\sin(kH_+z)}{kH_+z}\right]zQ(z)\psi(0,z),$$

$$I_6=\frac{k}{H_+}\int_0^x dz\,\sin(kH_+[x-z])[H_+^2-H(z)^2]\psi(0,z),$$

$$I_7=\frac{1}{kH_+}\int_0^x dz\,\sin(kH_+[x-z])[k^2\{H_+^2-H(z)^2\}+Q(z)][\psi(k,z)-\psi(0,z)].$$

The estimates

$$|\sin z|\leqslant\frac{C_1z}{1+z},\quad\left|1-\frac{\sin z}{z}\right|\leqslant\frac{C_1z^2}{(1+z)^2},\quad|1-\cos z|\leqslant\frac{C_1z^2}{(1+z)^2},$$

where $z\geqslant0$ and C_1 is a suitable constant, and the monotonicity of the function $z\mapsto z/(1+z)$ imply

$$|I_1|\leqslant C_1\left(\frac{kH_+x}{1+kH_+x}\right)^2\int_x^\infty dz\ z|Q(z)||\psi(0,z)|, \tag{A6}$$

$$|I_2|\leqslant C_1a\left(\frac{kH_+x}{1+kH_+x}\right)^2, \tag{A7}$$

$$|I_3|\leqslant C_1^2\left(\frac{kH_+x}{1+kH_+x}\right)^2\int_0^x dz\ z|Q(z)||\psi(0,z)|, \tag{A8}$$

$$|I_s|\leqslant C_1\left(\frac{kH_+x}{1+kH_+x}\right)^2\int_0^x dz\ z|Q(z)||\psi(0,z)|, \quad s=4,5, \tag{A9}$$

$$|I_6|\leqslant\frac{C_1k^2x}{1+kH_+x}\int_0^x dz|H_+^2-H(z)^2||\psi(0,z)|, \tag{A10}$$

$$|I_7|\leqslant\frac{C_1x}{1+kH_+x}\int_0^x dz[k^2|H_+^2-H(z)^2|+|Q(z)|]|\psi(k,z)-\psi(0,z)|. \tag{A11}$$

By combining Eqs. (A6)–(A10), for some constant C_2 we obtain

$$|I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|\leqslant C_2\left[\left(\frac{kH_+x}{1+kH_+x}\right)^2+\frac{k^2x}{1+kH_+x}\right].$$

Hence by using Eq. (A11), it follows that

$$u(k,x)\leqslant C_2+\frac{C_1x}{[1+kH_+x]h(k,x)}\left[\int_0^x dz\{k^2|H_+^2-H(z)^2|+|Q(z)|\}h(k,z)u(k,z)\right], \tag{A12}$$

where

$$h(k,x)=\left(\frac{kH_+x}{1+kH_+x}\right)^2+\frac{k^2x}{1+kH_+x}, \tag{A13}$$

$$u(k,x)=\frac{|\psi(k,x)-\psi(0,x)|}{h(k,x)}. \tag{A14}$$

Thus, from Eqs. (A12) and (A13) and the fact that $h(k,z)\leqslant h(k,x)$ for $0\leqslant z\leqslant x$, we obtain

$$\begin{aligned}u(k,x)\leqslant C_2&+\frac{C_1k}{H_+}\int_0^x dz|H_+^2-H(z)^2|u(k,z)\\&+\frac{C_1k^2x}{(1+kH_+x)h(k,x)}\left(\frac{H_+^2x}{1+kH_+x}+1\right)\int_0^x dz\ \frac{z|Q(z)|}{1+kH_+z}\,u(k,z).\end{aligned}$$

Hence

$$u(k,x)\leqslant C_2+\frac{C_1k}{H_+}\int_0^x dz|H_+^2-H(z)^2|u(k,z)+2C_1\int_0^x dz\ z|Q(z)|u(k,z).$$

Using Gronwall's inequality we obtain

$$u(k,x)\leq C_2 \exp\left(2C_1\int_0^\infty dz\left[\frac{k_0}{2H_+}|H_+^2-H(z)^2|+z|Q(z)|\right]\right),\quad 0\leq k<k_0. \tag{A15}$$

Now Eq. (A1) follows from Eqs. (A13), (A14), and (A15), where the right-hand side of Eq. (A15) and the factors H_+ in Eq. (A13) have been incorporated into the constant C_{k_0}. ■

