In generale nei metodi iterativi si considera lo splitting della matrice A

$$A = P - N$$

si ha quindi: $Ax = b \to (P-N)x = b \to$

$$\to Px - Nx = b \to Px = Nx + b$$

$$\to x = \underbrace{P^{-1}N}_{B}x + \underbrace{P^{-1}b}_{f} \to x = Bx + f$$

La forma generica è della forma

$$\boxed{x^{(k+1)} = Bx^{(k)} + f}$$ METODO ITERATIVO LINEARE STAZIONARIO DEL 1° ORDINE

Considerando la decomposizione

$$A = D - L - U$$

dove $D = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & 0 \\ & 0 & \ddots & \\ & & & a_{nn} \end{bmatrix}$ $L = \begin{bmatrix} 0 & & & \\ -a_{21} & 0 & & 0 \\ -a_{31} & -a_{32} & \ddots & \\ -a_{n1} & \cdots & -a_{nn-1} & 0 \end{bmatrix}$ $U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ & \ddots & & \vdots \\ & 0 & \ddots & -a_{n-1n} \\ & & & 0 \end{bmatrix}$

si possono avere i due metodi:

- $\boxed{\text{Jacobi}}$ $\to P = D \quad N = L + U$

  and $x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$

  cioè $\boxed{B_J = D^{-1}(L+U)}$ e $\boxed{f_J = D^{-1}b}$

- $\boxed{\text{Gauss - Seidel}}$ $\to P = D - L \quad N = U$

  Quindi $x^{(k+1)} = (D-L)^{-1}Ux^{(k)} + (D-L)^{-1}b$

  cioè $\boxed{B_{GS} = (D-L)^{-1}U}$ e $\boxed{f_{GS} = (D-L)^{-1}b}$

Th: Condizione necessaria e sufficiente per la convergenza di un metodo iterativo lineare è che il raggio spettrale $\rho(B)$ della matrice B sia minore di 1.

① Dimostrare che il metodo di Gauss-Seidel applicato al sistema $Ax=b$ con

$$A=\begin{bmatrix} 3 & 2 & -1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{bmatrix} \quad e \quad b=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

è convergente. Calcolare i primi due passi del metodo partendo dal vettore

$$x^{(0)}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

<u>Soluzione</u>: Individuiamo le matrici D, L, U:

$$D=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad L=\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \qquad U=\begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Nel metodi di G-S si ha $B_{GS}=(D-L)^{-1}U$.
Dobbiamo trovare quindi l'inversa di $D-L$

$$D-L=\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

che si trova risolvendo i tre sistemi lineari

$$(D-L)x=e_i \quad \text{con } i=1,2,3.$$

1° sistema:

$$\begin{cases} 3x_1 = 1 & \rightarrow x_1 = 1/3 \\ x_1+3x_2 = 0 & \rightarrow 1/3+3x_2=0 \rightarrow 3x_2=-\frac{1}{3} \rightarrow x_2=-\frac{1}{9} \\ 2x_2+3x_3=0 & \rightarrow -\frac{2}{9}+3x_3=0 \rightarrow 3x_3=\frac{2}{9} \rightarrow x_3=\frac{2}{27} \end{cases}$$

la 1ª colonna dell'inversa è quindi $\begin{bmatrix} 1/3 \\ -1/9 \\ 2/27 \end{bmatrix}$

2^o sistema:

$$\begin{cases} 3x_1 = 0 \to x_1 = 0 \\ x_1 + 3x_2 = 1 \to 3x_2 = 1 \to x_2 = 1/3 \\ 2x_2 + 3x_3 = 0 \to \frac{2}{3} + 3x_3 = 0 \to 3x_3 = -\frac{2}{3} \to x_3 = -\frac{2}{9} \end{cases}$$

la 2^a colonna dell'inversa è quindi $\begin{bmatrix} 0 \\ 1/3 \\ -2/9 \end{bmatrix}$

