# Algoritmi Numerici per l'Ingegneria
## Metodi alle differenze finite per equazioni ellittiche

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Sia

$$\Omega = \{(x, y) \quad t.c. \quad a \leq x \leq b, \qquad c \leq y \leq d\},$$

e si consideri il seguente problema ellittico

$$\begin{cases} u_{xx} + u_{yy} + p(x, y)u_x + q(x, y)u_y + r(x, y)u + s(x, y) = 0, \quad (x, y) \in \Omega \\ u(a, y) = f_1(y), \quad y \in [c, d] \\ u(b, y) = f_2(y), \quad y \in [c, d] \\ u(x, c) = g_1(x), \quad x \in [a, b] \\ u(x, d) = g_2(x), \quad x \in [a, b] \end{cases}$$

dove $p$, $q$, $r$, $s$, $f_0$, $f_1$, $g_0$ e $g_1$ sono funzioni assegnate e $u(x, y)$ è la funzione incognita.

Posto

$$x_0 = a, \qquad x_i = x_0 + ih,\ i = 1, ..., n \qquad x_{n+1} = b,\ h = x_{j+1} - x_j$$
$$y_0 = c, \qquad y_j = y_0 + jk,\ j = 1, ..., m \qquad y_{m+1} = d,\ k = y_{j+1} - y_j$$

uno schema di discretizzazione alle differenze finite centrali (schema a 5 punti) conduce al seguente problema

$$\begin{cases} c_{ij}\, u_{i,j-1} + b_{ij}\, u_{i-1,j} + a_{ij}\, u_{i,j} + \widehat{b_{ij}} u_{i+1,j} + \widehat{c_{ij}} u_{i,j+1} = -s(x_i, y_j), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, ..., n, j = 1, ..., m \\ u_{0,j} = f_1(y_j), \qquad u_{n+1,j} = f_2(y_j), \quad j = 1, ..., m \\ u_{i,0} = g_1(x_i), \qquad u_{i,m+1} = g_2(x_i), \quad i = 1, ..., n \end{cases}$$

dove

$$a_{ij} = -\frac{2}{h^2} - \frac{2}{k^2} + r(x_i, y_j)$$
$$b_{ij} = \frac{1}{h^2} - \frac{p(x_i, y_j)}{2h}, \qquad \widehat{b}_{ij} = \frac{1}{h^2} + \frac{p(x_i, y_j)}{2h}$$
$$c_{ij} = \frac{1}{k^2} - \frac{q(x_i, y_j)}{2k}, \qquad \widehat{c}_{ij} = \frac{1}{k^2} + \frac{q(x_i, y_j)}{2k}$$

Il sistema $n \times m$ ottenuto è un sistema pentadiagonale.
Ad esempio, nel caso $n = 4$ ed $m = 3$ il sistema è del tipo

$$A\mathbf{u} = \mathbf{b}$$

dove

$$A = \left[\begin{array}{cccc|cccc|cccc}
a_{11} & \hat{b}_{11} & 0 & 0 & \hat{c}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_{21} & a_{21} & \hat{b}_{21} & 0 & 0 & \hat{c}_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_{31} & a_{31} & \hat{b}_{31} & 0 & 0 & \hat{c}_{31} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b_{41} & a_{41} & 0 & 0 & 0 & \hat{c}_{41} & 0 & 0 & 0 & 0 \\
\hline
c_{12} & 0 & 0 & 0 & a_{12} & \hat{b}_{12} & 0 & 0 & \hat{c}_{12} & 0 & 0 & 0 \\
0 & c_{22} & 0 & 0 & b_{22} & a_{22} & \hat{b}_{22} & 0 & 0 & \hat{c}_{22} & 0 & 0 \\
0 & 0 & c_{32} & 0 & 0 & b_{32} & a_{32} & \hat{b}_{32} & 0 & 0 & \hat{c}_{32} & 0 \\
0 & 0 & 0 & c_{42} & 0 & 0 & b_{42} & a_{42} & 0 & 0 & 0 & \hat{c}_{42} \\
\hline
0 & 0 & 0 & 0 & c_{13} & 0 & 0 & 0 & a_{13} & \hat{b}_{13} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{23} & 0 & 0 & b_{23} & a_{23} & \hat{b}_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{33} & 0 & 0 & b_{33} & a_{33} & \hat{b}_{33} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{43} & 0 & 0 & b_{43} & a_{43}
\end{array}\right]$$

$$
\mathbf{u} = \left[\begin{array}{c}
u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ \hline
u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \\ \hline
u_{13} \\ u_{23} \\ u_{33} \\ u_{43}
\end{array}\right], \qquad
\mathbf{b} = \left[\begin{array}{l}
-s(x_1, y_1) - b_{11} f_1(y_1) - c_{11} g_1(x_1) \\
-s(x_2, y_1) \qquad\qquad\qquad - c_{21} g_1(x_2) \\
-s(x_3, y_1) \qquad\qquad\qquad - c_{31} g_1(x_3) \\
-s(x_4, y_1) - \widehat{b}_{41} f_2(y_1) - c_{41} g_1(x_4) \\ \hline
-s(x_1, y_2) - b_{12} f_1(y_2) \\
-s(x_2, y_2) \\
-s(x_3, y_2) \\
-s(x_4, y_2) - \widehat{b}_{42} f_2(y_2) \\ \hline
-s(x_1, y_3) - b_{13} f_1(y_3) - \widehat{c}_{13} g_2(x_1) \\
-s(x_2, y_3) \qquad\qquad\qquad - \widehat{c}_{23} g_2(x_2) \\
-s(x_3, y_3) \qquad\qquad\qquad - \widehat{c}_{33} g_2(x_3) \\
-s(x_4, y_3) - \widehat{b}_{43} f_2(y_3) - \widehat{c}_{43} g_2(x_4)
\end{array}\right]
$$