Proposition A.1 remains valid for $x\leq 0$ if $\psi(0,x)$ is bounded on $x\leq 0$. Then x has to be replaced by $|x|$ in Eq. (A1). In place of Eqs. (A4) and (A5), we then have

$$b-\int_{-\infty}^0 dz\ Q(z)\tilde{\psi}(0,z)=0, \tag{A16}$$

$$\psi(0,x)=a+\int_{-\infty}^0 dz\ zQ(z)\psi(0,z)+o(1),\quad x\to-\infty. \tag{A17}$$

Proof of Theorem 4.2 (ii): In the exceptional case it does not immediately follow from Eq. (2.1) that $\tau(k)$ is continuous at $k=0$. The situation is similar as in the case $H(x)=1$, which was studied in Ref. 35. Here we follow the method of Ref. 35 and generalize it to the case when $H(x)$ is not constant.

We continue to assume that k is real. Let $v(k,x)$ and $\varsigma(k,x)$ denote the solutions of Eq. (1.1) satisfying $v(k,0)=\varsigma'(k,0)=0$ and $v'(k,0)=\varsigma(k,0)=1$. Let $q_\pm(k,z)=k^2[H_\pm^2-H(z)^2]+Q(x)$. By using the definition of the Wronskian and the integral equations satisfied by $v(k,x)$ and $\varsigma(k,x)$ [see Eq. (A2)], we obtain the relations

$$[f_l(k,x);v(k,x)]=f_l(k,0)=1+\int_0^\infty dz\ e^{ikH_+z}q_+(k,z)v(k,z), \tag{A18}$$

$$[f_r(k,x);v(k,x)]=f_r(k,0)=1-\int_{-\infty}^0 dz\ e^{-ikH_-z}q_-(k,z)v(k,z), \tag{A19}$$

$$[f_l(k,x);\varsigma(k,x)]=-f_l'(k,0)=-ikH_++\int_0^\infty dz\ e^{ikH_+z}q_+(k,z)\varsigma(k,z), \tag{A20}$$

$$[f_r(k,x);\varsigma(k,x)]=-f_r'(k,0)=ikH_--\int_{-\infty}^0 dz\ e^{-ikH_-z}q_-(k,z)\varsigma(k,z). \tag{A21}$$

Let $\tilde{\psi}(k,x)$ denote the solution of Eq. (1.1) such that

$$\tilde{\psi}(k,0)=f_l(0,0),\quad \tilde{\psi}'(k,0)=f_l'(0,0). \tag{A22}$$

Thus

$$\tilde{\psi}(k,x)=f_l(0,0)\varsigma(k,x)+f_l'(0,0)v(k,x), \tag{A23}$$

and in particular $\tilde{\psi}(0,x)=f_l(0,x)$. Since we are in the exceptional case, $\tilde{\psi}(0,x)$ is a bounded solution of Eq. (1.1) for $k=0$. From Eqs. (A18)–(A23) it follows that

$$f_l(0,0)[f_l(k,x);f_r(k,x)]=f_r(k,0)\left[-ikH_+f_l(0,0)+f_l'(0,0)+\int_0^\infty dz\; e^{ikH_+z}q_+(k,z)\tilde{\psi}(k,z)\right]$$

$$-f_l(k,0)\left[ikH_-f_l(0,0)+f_l'(0,0)-\int_{-\infty}^0 dz\; e^{-ikH_-z}q_-(k,z)\tilde{\psi}(k,z)\right]. \tag{A24}$$

We first assume $f_l(0,0)\neq 0$. The case when $f_l(0,0)=0$ will be considered separately. In order to analyze the integral in the first bracket on the right-hand side of Eq. (A24), we write

$$\int_0^\infty dz\; e^{ikH_+z}q_+(k,z)\tilde{\psi}(k,z)=J_1+J_2+J_3+J_4+J_5,$$

where

$$J_1=\int_0^\infty dzQ(z)\tilde{\psi}(0,z),$$

$$J_2=\int_0^\infty dzQ(z)[e^{ikH_+z}-1]\tilde{\psi}(0,z),$$

$$J_3=k^2\int_0^\infty dz\; e^{ikH_+z}[H_+^2-H(z)^2]\tilde{\psi}(0,z),$$

$$J_4=\int_0^\infty dz\; e^{ikH_+z}Q(z)[\tilde{\psi}(k,z)-\tilde{\psi}(0,z)],$$

$$J_5=k^2\int_0^\infty dz\; e^{ikH_+z}[H_+^2-H(z)^2][\tilde{\psi}(k,z)-\tilde{\psi}(0,z)].$$