3^o sistema:

$$\begin{cases} 3x_1 = 0 \to x_1 = 0 \\ x_1 + 3x_2 = 0 \to x_2 = 0 \\ 2x_2 + 3x_3 = 1 \to 3x_3 = 1 \to x_3 = \frac{1}{3} \end{cases}$$

la 3^a colonna dell'inversa è quindi $\begin{bmatrix} 0 \\ 0 \\ 1/3 \end{bmatrix}$

Si ha:

$$(D-L)^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/9 & 1/3 & 0 \\ 2/27 & -\frac{2}{9} & 1/3 \end{bmatrix}$$

Quindi $B_{GS} = (D-L)^{-1} \cdot U = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/9 & 1/3 & 0 \\ 2/27 & -2/9 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} =$

$$= \begin{bmatrix} 0 & -2/3 & 1/3 \\ 0 & +2/9 & 2/9 \\ 0 & -4/27 & -4/27 \end{bmatrix}$$

Calcoliamo gli autovalori:

$$\det(B_{GS} - \lambda I) = \begin{bmatrix} -\lambda & -2/3 & 1/3 \\ 0 & 2/9-\lambda & 2/9 \\ 0 & -4/27 & -4/27-\lambda \end{bmatrix} =$$

$$= -\lambda \left[\left(\frac{2}{9} - \lambda \right) \left(-\frac{4}{27} - \lambda \right) + \frac{8}{243} \right] = -\lambda \left(-\cancel{\frac{8}{243}} + \frac{4}{27}\lambda - \frac{2}{9}\lambda + \lambda^2 + \cancel{\frac{8}{243}} \right)$$

$$= -\lambda \left(\lambda^2 - \frac{2}{27}\lambda \right) = -\lambda^2 \left(\lambda - \frac{2}{27} \right) = 0$$

$\lambda^2 = 0 \to \lambda = 0 \quad (\times 2)$

$\lambda - \frac{2}{27} = 0 \to \lambda = \frac{2}{27}$

Quindi $\sigma(B_{GS}) = \{0, 0, \frac{2}{27}\}$ e $\rho(B_{GS}) = \frac{2}{27}$

che è minore di 1 quindi il metodo converge.

Applichiamo il metodo partendo dal vettore

$$x^{(0)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

1° passo: $x^{(1)} = B_{GS} x^{(0)} + f_{GS}$

Calcoliamo $f_{GS} = (D-L)^{-1} \cdot b =$

$$= \begin{bmatrix} 1/3 & 0 & 0 \\ -1/9 & 1/3 & 0 \\ 2/27 & -2/9 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/9 \\ -\frac{7}{27} \end{bmatrix}$$

Quindi

$$x^{(1)} = \begin{bmatrix} 0 & -2/3 & 1/3 \\ 0 & 2/9 & 2/9 \\ 0 & -4/27 & -4/27 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/9 \\ -7/27 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/9 \\ 4/27 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/9 \\ -7/27 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/3 \\ -1/9 \end{bmatrix}$$

2° passo: $x^{(2)} = B_{GS} x^{(1)} + f_{GS}$

Qnd:

$$x^{(2)} = \begin{bmatrix} 0 & -2/3 & 1/3 \\ 0 & 2/9 & 2/9 \\ 0 & -4/27 & -4/27 \end{bmatrix} \begin{bmatrix} 1 \\ -1/3 \\ -1/9 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/9 \\ -7/27 \end{bmatrix} = \begin{bmatrix} 5/27 \\ -8/81 \\ 16/243 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/9 \\ -7/27 \end{bmatrix} = \begin{bmatrix} 14/27 \\ -17/81 \\ -47/243 \end{bmatrix}$$

② $A = \begin{bmatrix} 2 & 0 & 0 & \gamma \\ 0 & \gamma & 1 & 0 \\ 0 & 1 & \gamma & 0 \\ \gamma & 0 & 0 & 2 \end{bmatrix}$ trovare γ: ①

① A non singolare

② A def. positiva

③ Jacobi e Gauss-Seidel convergono

① $\det(A) = 2 \det \begin{bmatrix} \gamma & 1 & 0 \\ 1 & \gamma & 0 \\ 0 & 0 & 2 \end{bmatrix} - \gamma \det \begin{bmatrix} 0 & 0 & \gamma \\ \gamma & 1 & 0 \\ 1 & \gamma & 0 \end{bmatrix} =$