By Eq. (A4) with $b=f_l'(0,0)$, we have

$$J_1=-f_l'(0,0). \tag{A25}$$

Note that $f_l(0,x)\to 1$ as $x\to+\infty$; hence using Eq. (A5) with $a=f_l(0,0)$ and the estimate $|e^{ikH_+z}-ikzH_+-1|\leq C[k^2z^2/(1+|k|z)]$ [cf. Eq. (2.25)] we obtain

$$J_2=ikH_+[f_l(0,0)-1]+o(|k|^{1+\alpha}),\quad k\to 0. \tag{A26}$$

It is clear that

$$J_3=O(k^2),\quad k\to 0. \tag{A27}$$

Using Proposition A.1 we obtain

$$|J_4|\leq|k|^{1+\alpha}\int_0^\infty dz\;\frac{(|k|z)^{1-\alpha}}{(1+|k|z)^2}\,z^{1+\alpha}|Q(z)|+k^2\int_0^\infty dz\; z|Q(z)|=o(|k|^{1+\alpha}),\quad k\to 0. \tag{A28}$$

Finally, since $\alpha\in[0,1)$ we have

$$J_5=o(k^2),\quad k\to 0. \tag{A29}$$

Using Eqs. (A25)–(A29) in Eq. (A24) we obtain

$$\int_0^\infty dz\ e^{ikH_+z}q_+(k,z)\tilde{\psi}(k,z)=-f_l'(0,0)+ikH_+[f_l(0,0)-1]+o(|k|^{1+\alpha}),\quad k\to 0. \tag{A30}$$

Similarly, using Eqs. (A16) and (A17) which imply that $f_l(0,x)=\gamma$ as $x\to-\infty$, where γ is the constant in Eq. (4.5), we obtain

$$\int_{-\infty}^0 dz\ e^{-ikH_-z}q_-(k,z)\tilde{\psi}(k,z)=f_l'(0,0)-ikH_-[\gamma-f_l(0,0)]+o(|k|^{1+\alpha}),\quad k\to 0. \tag{A31}$$

Substituting Eqs. (A30) and (A31) in Eq. (A24) we get

$$f_l(0,0)[f_l(k,x);f_r(k,x)]=-ik[H_+f_r(k,0)+\gamma H_-f_l(k,0)]+o(|k|^{1+\alpha}),\quad k\to 0. \tag{A32}$$

Similarly, starting from

$$\begin{aligned}f_l(0,0)&[f_l(k,x);f_r(-k,x)]\\ &=f_r(-k,0)\left[-ikH_+f_l(0,0)+f_l'(0,0)+\int_0^\infty dz\ e^{ikH_+z}q_+(k,z)\tilde{\psi}(k,z)\right]\\ &\quad-f_l(k,0)\left[-ikH_-f_l(0,0)+f_l'(0,0)-\int_{-\infty}^0 dz\ e^{ikH_-z}q_-(k,z)\tilde{\psi}(k,z)\right],\end{aligned}$$

we deduce that

$$f_l(0,0)[f_l(k,x);f_r(-k,x)]=-ik[H_+f_r(-k,0)-\gamma H_-f_l(k,0)]+o(|k|^{1+\alpha}),\quad k\to 0. \tag{A33}$$

Using Eqs. (2.1), (2.2), (4.5), (A32), (A33), and Theorem 2.1 (ii) we obtain

$$T_l(k)=\frac{2H_-\gamma}{H_-\gamma^2+H_+}+o(|k|^\alpha),\quad k\to 0, \tag{A34}$$

$$T_r(k)=\frac{2H_+\gamma}{H_-\gamma^2+H_+}+o(|k|^\alpha),\quad k\to 0, \tag{A35}$$

$$R(k)=\frac{H_+-H_-\gamma^2}{H_-\gamma^2+H_+}+o(|k|^\alpha),\quad k\to 0, \tag{A36}$$

$$L(k)=\frac{H_-\gamma^2-H_+}{H_-\gamma^2+H_+}+o(|k|^\alpha),\quad k\to 0. \tag{A37}$$

The relations (A34)–(A37) are also valid if $f_l(0,0)=0$. To see this, assume $f_l(0,0)=0$ and consider Eq. (1.1) with $Q_\varpi(x)=Q(x+\varpi)$ and $H_\varpi(x)=H(x+\varpi)$, where ϖ will be specified later. Let $f_{l,\varpi}(k,x)$ and $f_{r,\varpi}(k,x)$ denote the Jost solutions and $T_{l,\varpi}(k)$, $T_{r,\varpi}(k)$, $R_\varpi(k)$, and $L_\varpi(k)$ denote the transmission and reflection coefficients associated with $Q_\varpi(x)$ and $H_\varpi(x)$. From Eqs. (2.1), (2.2), (2.10), and (2.11) it follows that

$$f_{l,\varpi}(k,x)=e^{-ikH_+\varpi}f_l(k,x+\varpi),\quad f_{r,\varpi}(k,x)=e^{ikH_-\varpi}f_r(k,x+\varpi), \tag{A38}$$

$$T_{l,\varpi}(k)=T_l(k)e^{-ik(H_--H_+)\varpi}, \quad T_{r,\varpi}(k)=T_r(k)e^{-ik(H_--H_+)\varpi}, \tag{A39}$$

$$R_{\varpi}(k)=R(k)e^{2ikH_+\varpi}, \quad L_{\varpi}(k)=L(k)e^{-2ikH_-\varpi}. \tag{A40}$$

Now pick ϖ such that $f_l(0,\varpi)\neq 0$. Then $f_{l,\varpi}(0,0)\neq 0$ and Eqs. (A34)–(A37) apply to $T_{l,\varpi}(k)$, $T_{r,\varpi}(k)$, $R_{\varpi}(k)$, and $L_{\varpi}(k)$. Note that by Eqs. (A38) and (A39), when considering $Q_{\varpi}(x)$ and $H_{\varpi}(x)$, we remain in the exceptional case, and that the constant γ defined in Eq. (4.5) does not change. It then follows from Eqs. (A39) and (A40) that Eqs. (A34)–(A37) remain valid even when $f_l(0,0)=0$.

Now Eqs. (4.9) and (4.10) follow from Eqs. (A36) and (A37) by using Eq. (4.1). Relation (4.8) also follows from Eqs. (A34) or (A35) and (4.1), but so far only when k is real. In order to extend Eq. (4.8) to complex $k\in\overline{\mathbf{C}^+}$, we fix $\delta>0$ and note that by Eq. (2.1) the estimate

$$\frac{1}{|T_l(k)|}\leqslant\frac{C}{|k|} \tag{A41}$$

holds on $\overline{\mathbf{C}^+}\cap\{|k|\leqslant\delta\}$. The validity of Eqs. (A34) and (A35) for real k together with Eq. (A41) allows us to appeal to theorems of Phragmén–Lindelöf (Ref. 25, Theorems 1.4.1 and 1.4.4) and to conclude that $T_l(k)$ and $T_r(k)$ approach finite limits [$T_l(0)$ and $T_r(0)$, respectively] as $k\to 0$ uniformly in $0\leqslant\arg(k)\leqslant\pi$. By considering $[T_{l,r}(k)-T_{l,r}(0)]/k^{\alpha}$ we can see that also the error term $o(|k|^{\alpha})$ is valid for $k\in\overline{\mathbf{C}^+}$. Hence Eq. (4.8) holds if $k\to 0$ in $\overline{\mathbf{C}^+}$. ■

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[1] N. I. Grinberg, "The one-dimensional inverse scattering problem for the wave equation," Math. USSR Sb. **70**, 557–572 (1991) [Mat. Sb. **181** (8), 1114–1129 (1990) (Russian)].
[2] N. I. Grinberg, "Inverse scattering problem for an elastic layered medium," Inverse Problems **7**, 567–576 (1991).
[3] T. Aktosun, M. Klaus, and C. van der Mee, "Inverse scattering in 1-D nonhomogeneous media and the recovery of the wave speed," J. Math. Phys. **33**, 1395–1402 (1992).
[4] T. Aktosun and C. van der Mee, "Nonuniqueness in inverse acoustic scattering on the line," J. Math. Phys. **35**, 693–709 (1994).
[5] R. G. Babadzhanyan and V. S. Rabinovich, "On the factorization of almost periodic operator functions," in *Differential and Integral Equations and Complex Analysis* (Kalmytsk. Gos. University, Elitsa, 1986), pp. 13–22 (Russian).
[6] I. M. Spitkovskii, "Factorization of almost-periodic matrix-valued functions," Math. Notes **45**, 482–488 (1989) [Matem. Zametki **45**, 74–82, 111 (1989) (Russian)].
[7] R. G. Newton, "On a generalized Hilbert problem," J. Math. Phys. **23**, 2257–2265 (1982).
[8] A. I. Nachman and M. J. Ablowitz, "A multidimensional inverse scattering method," Stud. Appl. Math. **71**, 243–250 (1984).
[9] R. Beals and R. R. Coifman, "The *D*-bar approach to inverse scattering and nonlinear evolutions," Phys. D **18**, 242–249 (1986).
[10] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed. (Springer, New York, 1989).
[11] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations, and Inverse Scattering* (Cambridge University, Cambridge, 1991).
[12] I. A. Feldman, "On an effective solution of certain integral equations on the line and on the half line," Izv. Akad. Nauk Mold. SSR **10** (88), 16–26 (1961) (Russian).
[13] P. C. Sabatier and B. Dolveck-Guilpard, "On modeling discontinuous media. One-dimensional approximations," J. Math. Phys. **29**, 861–868 (1988).
[14] P. C. Sabatier, "On modeling discontinuous media. Three-dimensional scattering," J. Math. Phys. **30**, 2585–2598 (1989).
[15] F. Dupuy and P. C. Sabatier, "Discontinuous media and undetermined scattering problems," J. Phys. A **25**, 4253–4268 (1992).