$= 2 \cdot 2(\gamma^2 - 1) - \gamma \cdot \gamma(\gamma^2 - 1) =$

$4(\gamma^2 - 1) - \gamma^2(\gamma^2 - 1) = (\gamma^2 - 1)(\gamma^2 - 4) \neq 0$

$\gamma \neq \pm 1 \quad \gamma \neq \pm 2$

② $\sigma(A) > 0 \quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 & \gamma \\ 0 & \gamma-\lambda & 1 & 0 \\ 0 & 1 & \gamma-\lambda & 0 \\ \gamma & 0 & 0 & 2-\lambda \end{vmatrix} =$

$= (2-\lambda) \begin{vmatrix} \gamma-\lambda & 1 & 0 \\ 1 & \gamma-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} - \gamma \begin{vmatrix} 0 & 0 & \gamma \\ \gamma-\lambda & 1 & 0 \\ 1 & \gamma-\lambda & 0 \end{vmatrix} =$

$= (2-\lambda)^2[(\gamma-\lambda)^2 - 1] - \gamma^2[(\gamma-\lambda)^2 - 1] = [(\gamma-\lambda)^2 - 1] \cdot [(2-\lambda)^2 - \gamma^2] =$

$= [(\gamma-\lambda-1)(\gamma-\lambda+1)][(2-\lambda-\gamma)(2-\lambda+\gamma)] = 0$

$\gamma - \lambda - 1 = 0 \rightarrow \lambda = \gamma - 1$

$\gamma - \lambda + 1 = 0 \rightarrow \lambda = \gamma + 1$

$2 - \lambda - \gamma = 0 \rightarrow \lambda = 2 - \gamma$

$2 - \lambda + \gamma = 0 \rightarrow \lambda = 2 + \gamma$

$\sigma(A) = \{\gamma \pm 1, 2 \pm \gamma\}$

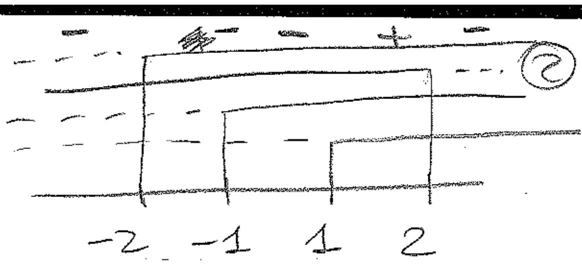
La matrice è definita positiva se tutti gli autovalori sono maggiori di zero.

$\gamma - 1 > 0 \rightarrow \gamma > 1$

$\gamma + 1 > 0 \rightarrow \gamma > -1$

$2 - \gamma > 0 \rightarrow -\gamma > -2 \rightarrow \gamma < 2$

$2 + \gamma > 0 \rightarrow \gamma > -2$



$1 < \gamma < 2$

③
Jacobi

$A = D - L - U$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -\gamma & 0 & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 0 & 0 & -\gamma \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$P = D \qquad N = L + U$

$B_J = D^{-1}(L+U)$

$$D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma} & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Infatti D è diagonale

$$L + U = \begin{bmatrix} 0 & 0 & 0 & -\gamma \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -\gamma & 0 & 0 & 0 \end{bmatrix}$$

$$B_J = \begin{bmatrix} 0 & 0 & 0 & -\frac{\gamma}{2} \\ 0 & 0 & -\frac{1}{\gamma} & 0 \\ 0 & -\frac{1}{\gamma} & 0 & 0 \\ -\frac{\gamma}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\det(B_J - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & -\frac{\gamma}{2} \\ 0 & -\lambda & -\frac{1}{\gamma} & 0 \\ 0 & -\frac{1}{\gamma} & -\lambda & 0 \\ -\frac{\gamma}{2} & 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -\frac{1}{\gamma} & 0 \\ -\frac{1}{\gamma} & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} + \frac{\gamma}{2} \begin{vmatrix} 0 & 0 & -\frac{\gamma}{2} \\ -\lambda & -\frac{1}{\gamma} & 0 \\ -\frac{1}{\gamma} & -\lambda & 0 \end{vmatrix} =$$

$$= \lambda^2\left(\lambda^2 - \frac{1}{\gamma^2}\right) - \frac{\gamma^2}{4}\left(\lambda^2 - \frac{1}{\gamma^2}\right) = \left(\lambda^2 - \frac{1}{\gamma^2}\right)\left(\lambda^2 - \frac{\gamma^2}{4}\right) =$$

$$= \left(\lambda - \frac{1}{\gamma}\right)\left(\lambda + \frac{1}{\gamma}\right)\left(\lambda - \frac{\gamma}{2}\right)\left(\lambda + \frac{\gamma}{2}\right) = 0$$ ③

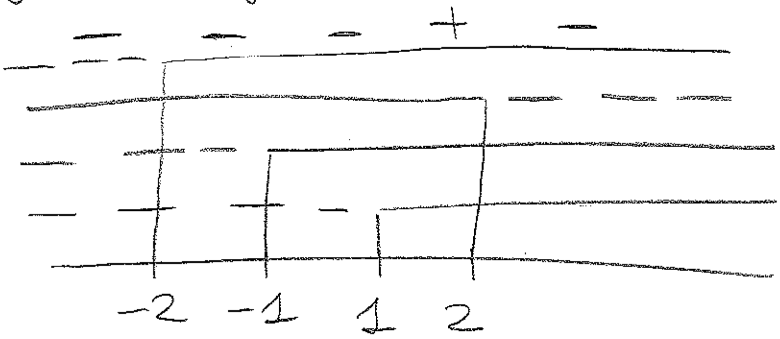
$\lambda = \frac{1}{\gamma} \quad \lambda = -\frac{1}{\gamma} \quad \lambda = \frac{\gamma}{2} \quad \lambda = -\frac{\gamma}{2} \qquad \sigma(B_J) = \left\{ \pm\frac{1}{\gamma}, \pm\frac{\gamma}{2} \right\}$

Il metodo converge se tutti gli autovalori sono minori di 1 in modulo

$\frac{1}{\gamma} < 1 \qquad -\frac{1}{\gamma} < 1 \qquad \frac{\gamma}{2} < 1 \qquad -\frac{\gamma}{2} < 1$

$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$

$\gamma > 1 \qquad \gamma > -1 \qquad \gamma < 2 \qquad \gamma > -2$



$1 < \gamma < 2$

Gauss-Seidel $\qquad B_{GS} = (D - L)^{-1} \cdot U$

$$D - L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 1 & \gamma & 0 \\ \gamma & 0 & 0 & 2 \end{bmatrix} \qquad U = \begin{bmatrix} 0 & 0 & 0 & -\gamma \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(D-L)^{-1}$ si trova risolvendo i 4 sistemi lineari

$$(D-L)x = e_i \quad , i = 1, 2, 3, 4$$

1º sistema:

$$\begin{cases} 2x_1 = 1 & \rightarrow x_1 = \frac{1}{2} \\ \gamma x_2 = 0 & \rightarrow x_2 = 0 \\ x_2 + \gamma x_3 = 0 & \rightarrow x_3 = 0 \\ \gamma x_1 + 2x_4 = 0 & \rightarrow \frac{1}{2}\gamma + 2x_4 = 0 \rightarrow 2x_4 = -\frac{\gamma}{2} \rightarrow x_4 = -\frac{\gamma}{4} \end{cases}$$

La 1^a colonna dell'inversa è quindi $\begin{bmatrix} 1/2 \\ 0 \\ 0 \\ -\gamma/4 \end{bmatrix}$ ④

2^o sistema:

$$\begin{cases} 2x_1 = 0 & \to x_1 = 0 \\ \gamma x_2 = 1 & \to x_2 = 1/\gamma \\ x_2 + \gamma x_3 = 0 & \to \frac{1}{\gamma} + \gamma x_3 = 0 \to \gamma x_3 = -\frac{1}{\gamma} \to x_3 = -\frac{1}{\gamma^2} \\ \gamma x_1 + 2x_4 = 0 & \to x_4 = 0 \end{cases}$$