[16] F. R. Molino and P. C. Sabatier, "Elastic waves in discontinuous media: Three-dimensional scattering," J. Math. Phys. **35**, 4594–4635 (1994).
[17] R. J. Krueger, "An inverse problem for a dissipative hyperbolic equation with discontinuous coefficients," Q. Appl. Math. **34**, 129–147 (1976).
[18] R. J. Krueger, "An inverse problem for an absorbing medium with multiple discontinuities," Q. Appl. Math. **36**, 235–253 (1978).
[19] R. J. Krueger, "Inverse problems for nonabsorbing media with discontinuous material properties," J. Math. Phys. **23**, 396–404 (1982).
[20] T. Aktosun, M. Klaus, and C. van der Mee, "Scattering and inverse scattering in one-dimensional nonhomogeneous media," J. Math. Phys. **33**, 1717–1744 (1992).
[21] L. D. Faddeev, "Properties of the S-matrix of the one-dimensional Schrödinger equation," Am. Math. Soc. Transl., Series **2**, **2**, 139–166 (1964) [Tr. Mat. Inst. Steklova **73**, 314–336 (1964) (Russian)].
[22] P. Deift and E. Trubowitz, "Inverse scattering on the line," Commun. Pure Appl. Math. **32**, 121–251 (1979).
[23] V. A. Marchenko, *Sturm–Liouville Operators and Applications* (Birkhäuser, Basel, 1986), OT 22.
[24] T. Aktosun, M. Klaus, and C. van der Mee, "On the Riemann–Hilbert problem for the one-dimensional Schrödinger equation," J. Math. Phys. **34**, 2651–2690 (1993).
[25] R. P. Boas, *Entire Functions* (Academic, New York, 1954).
[26] I. M. Gel'fand, D. A. Raikov, and G. E. Shilov, *Commutative Normed Rings* (Chelsea, New York, 1964).
[27] T. Aktosun, M. Klaus, and C. van der Mee, "On the recovery of a discontinuous wave speed in wave scattering in a nonhomogeneous medium," in Proceedings of the 20th Summer School "Applications of Mathematics in Engineering," Varna, August 26–September 2, 1994, edited by D. Ivanchev and D. Mishev.
[28] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Operators* (Birkhäuser, Basel, 1992), Vol. I, OT 53.
[29] M. Klaus, "Exact behavior of Jost functions at low energy," J. Math. Phys. **29**, 148–154 (1988).
[30] K. Hoffman, *Banach Spaces of Analytic Functions* (Constable, London, 1962) [also: Dover, New York, 1988].
[31] S. C. Power, *Hankel Operators on Hilbert Space*, Research Notes in Mathematics, Vol. 64 (Pitman, New York, 1982).
[32] J. R. Partington, *An Introduction to Hankel Operators*, London Mathematical Society Student Texts, Vol. 13 (Cambridge University, Cambridge, 1988).
[33] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators* (Birkhäuser, Basel, 1981), OT 3.
[34] T. Aktosun, M. Klaus, and C. van der Mee, "Inverse scattering problem for the wave equation with discontinuous wave speed," in Proceedings of the Fifth International Colloquium on Differential Equations, Plovdiv, August 18–23, 1994, edited by D. Bainov and V. Covachev (VSP International Science, Zeist, The Netherlands, 1995), pp. 3–22.
[35] M. Klaus, "Low-energy behavior of the scattering matrix for the Schrödinger equation on the line," Inverse Problems **4**, 505–512 (1988).