La 2^a colonna dell'inversa è quindi $\begin{bmatrix} 0 \\ 1/\gamma \\ -1/\gamma^2 \\ 0 \end{bmatrix}$

3^o sistema:

$$\begin{cases} 2x_1 = 0 & \to x_1 = 0 \\ \gamma x_2 = 0 & \to x_2 = 0 \\ x_2 + \gamma x_3 = 1 & \to x_3 = 1/\gamma \\ \gamma x_1 + 2x_4 = 0 & \to x_4 = 0 \end{cases}$$

3^a colonna dell'inversa $\begin{bmatrix} 0 \\ 0 \\ 1/\gamma \\ 0 \end{bmatrix}$

4^o sistema:

$$\begin{cases} 2x_1 = 0 & \to x_1 = 0 \\ \gamma x_2 = 0 & \to x_2 = 0 \\ x_2 + \gamma x_3 = 0 & \to x_3 = 0 \\ \gamma x_1 + 2x_4 = 1 & \to x_4 = 1/2 \end{cases}$$

4^a colonna dell'inversa $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$

Quindi

$$(D-L)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/\gamma & 0 & 0 \\ 0 & -1/\gamma^2 & 1/\gamma & 0 \\ -\gamma/4 & 0 & 0 & 1/2 \end{bmatrix}$$

(3) $A = \begin{bmatrix} \gamma & \frac{1}{2} & 0 \\ \frac{1}{2} & \gamma & \frac{1}{2} \\ 0 & \frac{1}{2} & \gamma \end{bmatrix}$

Determinare γ tale che
1) A non singolare
2) A def positiva
3) Jacobi converge

(1) A è non singolare se il suo determinante è $\neq 0$:

$$\det(A) = \begin{vmatrix} \gamma & \frac{1}{2} & 0 \\ \frac{1}{2} & \gamma & \frac{1}{2} \\ 0 & \frac{1}{2} & \gamma \end{vmatrix} = \gamma\left(\gamma^2 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2}\gamma\right) =$$

$$= \gamma\left(\gamma^2 - \frac{1}{4} - \frac{1}{4}\right) = \gamma\left(\gamma^2 - \frac{1}{2}\right) \neq 0$$

$$\gamma \neq 0$$

$$\gamma^2 - \frac{1}{2} \neq 0 \rightarrow \gamma^2 \neq \frac{1}{2} \rightarrow \gamma \neq \pm\frac{1}{\sqrt{2}}$$

(2) A è definita positiva se i suoi autovalori sono >0:

$$\det(A - \lambda I) = \begin{vmatrix} \gamma-\lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & \gamma-\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & \gamma-\lambda \end{vmatrix} = (\gamma-\lambda)\left[(\gamma-\lambda)^2 - \frac{1}{4}\right] - \frac{1}{2}\left[\frac{1}{2}(\gamma-\lambda)\right] =$$

$$= (\gamma-\lambda)\left[(\gamma-\lambda)^2 - \frac{1}{4} - \frac{1}{4}\right] =$$

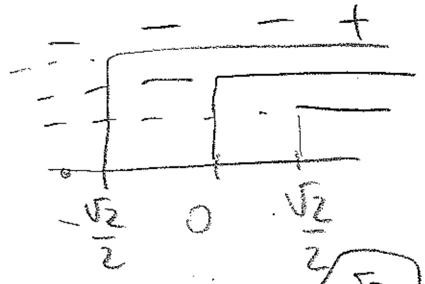
$$= (\gamma-\lambda)\left[(\gamma-\lambda)^2 - \frac{1}{2}\right] = 0$$

$$\gamma - \lambda = 0 \rightarrow \lambda = \gamma$$

$$(\gamma-\lambda)^2 - \frac{1}{2} = 0 \rightarrow (\gamma-\lambda)^2 = \frac{1}{2} \rightarrow \gamma - \lambda = \pm\frac{\sqrt{2}}{2} \rightarrow$$

$$\rightarrow \lambda = \gamma \pm \frac{\sqrt{2}}{2}$$

$$\sigma(A) = \left\{\gamma, \gamma - \frac{\sqrt{2}}{2}, \gamma + \frac{\sqrt{2}}{2}\right\}$$



$\gamma > 0$ $\quad \gamma - \frac{\sqrt{2}}{2} > 0 \rightarrow \gamma > \frac{\sqrt{2}}{2}$

$\gamma + \frac{\sqrt{2}}{2} > 0 \rightarrow \gamma > -\frac{\sqrt{2}}{2}$

$$\boxed{\gamma > \frac{\sqrt{2}}{2}}$$

$B_J \overset{③}{=} D^{-1} \cdot (L+U)$ ②

$$D = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \to D^{-1} \begin{bmatrix} 1/\gamma & 0 & 0 \\ 0 & 1/\gamma & 0 \\ 0 & 0 & 1/\gamma \end{bmatrix}$$

Infatti D è diagonale

$$L+U = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$B_J = \begin{bmatrix} 0 & -\frac{1}{2\gamma} & 0 \\ -\frac{1}{2\gamma} & 0 & -\frac{1}{2\gamma} \\ 0 & -\frac{1}{2\gamma} & 0 \end{bmatrix}$$

$$\det(B_J - \lambda I) = \begin{vmatrix} \boxed{\begin{matrix} -\lambda \\ -\frac{1}{2\gamma} \\ 0 \end{matrix}} & \begin{matrix} -\frac{1}{2\gamma} \\ -\lambda \\ -\frac{1}{2\gamma} \end{matrix} & \begin{matrix} 0 \\ -\frac{1}{2\gamma} \\ -\lambda \end{matrix} \end{vmatrix} =$$

$$= -\lambda\left(\lambda^2 - \frac{1}{4\gamma^2}\right) + \frac{1}{2\gamma}\left(\frac{\lambda}{2\gamma}\right) = -\lambda\left(\lambda^2 - \frac{1}{4\gamma^2} - \frac{1}{4\gamma^2}\right) =$$

$$= -\lambda\left(\lambda^2 - \frac{1}{2\gamma^2}\right) = 0 \qquad \lambda = 0$$

$$\lambda^2 - \frac{1}{2\gamma^2} = 0 \to \lambda^2 = \frac{1}{2\gamma^2} \to \lambda = \pm\frac{1}{\gamma\sqrt{2}}$$

$\sigma(B_J) = \left\{0, \pm\frac{1}{\gamma\sqrt{2}}\right\}$. Gli autovalori devono essere <1:

$$\frac{1}{\gamma\sqrt{2}} < 1 \qquad -\frac{1}{\gamma\sqrt{2}} < 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$\gamma\sqrt{2} > 1 \qquad \gamma\sqrt{2} > -1 \qquad \to \gamma > \frac{1}{\sqrt{2}} \quad > \quad \gamma > \frac{1}{\sqrt{2}}$$

$$\to \gamma > -\frac{1}{\sqrt{2}}$$

Fissati $\gamma = 2$ $\quad b = \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix}$ $\quad x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ③

Calcolare con il metodo di Jacobi $x^{(1)}, x^{(2)}$

$$A = \begin{pmatrix} 2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & 2 \end{pmatrix}$$

$$x^{(k+1)} = B_J x^{(k)} + f$$

$$B_J = D^{-1}(L+U)$$

$$f = D^{-1} b$$

$$B_J = \begin{pmatrix} 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 0 \end{pmatrix} \qquad D^{-1} b = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 \\ 3/2 \\ 2 \end{pmatrix}$$

Primo passo:

$$x^{(1)} = B_J x^{(0)} + f = \begin{pmatrix} 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 3/2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 3/2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3/2 \\ 2 \end{pmatrix}$$

Secondo passo:

$$x^{(2)} = B_J x^{(1)} + f = \begin{pmatrix} -\frac{3}{8} \\ -\frac{3}{2} \\ -\frac{3}{8} \end{pmatrix} + \begin{pmatrix} 4 \\ 3/2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{29}{8} \\ 0 \\ \frac{13}{8} \end{pmatrix}$$